On linear threshold functions and local Chernoff inequalities

Abstract:

A linear threshold function (LTF) is a Boolean function \( f: \{-1,1\}^n \rightarrow \{0,1\} \) of the form \( f(x) = \mathbf{1}_{\sum a_i x_i > t} \), for some fixed coefficients \( a_i \) and some threshold \( t \). LTFs play an important role in complexity theory, machine learning, and other fields.

In this talk we present a new approach that allows us obtaining sharp results on Fourier-theoretic properties of biased LTFs. In particular, we determine the exact asymptotic order of the total influence and of the degree-1 Fourier weight of any biased LTF, in terms of its maximal (normalized) coefficient and its expectation. This provides a sharp generalization of theorems proved by Matulef, O'Donnell, Rubinfeld, and Servedio (in the context of property testing), and settles a conjecture posed by Kalai et al.

Our main tools are 'local' forms of the classical Chernoff inequality, like the following one proved by Devroye and Lugosi (2008): Let \( \{x_i\} \) be independent random variables uniformly distributed in \( \{-1, 1\} \), and let \( a_i \) be nonnegative numbers such that \( \sum a_i^2 = 1 \). If for some \( t > 0 \), we have \( \Pr[\sum a_i x_i > t] = b \), then \( \Pr[\sum a_i x_i > t + \delta] < b/2 \) holds for \( \delta < c/\sqrt{\log(1/b)} \), where \( c \) is a universal constant. Such inequalities seem to be little-known and probably can be useful in other contexts as well.

Joint work with Ohad Klein.