Random graphs with constant r-balls

Abstract:

Let F be a fixed infinite, vertex-transitive graph. We say a graph G is `r-locally F' if for every vertex v of G, the ball of radius r and centre v in G is isometric to the ball of radius r in F. For each positive integer n, let $G_n$ be a graph chosen uniformly at random from the set of all unlabelled, n-vertex graphs that are r-locally F. We investigate the properties that the random graph $G_n$ has with high probability --- i.e., how these properties depend on the fixed graph F.

We show that if F is a Cayley graph of a torsion-free group of polynomial growth, then there exists a positive integer $r_0$ such that for every integer r at least $r_0$, with high probability the random graph $G_n = G_n(F, r)$ defined above has largest component of size between $n^{c_1}$ and $n^{c_2}$, where $0 < c_1 < c_2 < 1$ are constants depending upon F alone, and moreover that $G_n$ has a rather large automorphism group. This contrasts sharply with the random d-regular graph $G_n(d)$ (which corresponds to the case where F is replaced by the infinite d-regular tree).

Our proofs use a mixture of results and techniques from group theory, geometry and combinatorics.

We obtain somewhat more precise results in the case where F is $L^d$ (the standard Cayley graph of $Z^d$): for example, we obtain quite precise estimates on the number of n-vertex graphs that are r-locally $L^d$, for r at least linear in d.

Many intriguing open problems remain: concerning groups with torsion, groups with faster than polynomial growth, and what happens for more general structures than graphs.

This is joint work with Itai Benjamini (WIS).