The values of quadratic forms on difference sets, measure rigidity and equidistribution

Abstract:

Given a quadratic form Q in d variables over the integers, e.g. \( Q(x,y,z) = xy - z^2 \), and a set of positive density \( E \) in \( \mathbb{Z}^d \), we investigate what kind of structure can be found in the set \( Q(E-E) \).

We will see that if \( d \geq 3 \), and \( Q \) is indefinite, then the measure rigidity, due to Bourgain-Furman-Lindenstrauss-Mozes or Benoist-Quint, of the action of the group of the symmetries of \( Q \) implies that there exists \( k \geq 1 \) such that \( k^2 Q(\mathbb{Z}^d) \) is a subset of \( Q(E-E) \).

We will give an alternative proof of the theorem for the case \( Q(x,y,z) = xy - z^2 \) that uses more classical equidistribution results of Vinogradov, and Weyl, as well as a more recent result by Frantzikinakis-Kra. The latter proof extends the theorem to other polynomials having a much smaller group of symmetries. Based on joint works with M. Bjorklund (Chalmers), and K. Bulinski (Sydney).