Abstract:

In classical algebra, a commutative ring \( R \), or more generally, a scheme, has two related invariants: the group of units and the Picard group. The first consists of invertible elements in \( R \), while the second consists of invertible modules of \( R \). In higher algebra, one is interested in generalizations of abelian groups and commutative rings, which allow the underlying collection of elements to be a homotopy type rather than a set. Moreover, all the identities required in classical algebra should be replaced by (a hierarchy of) homotopies between the two sides of the identity; for instance, the associativity relation \((xy)z = x(yz)\) is replaced by a chosen homotopy between these two expressions, depending continuously on \( x, y, \) and \( z \).

Such a homotopical version of the category of abelian groups is the category of spectra, and the corresponding notion of commutative rings is that of commutative ring spectra. Accordingly, one can assign spectra of units and Picard spectra to commutative ring spectra, generalizing the classical units and Picard groups. However, some of the constructions carried out in classical algebra, such as the construction of the line bundle associated with an element of the Picard group of a scheme, require “strict” variants of these spectra, in which multiplication can be made genuinely commutative (not only up to homotopy). These variants are called the “strict units” and “strict Picard” spectra. However, despite their theoretical advantages, there are only a few cases in which the strict units and Picard spectra have been determined beyond classical commutative rings.

The sphere spectrum is one of the most important examples of a commutative ring spectrum. As the initial commutative ring spectrum, it plays a similar role to the ring of integers in classical algebra. Moreover, its homotopy groups are the stable homotopy groups of spheres, whose computation is the primary motivation for the development of stable homotopy theory.

In my talk, I will discuss the notions of spectra, commutative ring spectra, and their strict units and Picard spectra. Then, I will present a recent computation of the strict units and Picard spectra of the sphere spectrum and other related commutative ring spectra.