A Generalized Turan Problem and Its Applications

Abstract:
Our first theorem is a hierarchy theorem for the query complexity of testing graph properties with one-sided error; more precisely, we show that for every sufficiently fast-growing function $f$ from $(0,1)$ to the natural numbers, there is a graph property whose one-sided-error query complexity is precisely $f(\Theta(\epsilon))$. No result of this type was previously known for any $f$ which is super-polynomial. Goldreich [ECCC 2005] asked to exhibit a graph property whose query complexity is exponential in $1/\epsilon$. Our hierarchy theorem partially resolves this problem by exhibiting a property whose one-sided-error query complexity is exponential in $1/\epsilon$. We also use our hierarchy theorem in order to resolve a problem raised by Alon and Shapira [STOC 2005] regarding testing relaxed versions of bipartiteness.

Our second theorem states that for any function $f$ there is a graph property whose one-sided-error query complexity is at least $f(\epsilon)$ while its two-sided-error query complexity is only polynomial in $1/\epsilon$. This is the first indication of the surprising power that two-sided-error testing algorithms have over one-sided-error ones, even when restricted to properties that are testable with one-sided error. Again, no result of this type was previously known for any $f$ that is super-polynomial.

The above theorems are derived from a graph theoretic result which we think is of independent interest, and might have further applications. Alon and Shikhelman [JCTB 2016] introduced the following generalized Turan problem: for fixed graphs $H$ and $T$, and an integer $n$, what is the maximum number of copies of $T$, denoted by $ex(n,T,H)$, that can appear in an $n$-vertex $H$-free graph? This problem received a lot of attention recently, with an emphasis on $T = C_3$, $H=C_{2m+1}$. Our third theorem gives tight bounds for $ex(n,C_k,C_m)$ for all the remaining values of $k$ and $m$. Joint work with Asaf Shapira.