A Generalized Turan Problem and Its Applications

Abstract:

Our first theorem is a hierarchy theorem for the query complexity of testing graph properties with one-sided error; more precisely, we show that for every sufficiently fast-growing function \( f \) from \((0,1)\) to the natural numbers, there is a graph property whose one-sided-error query complexity is precisely \( f(\Theta(\epsilon)) \). No result of this type was previously known for any \( f \) which is super-polynomial. Goldreich \[ECCC 2005\] asked to exhibit a graph property whose query complexity is exponential in \( 1/\epsilon \). Our hierarchy theorem partially resolves this problem by exhibiting a property whose one-sided-error query complexity is exponential in \( 1/\epsilon \). We also use our hierarchy theorem in order to resolve a problem raised by Alon and Shapira \[STOC 2005\] regarding testing relaxed versions of bipartiteness.

Our second theorem states that for any function \( f \) there is a graph property whose one-sided-error query complexity is at least \( f(\epsilon) \) while its two-sided-error query complexity is only polynomial in \( 1/\epsilon \). This is the first indication of the surprising power that two-sided-error testing algorithms have over one-sided-error ones, even when restricted to properties that are testable with one-sided error. Again, no result of this type was previously known for any \( f \) that is super-polynomial.

The above theorems are derived from a graph theoretic result which we think is of independent interest, and might have further applications. Alon and Shikhelman \[JCTB 2016\] introduced the following generalized Turan problem: for fixed graphs \( H \) and \( T \), and an integer \( n \), what is the maximum number of copies of \( T \), denoted by \( \text{ex}(n,T,H) \), that can appear in an \( n \)-vertex \( H \)-free graph? This problem received a lot of attention recently, with an emphasis on \( T = C_3 \), \( H = C_{2m+1} \). Our third theorem gives tight bounds for \( \text{ex}(n, C_k, C_m) \) for all the remaining values of \( k \) and \( m \). Joint work with Asaf Shapira.