On linear threshold functions and local Chernoff inequalities

Abstract:

A linear threshold function (LTF) is a Boolean function $f: \{-1,1\}^n \to \{0,1\}$ of the form $f(x) = 1_{\sum a_i x_i > t}$, for some fixed coefficients $a_i$ and some threshold $t$. LTFs play an important role in complexity theory, machine learning, and other fields.

In this talk we present a new approach that allows us obtaining sharp results on Fourier-theoretic properties of biased LTFs. In particular, we determine the exact asymptotic order of the total influence and of the degree-1 Fourier weight of any biased LTF, in terms of its maximal (normalized) coefficient and its expectation. This provides a sharp generalization of theorems proved by Matulef, O'Donnell, Rubinfeld, and Servedio (in the context of property testing), and settles a conjecture posed by Kalai et al.

Our main tools are 'local' forms of the classical Chernoff inequality, like the following one proved by Devroye and Lugosi (2008): Let $\{x_i\}$ be independent random variables uniformly distributed in $\{-1,1\}$, and let $a_i$ be nonnegative numbers such that $\sum a_i^2 = 1$. If for some $t > 0$, we have $\Pr[\sum a_i x_i > t] = b$, then $\Pr[\sum a_i x_i > t + \delta] < b/2$ holds for $\delta < c/\sqrt{\log(1/b)}$, where $c$ is a universal constant. Such inequalities seem to be little-known and probably can be useful in other contexts as well.

Joint work with Ohad Klein.