Oka Principles and the Linearization Problem

Abstract:

Let $Q$ be a Stein space and $L$ a complex Lie group. Then Grauert’s Oka Principle states that the canonical map of the isomorphism classes of holomorphic principle $L$-bundles over $Q$ to the isomorphism classes of topological principle $L$-bundles over $Q$ is an isomorphism. In particular he showed that if $P$, $P'$ are holomorphic principle $L$-bundles and $\Phi : P \to P'$ a topological isomorphism, then there is a homotopy $\Phi_t$ of topological isomorphisms with $\Phi_0 = \Phi$ and $\Phi_1 : P \to P'$ a holomorphic isomorphism.

Let $X$ and $Y$ be Stein $G$-manifolds where $G$ is a reductive complex Lie group. Then there is a quotient Stein space $Q_X$ and a morphism $\pi_X : X \to Q_X$ such that $(\pi_X)^* \mathcal{O}(Q_X) = \mathcal{O}(X)^G$. Similarly we have $p_Y : Y \to Q_Y$.

Suppose that $\Phi : X \to Y$ is a $G$-biholomorphism. Then the induced mapping $\phi : Q_X \to Q_Y$ has the following property: for any $z \in Q_X$, $X_z := \pi_X^{-1}(z)$ is $G$-isomorphic to $Y_{\phi(z)}$ (the fibers are actually affine $G$-varieties). We say that $\Phi$ is admissible. Now given an admissible $\Phi$, assume that we have a $G$-equivariant homeomorphism $\Phi : X \to Y$ lifting $\phi$. Our goal is to establish an Oka principle, saying that $\Phi$ has a deformation $\Phi_t$ with $\Phi_0 = \Phi$ and $\Phi_1$ biholomorphic.

We establish this in two main cases. One case is where $\Phi$ is a diffeomorphism that restricts to $G$-isomorphisms on the reduced fibers of $\pi_X$ and $\pi_Y$. The other case is where $\Phi$ restricts to $G$-isomorphisms on the fibers and $X$ satisfies an auxiliary condition, which usually holds. Finally, we give applications to the Holomorphic Linearization Problem. Let $G$ act holomorphically on $X = \mathbb{C}^n$. When is there a change of coordinates such that the action of $G$ becomes linear? We prove that this is true, for $X$ satisfying the same auxiliary condition as before, if and only if the quotient $Q_X$ is admissibly biholomorphic to the quotient of a $G$-module $\mathcal{V}$. 