Oka Principles and the Linearization Problem

Abstract:

Let \( Q \) be a Stein space and \( L \) a complex Lie group. Then Grauert’s Oka Principle states that the canonical map of the isomorphism classes of holomorphic principle \( L \)-bundles over \( Q \) to the isomorphism classes of topological principle \( L \)-bundles over \( Q \) is an isomorphism. In particular he showed that if \( P, P' \) are holomorphic principle \( L \)-bundles and \( \Phi : P \to P' \) a topological isomorphism, then there is a homotopy \( \Phi_t \) of topological isomorphisms with \( \Phi_0 = \Phi \) and \( \Phi_1 : P \to P' \) a holomorphic isomorphism.

Let \( X \) and \( Y \) be Stein \( G \)-manifolds where \( G \) is a reductive complex Lie group. Then there is a quotient Stein space \( Q_X \), and a morphism \( \pi_X : X \to Q_X \) such that \( (\pi_X)^* \mathcal{O}(Q_X) = \mathcal{O}(X)^G \). Similarly we have \( p_Y : Y \to Q_Y \).

Suppose that \( \Phi : X \to Y \) is a \( G \)-biholomorphism. Then the induced mapping \( \phi : Q_X \to Q_Y \) has the following property: for any \( z \in Q_X \), \( X_z := \pi_X^{-1}(z) \) is \( G \)-isomorphic to \( Y_{\#(z)} \) (the fibers are actually affine \( G \)-varieties). We say that \( \phi \) is admissible. Now given an admissible \( \phi \), assume that we have a \( G \)-equivariant homeomorphism \( \Phi : X \to Y \) lifting \( \phi \). Our goal is to establish an Oka principle, saying that \( \phi \) has a deformation \( \Phi_t \) with \( \Phi_0 = \Phi \) and \( \Phi_1 \) biholomorphic.

We establish this in two main cases. One case is where \( \Phi \) is a diffeomorphism that restricts to \( G \)-isomorphisms on the reduced fibers of \( \pi_X \) and \( \pi_Y \). The other case is where \( \Phi \) restricts to \( G \)-isomorphisms on the fibers and \( X \) satisfies an auxiliary condition, which usually holds. Finally, we give applications to the Holomorphic Linearization Problem. Let \( G \) act holomorphically on \( X = \mathbb{C}^n \). When is there a change of coordinates such that the action of \( G \) becomes linear? We prove that this is true, for \( X \) satisfying the same auxiliary condition as before, if and only if the quotient \( Q_X \) is admissibly biholomorphic to the quotient of a \( G \)-module \( V \).