The values of quadratic forms on difference sets, measure rigidity and equidistribution

Abstract:

Given a quadratic form $Q$ in $d$ variables over the integers, e.g. $Q(x,y,z) = xy - z^2$, and a set of positive density $E$ in $Z^d$, we investigate what kind of structure can be found in the set $Q(E-E)$. We will see that if $d \geq 3$, and $Q$ is indefinite, then the measure rigidity, due to Bourgain-Furman-Lindenstrauss-Mozes or Benoist-Quint, of the action of the group of the symmetries of $Q$ implies that there exists $k \geq 1$ such that $k^2 Q(Z^d)$ is a subset of $Q(E-E)$.

We will give an alternative proof of the theorem for the case $Q(x,y,z) = xy - z^2$ that uses more classical equidistribution results of Vinogradov, and Weyl, as well as a more recent result by Frantzikinakis-Kra. The latter proof extends the theorem to other polynomials having a much smaller group of symmetries. Based on joint works with M. Bjorklund (Chalmers), and K. Bulinski (Sydney).