Lower bounds on the size of semi-definite programs

Abstract:

Much of the theory of mathematical programs for combinatorial optimization can be described in the following way: A polytope of interest has exponentially many (in the dimension) facets, but can be written as the linear projection of a simpler convex body in a higher-dimensional space. Simple might mean a polytope with a much smaller number of facets, or a spectrahedron (the intersection of an affine subspace with the PSD cone) of small dimension. This allows one to optimize linear functionals over the base polytope by instead optimizing a lifted functional over the lifted body.

Unless $P=NP$, one does not expect certain polytopes--like the convex hull of indicators of traveling salesman tours in a graph--to have a small lift. But it remained open to prove any non-trivial lower bound on the necessary dimension for a spectrahedral lift, i.e. to prove that semi-definite programs do not yield efficient optimization procedures over these polytopes.

We show that the cut, TSP, and stable set polytopes on $n$-vertex graphs are not the linear image of a spectrahedron of dimension less than $\exp(n^{\epsilon})$ for some constant $\epsilon > 0$. In the process, many interesting phenomena emerge: Factorization of operators through the PSD cone, quantum information theory, discrete Fourier analysis, and real algebraic geometry.

This is based joint work with Prasad Ragahvendra and David Steurer.