Random graphs with constant $r$-balls

Abstract:

Let $F$ be a fixed infinite, vertex-transitive graph. We say a graph $G$ is `$r$-locally $F$' if for every vertex $v$ of $G$, the ball of radius $r$ and centre $v$ in $G$ is isometric to the ball of radius $r$ in $F$. For each positive integer $n$, let $G_n$ be a graph chosen uniformly at random from the set of all unlabelled, $n$-vertex graphs that are $r$-locally $F$. We investigate the properties that the random graph $G_n$ has with high probability --- i.e., how these properties depend on the fixed graph $F$.

We show that if $F$ is a Cayley graph of a torsion-free group of polynomial growth, then there exists a positive integer $r_0$ such that for every integer $r$ at least $r_0$, with high probability the random graph $G_n = G_n(F,r)$ defined above has largest component of size between $n^{c_1}$ and $n^{c_2}$, where $0 < c_1 < c_2 < 1$ are constants depending upon $F$ alone, and moreover that $G_n$ has a rather large automorphism group. This contrasts sharply with the random $d$-regular graph $G_n(d)$ (which corresponds to the case where $F$ is replaced by the infinite $d$-regular tree).

Our proofs use a mixture of results and techniques from group theory, geometry and combinatorics.

We obtain somewhat more precise results in the case where $F$ is $L^d$ (the standard Cayley graph of $Z^d$): for example, we obtain quite precise estimates on the number of $n$-vertex graphs that are $r$-locally $L^d$, for $r$ at least linear in $d$.

Many intriguing open problems remain: concerning groups with torsion, groups with faster than polynomial growth, and what happens for more general structures than graphs.

This is joint work with Itai Benjamini (WIS).