

ADIC FLOWS, TRANSVERSAL FLOWS, AND HOROCYCLE FLOWS

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ABSTRACT. We give a symbolic construction of Shunji Ito's "transversal flow for a subshift of finite type" [I], along the lines of Vershik's construction of an adic transformation. We then show how these flows arise naturally in the symbolic coding of horocycle flows on non-compact hyperbolic surfaces with finite area.

1. INTRODUCTION

This paper grew out of the study of the following classical constructions in dynamical systems.

The horocycle flow [He], [Ho]. In their study of the geodesic flow g^s on the unit tangent bundle T^1M of a hyperbolic surface M , Hedlund and Hopf used a continuous flow h^t with the following two properties:

- 1.1. Orbit property: $\{h^t(\omega) : t \in \mathbb{R}\} = \{\omega' \in T^1M : d(g^s(\omega'), g^s(\omega)) \xrightarrow{s \rightarrow \infty} 0\}$;
- 1.2. Renormalization Property: $g^{-s} \circ h^t \circ g^s = h^{te^s}$.

The flow h^t is called the (stable) *horocycle flow*. When lifted to the universal cover $T^1\mathbb{D}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, its orbits trace horocycles (circles tangent to $\partial\mathbb{D}$).

The adding machine. Let $\Sigma_2^+ := \{0, 1\}^{\mathbb{N} \cup \{0\}}$, equipped with the metric $d(\underline{y}, \underline{z}) := \exp[-\min\{i \geq 0 : y_i \neq z_i\}]$, and let $\sigma : \Sigma_2^+ \rightarrow \Sigma_2^+$ denote the left shift map $\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The *adding machine* (apparently first constructed by von Neumann) is the map $\tau : \Sigma_2^+ \rightarrow \Sigma_2^+$ given by

$$\tau : (\underbrace{1 \cdots 1}_n, 0, x_{n+1}, x_{n+2}, \dots) \mapsto (\underbrace{0 \cdots 0}_n, 1, x_{n+1}, x_{n+2}, \dots), \quad \tau : 1^\infty \mapsto 0^\infty.$$

(Addition of one with "carry to the right.") Again, we have

- 2.1. Orbit property: $\{\tau^n(\underline{x}) : n \in \mathbb{Z}\} = \{\underline{y} : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow{n \rightarrow \infty} 0\}$ for all \underline{x} not eventually constant;
- 2.2. Renormalization property: $\tau^m \circ \sigma^n = \sigma^n \circ \tau^{2^m m}$ ($m, n \in \mathbb{Z}$).

The adic transformation [V]. Vershik generalized von Neumann's construction to more general Markov compacta. We focus on the special case of subshifts of finite type ("stationary adic transformations"). We review the necessary definitions.

Subshifts of finite type. Let $N > 1$, $S := \{1, \dots, N\}$, and suppose $A := (t_{ij})_{N \times N}$ is a matrix of zeroes and ones without rows or columns made entirely of zeroes.

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The one-sided *subshift of finite type* with states S and transition matrix A is the left shift map $\sigma : \Sigma^+ \rightarrow \Sigma^+$ where

$$\Sigma^+ = \{(x_0, x_1, \dots) \in S^{\mathbb{N}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

We equip Σ^+ with the metric $d(\underline{y}, \underline{z}) := \exp[-\min\{i \geq 0 : y_i \neq z_i\}]$. It is well known that $h_{\text{top}}(\sigma) = \ln \lambda$ where λ is the leading eigenvalue of A (see e.g. [PP]).

Stable tail relation. One sees that $\{\underline{y} \in \Sigma^+ : d(\sigma^n \underline{x}, \sigma^n \underline{y}) \xrightarrow{n \rightarrow \infty} 0\} = \{\underline{y} : \underline{y} \sim \underline{x}\}$, where \sim is the *stable tail relation*,

$$(1.1) \quad (\underline{x} \sim \underline{y}) \iff \exists p \ (\forall i \geq p \ (x_i = y_i)).$$

Reverse lexicographic orders. We would like to define a total order on every equivalence class of \sim . To do this, fix for every $a \in S$ a total order \leq_a on S , and let

$$(1.2) \quad \underline{x} \prec_a \underline{y} \quad \text{if and only if} \quad \exists p \ (x_p \leq_{a_{p+1}} y_p \text{ and } \forall i \geq p+1 \ (x_i = y_i)).$$

We write \preceq for “ \prec or $=$ ”. The symbols \prec and \preceq mean the same thing.

If \leq_a does not depend on a and equals the natural order on S , then we get the *standard reverse lexicographic order*. If we define \leq_a to be the cyclic order on S with a as the minimal element, then we get the *cyclic reverse lexicographic order*.

Eventually extremal sequences. No matter what total orders \leq_a we choose, there are only finitely many \prec -maximal or \prec -minimal sequences, all periodic. To see this let $P_{\max}, P_{\min} : S \rightarrow S$ denote the functions

$$\begin{aligned} P_{\max}(a) &:= \max\{b \in S : t_{ba} = 1\} \\ P_{\min}(a) &:= \min\{b \in S : t_{ba} = 1\} \end{aligned} \quad \text{calculated using } \leq_a.$$

Every \prec -maximal (resp. \prec -minimal) point \underline{x} must satisfy $x_i = P_{\max}(x_{i+1})$ (resp. $x_i = P_{\min}(x_{i+1})$) for all i . Since S is finite, every iteration of P_{\max} and P_{\min} ends at a cycle. It follows that the extremal points for \prec are periodic and finite in number.

Sequences which are tail equivalent to a maximal or minimal sequence are called *eventually extremal*. Let

$$\Sigma_0^+ := \Sigma^+ \setminus \{\text{eventually extremal sequences}\}.$$

Vershik's adic transformation. This is $\tau : \Sigma_0^+ \rightarrow \Sigma_0^+$ given by

$$(1.3) \quad \tau(\underline{x}) := \min\{\underline{y} \sim \underline{x} : \underline{y} \succneq \underline{x}\}.$$

The definition depends on the choice of \prec . The *standard Vershik adic transformation* is obtained from the standard reverse lexicographic order. The *cyclic Vershik adic transformation* is obtained from the cyclic lexicographic order. In any case, we have

3.1. Orbit property: $\{\tau^n(\underline{x}) : x \in \mathbb{Z}\} = \{\underline{y} : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow{n \rightarrow \infty} 0\}$ for all \underline{x} non-eventually extremal;

3.2. Renormalization property: $\exists p_{m,n} : \Sigma_0^+ \rightarrow \mathbb{Z}$ s.t.

$$(\tau^m \circ \sigma^n)(\underline{x}) = (\sigma^n \circ \tau^{p_{m,n}(\underline{x})})(\underline{x})$$

and $p_{m,n}(\underline{x}) = e^{n \cdot h_{\text{top}}(\sigma)}[m + O(1)]$ uniformly on Σ^+ .

Ito's transversal flow and this paper [I]. Ito found a way to realize the natural extension of $\sigma : \Sigma^+ \rightarrow \Sigma^+$ as a map T of $[0, 1) \times [0, 1)$, and constructed a flow τ^t on $[0, 1) \times [0, 1)$ with the property $T^n \circ \tau^t = \tau^{\lambda^{-n}t} \circ T^n$, $\lambda = e^{h_{\text{top}}(\sigma)}$.

In this paper we explain how Ito's flow appears naturally in the symbolic coding of the horocycle flow of a non-compact hyperbolic surface of finite area.

To do this, we first represent Ito's flow explicitly as a flow on the symbolic space (the "adic flow"), and then we observe that the action of the horocycle flow on the space of geodesics in the Poincaré disk takes a similar form if the endpoints of the geodesics are coded using boundary expansions.

The symbolic representation of Ito's flow appeared before in the work of Bufetov, who applied it to translation flows on flat surfaces [Bu1], [Bu2]. It was also known to A. Fisher [F1] and appears naturally in the study of scenery flows [F2].

2. ADIC FLOWS

These will be flows on two-sided subshifts of finite type

$$\Sigma := \{(x_i)_{i \in \mathbb{Z}} \in S^{\mathbb{Z}} : t_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\},$$

metrized by $d(\underline{x}, \underline{y}) = \exp[-\min\{|n| : x_n \neq y_n\}]$. As in the one-sided case, the left shift map $\sigma : \Sigma \rightarrow \Sigma$, $\sigma(\underline{x})_i = x_{i+1}$, has topological entropy $h_{\text{top}}(\Sigma) = \ln \lambda$ where λ is the leading eigenvalue of the transition matrix $(t_{ij})_{N \times N}$.

It is known that the left shift map is topologically mixing iff there exists some m s.t. all the entries of the m -th power of $(t_{ij})_{N \times N}$ are positive (see e.g. [B]).

Fix once and for all a reverse lexicographic order on Σ^+ (see (1.2)), and let τ be the corresponding adic transformation. (1.2) can also be used to define a partial order on Σ , and we use the same symbol \prec for the two orders. If $\pi : \Sigma \rightarrow \Sigma^+$ is the projection onto the non-negative coordinates, then $(\underline{x} \prec \underline{y}) \Rightarrow (\pi(\underline{x}) \preceq \pi(\underline{y}))$.

Let $(\underline{x}, \underline{y}) := \{\underline{z} : \underline{x} \not\prec \underline{z} \not\prec \underline{y}\}$ and define

$$\Sigma_0 := \{\underline{x} \in \Sigma : \pi(\underline{x}) \in \Sigma_0^+ \text{ and } \forall \underline{y} \succ \underline{x} [(\underline{x}, \underline{y}) \neq \emptyset]\}.$$

This excludes sequences with a right maximal tail ($x_i = P_{\max}(x_{i+1})$ for all i large); sequences with a right minimal tail ($x_i = P_{\min}(x_{i+1})$ for all i large); and sequences with a left maximal tail ($x_i = P_{\max}(x_{i+1})$ for all i sufficiently negative).

Sequences with left minimal tails ($x_i = P_{\min}(x_{i+1})$ for all i sufficiently negative) are not excluded.

Theorem 1. *Suppose (Σ, σ) is topologically mixing. There exists a measurable flow $\tau_s^t : \Sigma_0 \rightarrow \Sigma_0$ such that:*

- (a) Orbit property: $\{\tau_s^t(\underline{x}) : t \in \mathbb{R}\} = \{\underline{y} \in \Sigma_0 : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow[t \rightarrow \infty]{} 0\}$ for every $\underline{x} \in \Sigma_0$.
- (b) Renormalization property: $\tau_s^t \circ \sigma^n = \sigma^n \circ \tau_s^{\lambda^n t}$.
- (c) Order property: if $t_1 < t_2$ then $\tau_s^{t_1}(\underline{x}) \prec \tau_s^{t_2}(\underline{x})$.
- (d) Almost Continuity: Suppose $\tau_s^t(\underline{x})$ has no left minimal tails, then $d(\tau_s^{t_n}(\underline{x}^{(n)}), \tau_s^t(\underline{x})) \rightarrow 0$ for all sequences $(\underline{x}^{(n)}, t_n) \rightarrow (\underline{x}, t)$.

This flow is unique up to a change of time scale.

Existence follows from Ito's work [I].

To determine a flow one has to specify the orbits, the order of movement on the orbits, and the time it takes to flow from one point to another. We show that conditions (a),(b),(c) and (d) dictate these choices, and that these choices lead to a well defined flow.

The orbits. $\{\underline{y} \in \Sigma_0 : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \rightarrow 0\} = \{\underline{y} \in \Sigma_0 : \exists p(\forall i > p \ y_i = x_i)\}$, therefore property (a) forces the orbits to equal the equivalence classes of the (stable) tail equivalence relation on Σ_0 :

$$\underline{x} \sim \underline{y} \Leftrightarrow \exists p \in \mathbb{Z} \text{ s.t. } \forall i > p \ (x_i = y_i).$$

We denote these equivalence classes by $[\underline{x}]$.

The order of movement. Property (c) says that the order on the orbits of τ_s is \prec . The question arises whether such an order can be realized by a flow. This is guaranteed by the following lemma.

Lemma 1. $([\underline{x}], \prec)$ is order isomorphic to $(\mathbb{R}, <)$ for every $x \in \Sigma_0$.

Proof. By [C], it is enough to check that

- (i) $[\underline{x}]$ is totally ordered by \prec .
- (ii) $[\underline{x}]$ contains a countable set Z which is order isomorphic to \mathbb{Z} , and such that for every $\underline{y} \in [\underline{x}]$, there exist $\underline{z}_1, \underline{z}_2 \in Z$ s.t. $\underline{z}_1 \prec \underline{y} \prec \underline{z}_2$.
- (iii) $[\underline{x}]$ contains a countable set Q with the property that for any $\underline{y}_1 \not\prec \underline{y}_2$ in $[\underline{x}]$, there exists some $\underline{z} \in Q$ s.t. $\underline{y}_1 \not\prec \underline{z} \not\prec \underline{y}_2$.
- (iv) Every bounded monotonic sequence of elements in $[\underline{x}]$ has a limit in $[\underline{x}]$ (convergence w.r.t. the order topology).
- (v) Every element of $[\underline{x}]$ is the limit of a non-eventually constant increasing (resp. decreasing) sequence in $[\underline{x}]$.

The first property is trivial. (ii) holds with $Z := \{(\Phi \circ \tau^n \circ \pi)(\underline{x}) : n \in \mathbb{Z}\}$, where $\pi : \Sigma \rightarrow \Sigma^+$ is the natural projection, $\tau : \Sigma_0^+ \rightarrow \Sigma_0^+$ is Vershik's adic transformation, and $\Phi : \Sigma_0^+ \rightarrow \Sigma$ is $\Phi(y_0, y_1, y_2, \dots) := (\dots, y_{-2}, y_{-1}, \dot{y}_0, y_1, y_2, \dots)$ with $y_{-i} := P_{\min}^i(y_0)$ for $i \in \mathbb{N}$. The dot above y_0 signifies the position of the zeroth coordinate. (iii) holds with $Q := \{\underline{y} \in [\underline{x}] : \exists n \text{ s.t. } \forall i \leq n \ (y_{i-1} = P_{\min}(y_i))\}$.

Here is the proof of (iv). We do the increasing case, and leave the decreasing case to the reader. Let $\underline{y}^{(n)}$ be an increasing sequence in $[\underline{x}]$ and suppose $\underline{y}^{(n)} \prec \underline{y}$ for all n . Since $\underline{y}^{(1)} \sim \underline{y}$, there exists some p s.t. $y_i^{(1)} = y_i$ for all $i \geq p$. Since $\underline{y}^{(1)} \preceq \underline{y}^{(n)} \preceq \underline{y}$, (1.2) forces $y_i^{(n)} = y_i$ for all $n \in \mathbb{N}$ and $i \geq p$. It also forces $\{y_{p-1}^{(n)}\}_{n \geq 1}$ to be an increasing sequence in $(S, <)$. Since S is finite, $\{y_{p-1}^{(n)}\}_{n \geq 1}$ is eventually constant. Suppose $y_{p-1}^{(n)}$ equals c for all $n > N_1$, then $\{y_{p-2}^{(n)}\}_{n \geq N_1}$ must be an increasing sequence in $(S, <)$. Since S is finite, it also eventually stabilizes.

Continuing in this way, we see that $\exists \underline{z} \in \Sigma$ s.t. $d(\underline{y}^{(n)}, \underline{z}) \rightarrow 0$.

The limiting sequence belongs to $[\underline{x}]$, because $z_i = y_i$ for all $i \geq p$ and $\underline{y} \in [\underline{x}]$. If $\underline{z} \in \Sigma_0$, then we are done. Otherwise for some $q \in \mathbb{Z}$

$$z_{i-1} = P_{\max}(z_i) \text{ for all } i \leq q.$$

Suppose q is maximal with this property (a maximum exists, since $\underline{z} \sim \underline{x}$ and $\underline{x} \in \Sigma_0$). Then $z_q < P_{\max}(z_{q+1})$ and $\underline{y}^{(n)}$ converges in the order topology of $[\underline{x}]$ to $\underline{w} := (\dots, P_{\min}^2(w_q), P_{\min}(w_q), w_q, z_{q+1}, z_{q+2}, \dots)$, where the q -th coordinate is

$$w_q := \min\{w \in S : t_{wz_{q+1}} = 1 \text{ and } w > z_q\} \text{ w.r.t. } \leq_{z_{q+1}}.$$

This is an element of $[\underline{x}]$.

To prove (v), fix some $\underline{y} \in [\underline{x}]$. We construct a non eventually constant decreasing sequence in $[\underline{x}]$ with limit \underline{y} . Since $\underline{y} \in \Sigma_0$, it has no left maximal tails, so $\exists i_n \downarrow -\infty$ s.t. $y_{i_n} \neq P_{\max}(y_{i_n+1})$. Let

$$w_{i_n} := \min\{w \in S : t_{wy_{i_n+1}} = 1 \text{ and } w > y_{i_n}\} \text{ w.r.t. } \leq_{y_{i_n+1}}.$$

Define $\underline{y}^{(n)} := (\dots, P_{\min}^2(w_{i_n}), P_{\min}(w_{i_n}), w_{i_n}, y_{i_n+1}, y_{i_n+2}, \dots)$. Then $\underline{y}^{(n)}$ is a decreasing sequence of points in $\Sigma_0 \cap [\underline{x}]$ whose limit is \underline{y} . The sequence is not eventually constant, because the starting point of the longest left minimal tail of $\underline{y}^{(n)}$ is i_n , and i_n is not eventually constant.

Next we construct a non-eventually constant increasing sequence in $[\underline{x}]$ with limit \underline{y} . Suppose first that \underline{y} does not have a left minimal tail. Let

$$\underline{y}^{(n)} := (\dots, P_{\min}^2(y_{-n}), P_{\min}(y_{-n}), y_{-n}, y_{-n+1}, y_{-n+2}, \dots).$$

This is an increasing sequence in $[\underline{x}]$ which converges to \underline{y} , and it is not eventually constant because there are $n_k \rightarrow -\infty$ s.t. $y_{n_k} \neq P_{\min}(y_{n_k+1})$.

Now suppose \underline{y} does have a left minimal tail. There is a maximal q s.t. $y_{i-1} = P_{\min}(y_i)$ for all $i \leq q$, otherwise $\underline{y} \notin \Sigma_0$. By maximality, $y_q > P_{\min}(y_{q+1})$, therefore the following definitions are proper:

$$w_q := \max\{w \in S : t_{wy_{q+1}} = 1 \text{ and } w < y_q\} \text{ w.r.t. } \leq_{y_{q+1}}$$

$$\begin{aligned} \underline{y}^{(n)} := & (\dots, P_{\min}^3(P_{\max}^n(w_q)), P_{\min}^2(P_{\max}^n(w_q)), P_{\min}(P_{\max}^n(w_q)), \\ & P_{\max}^n(w_q), P_{\max}^{n-1}(w_q), \dots, P_{\max}(w_q), w_q, y_{q+1}, y_{q+2}, \dots). \end{aligned}$$

The sequence $\{\underline{y}^{(n)}\}_{n \geq 1}$ is increasing sequence. It is not difficult to see using the topological mixing of (Σ, σ) that there are infinitely many n 's s.t. $P_{\max}^{n+1}(w_q) \neq P_{\min}[P_{\max}^n(w_q)]$, and this implies as above that $\{\underline{y}^{(n)}\}_{n \geq 1}$ is not eventually constant. In (Σ, d) , $\lim \underline{y}^{(n)} = \underline{z} := (\dots, P_{\max}^3(w_q), P_{\max}^2(w_q), P_{\max}(w_q), w_q, y_{q+1}, y_{q+2}, \dots)$ which lies outside of Σ_0 . With respect to the order topology, $\lim \underline{y}^{(n)}$ is the smallest element of $[\underline{x}] \cap \Sigma_0$ above \underline{z} , which is exactly \underline{y} . \square

Time parametrization. The time parametrization of the orbits of a flow can be encoded in terms of a measure as follows. Suppose there were a flow $\tau_s^t : \Sigma_0 \rightarrow \Sigma_0$ satisfying the requirements of the theorem. For every $\underline{x} \in \Sigma_0$ and $\underline{y} \prec \underline{z}$ in $[\underline{x}]$ let $[\underline{y}, \underline{z}] := \{\underline{w} : \underline{y} \preceq \underline{w} \preceq \underline{z}\}$ and

$$(2.1) \quad \mu_{[\underline{x}]}([\underline{y}, \underline{z}]) := \text{the unique } t \text{ s.t. } \underline{z} = \tau_s^t(\underline{y}).$$

This makes sense, since $\underline{y}, \underline{z}$ belong to the same orbit of τ_s . Since the orbit order is \prec , $\mu_{[\underline{x}]}([\underline{y}, \underline{z}]) > 0$. Since τ_s^t is a flow, $\mu_{[\underline{x}]}$ is σ -additive on the semi-algebra of

left-closed right-open intervals in $[\underline{x}]$. By Carathéodory's Extension Theorem, $\mu_{[\underline{x}]}$ extends to a measure on the Borel σ -algebra of $[\underline{x}]$ (Borel w.r.t. the order topology). We denote this measure by $\mu_{[\underline{x}]}$.

The measures $\{\mu_{[\underline{x}]} : \underline{x} \in \Sigma_0\}$ contain complete information on the speed of movement on the orbits, and they must satisfy the following properties:

- (M1) $\mu_{[\underline{x}]}$ is a non-atomic measure on $[\underline{x}]$.
 - (M2) In $[\underline{x}]$, non-empty bounded open intervals have finite positive measure and infinite rays (sets of the form $\{\underline{z} : \underline{z} \preceq \underline{y}\}$ or $\{\underline{z} : \underline{z} \succeq \underline{y}\}$, $\underline{y} \in [\underline{x}]$) have infinite measure.
 - (M3) $\mu_{[\sigma(\underline{x})]} \circ \sigma = \lambda^{-1} \mu_{[\underline{x}]}$.
 - (M4) for every continuous function $f : \Sigma \rightarrow \mathbb{R}$, the real valued function $\underline{x} \mapsto \int_{[\underline{x}]} f(\underline{y}) \delta_{x_0^\infty}(y_0^\infty) d\mu_{[\underline{x}]}(\underline{y})$ is Borel measurable. Here $\delta_{x_0^\infty}(y_0^\infty)$ is the function which is equal to one if $y_0^\infty := (y_0, y_1, \dots) = (x_0, x_1, \dots) =: x_0^\infty$ and equal to zero otherwise.
- (M1) and (M2) are obvious. (M3) is because of the renormalization property (c). (M4) is because of the measurability of the flow: the integral can be rewritten as $\sup \int_\alpha^\beta (f \circ \tau_s^t)(\underline{x}) dt$ where the supremum is taken over all rational numbers $\alpha < \beta$ s.t. $(\tau_s^\alpha(\underline{x}), \tau_s^\beta(\underline{x})) \subset \{\underline{y} \in \Sigma : y_0^\infty = x_0^\infty\}$.

Lemma 2 (Bowen & Marcus). *There is a measurable family of measures $\{\mu_{[\underline{x}]} : \underline{x} \in \Sigma_0\}$ satisfying (M1), (M2), (M3), and (M4).*

Proof. This is essentially in [BM], albeit in different notation. We begin with a simple, but useful, description of $[\underline{x}]$. Abusing notation, we use the same symbol \sim for the tail relation on Σ_0 and Σ_0^+ . Then $[\underline{x}] = \{\underline{y} \in \Sigma_0 : \underline{y} \sim \underline{x}\}$ and $\pi[\underline{x}] = \{\underline{y}^+ \in \Sigma^+ : \underline{y}^+ \sim \pi(\underline{x})\}$. The last set is countable. We obtain the following countable disjoint decomposition of $[\underline{x}]$:

$$[\underline{x}] = \bigsqcup_{\underline{y}^+ \sim \pi(\underline{x})} A(\underline{y}^+), \text{ where } A(\underline{y}^+) := \{\underline{z} \in \Sigma_0 : z_0^\infty = y_0^\infty\}.$$

Each of the sets $A(\underline{y}^+)$ is a bounded open interval in $[\underline{x}]$, and therefore must have finite measure w.r.t. $\mu_{[\underline{x}]}$. So instead of describing the infinite measure $\mu_{[\underline{x}]}$, we will describe the finite measures

$$\mu_{\underline{y}^+}(E) := \mu_{[\underline{x}]}(E \cap A(\underline{y}^+)) \text{ for every } \underline{y}^+ \sim \pi(\underline{x}).$$

It is enough to determine $\mu_{\underline{y}^+}$ consistently on the family of cylinders

$$C(a_{-n}, \dots, a_{-1}; \underline{y}^+) := \{\underline{z} \in \Sigma_0 : z_{-n}^{-1} = a_{-n}^{-1}, z_0^\infty = \underline{y}^+\}, \text{ where } z_m^n := (z_m, \dots, z_n).$$

Let $A = (t_{ij})_{S \times S}$ be the transition matrix of Σ . Construct using the Perron-Frobenius theorem a row vector $\ell = \langle \ell_a : a \in S \rangle$ with positive entries such that $\ell A = \lambda \ell$. The eigenvalue λ is bigger than one, and is the leading eigenvalue of A . We have

$$\sum_{p \in S} \ell_p t_{pa} = \lambda \cdot \ell_a, \text{ in particular } \sum_{p \in S} g_{pa} = 1 \text{ where } g_{pa} := \frac{\ell_p t_{pa}}{\lambda \ell_a}.$$

Just as in the case of Markov chains, the set function

$$\mu_{\underline{y}^+}[C(a_{-n}, \dots, a_{-1}; \underline{y}^+)] := g_{a_{-n} a_{-n+1}} g_{a_{-n+1} a_{-n+2}} \cdots g_{a_{-2} a_{-1}} g_{a_{-1} y_0} \ell_{y_0}$$

extends to a σ -additive measure on $A(\underline{y}^+)$ (with total mass ℓ_{y_0}). This is our measure $\mu_{\underline{y}^+}$. Having $\mu_{\underline{y}^+}$, we proceed to define $\mu_{[\underline{x}]} := \sum_{\underline{y}^+ \sim \pi(\underline{x})} \mu_{\underline{y}^+}$.

It is easy to check that $\mu_{[\underline{x}]}$ satisfies the following identity:

$$(2.2) \quad \mu_{[\underline{x}]}[C(a_{-n}, \dots, a_{-1}; \underline{y}^+)] = \ell_{a_{-n}} \lambda^{-n}.$$

We can use this to show that $\mu_{[\underline{x}]}$ is non-atomic: For every $\underline{y} \in [\underline{x}]$,

$$\mu_{[\underline{x}]}(\{\underline{y}\}) \leq \mu_{[\underline{x}]}[C(y_{-n}, \dots, y_{-1}; \pi(\underline{y}))] = O(\lambda^{-n}) \xrightarrow{n \rightarrow \infty} 0.$$

Another consequence of (2.2) is that every non-empty cylinder has finite positive measure. Since every non-empty bounded open interval contains a non-empty cylinder, and is contained in a finite union of cylinders, every non-empty bounded open interval has finite positive measure. Finally, infinite rays have infinite measure, because they contain infinitely many cylinders of the form $C(a_{-1}^i; \underline{y}^+(i))$ with a_{-1}^i constant. In summary, (M1) and (M2) hold.

To verify (M3), it is enough to show that $\mu_{[\underline{x}]} \circ \sigma = \lambda^{-1} \mu_{[\underline{x}]}$ when evaluated on cylinders. This holds because

$$\begin{aligned} \frac{(\mu_{[\underline{x}]} \circ \sigma)[C(a_{-n}, \dots, a_{-1}; y_0^\infty)]}{\mu_{[\underline{x}]}[C(a_{-n}, \dots, a_{-1}; y_0^\infty)]} &= \frac{\mu_{[\underline{x}]}[C(a_{-n}, \dots, a_{-1}, y_0; y_1^\infty)]}{\mu_{[\underline{x}]}[C(a_{-n}, \dots, a_{-1}; y_0^\infty)]} = \\ &= \frac{g_{a_{-n}a_{-n+1}} \cdots g_{a_{-1}y_0} g_{y_0 y_1} \ell_{y_1}}{g_{a_{-n}a_{-n+1}} \cdots g_{a_{-1}y_0} \ell_{y_0}} = \frac{g_{y_0 y_1} \ell_{y_1}}{\ell_{y_0}} = \frac{1}{\lambda}. \end{aligned}$$

(M4) is obvious. \square

Lemma 3. *Let m denote the measure of maximal entropy of $\sigma : \Sigma \rightarrow \Sigma$. For any two families $\mu_{[\underline{x}]}, \nu_{[\underline{x}]}$ which satisfy (M1), (M2), (M3), and (M4) there exists a positive constant c and a set $\Omega \subset \Sigma$ of full m -measure such that $\mu_{[\underline{x}]} = c\nu_{[\underline{x}]}$ for every $x \in \Omega$.*

Proof. Define for a function $f : \Sigma^+ \rightarrow \Sigma^+$ a new function $(Lf)(\underline{x}) = \sum_{a \in S} t_{ax_0} f(a, \underline{x})$.

Ruelle's Perron-Frobenius Theorem (see e.g. [B]) states that there exists a unique probability measure ν_0 on Σ^+ and a unique positive continuous function h_0 on Σ^+ s.t. for some $\lambda > 0$, $Lh = \lambda h$, $L^* \nu_0 = \lambda \nu_0$, $\int h_0 d\nu_0 = 1$. We have already met the function h_0 : it is $h_0(\underline{x}) = \ell_{x_0}$ where ℓ is the left eigenvector of $(t_{ij})_{N \times N}$.

The measure ν_0 is a Markov measure, $h_0(x)$ only depends on x_0 , and λ is the leading eigenvalue of the transition matrix A . The measure $h_0 d\nu_0$ is globally supported, invariant, and ergodic. It is the measure of maximal entropy for $\sigma : \Sigma^+ \rightarrow \Sigma^+$ [P]. The measure of maximal entropy for the two sided shift is its natural extension.

Suppose $\mu_{[\underline{x}]}$ is a family of measures satisfying (M1), (M2), (M3), and (M4). Fix $p \geq 1$, and define a function $\varphi^{(p)} : \Sigma_0^+ \rightarrow \mathbb{R}$ by

$$\varphi^{(p)}(x_0, x_1, \dots) = \mu_{[\sigma^p(\underline{x}^*)]} \{ \underline{y} \in \Sigma_0 : y_{-p}^{-1} = x_0^{p-1} ; y_0^\infty = x_p^\infty \}.$$

where \underline{x}^* is some (any) extension of $\underline{x} = (x_0, x_1, \dots)$ to a two sided sequence in Σ_0 . The definition is proper, since $[\sigma^p(\underline{x}^*)]$ is the same for all possible extensions.

We claim that $L\varphi^{(p)} = \lambda\varphi^{(p)}$ on Σ_0^+ . For every $\underline{x} \in \Sigma$

$$(L\varphi^{(p)})(\underline{x}) = \sum_{a \in S} t_{ax_0} \varphi^{(p)}(a, \underline{x}) =$$

$$\begin{aligned}
&= \sum_{a \in S} t_{ax_0} \mu_{[\sigma^p((a, \underline{x})^*)]} \{ \underline{y} \in \Sigma_0 : y_{-p}^{-1} = (a\underline{x})_0^{p-1} ; y_0^\infty = (a\underline{x})_p^\infty \} \\
&= \sum_{a \in S} t_{ax_0} \mu_{[\sigma^{p-1}(\underline{x}^*)]} \{ \underline{y} \in \Sigma_0 : y_{-p} = a, y_{-p+1}^{-1} = x_0^{p-2} ; y_0 = x_{p-1}, y_1^\infty = x_p^\infty \} \\
&= \lambda \sum_{a \in S} t_{ax_0} \mu_{[\sigma^p(\underline{x}^*)]} \{ \sigma(\underline{y}) : \underline{y} \in \Sigma_0, y_{-p} = a, y_{-p+1}^{-1} = x_0^{p-2} ; y_0 = x_{p-1}, y_1^\infty = x_p^\infty \} \\
&= \lambda \sum_{a \in S} t_{ax_0} \mu_{[\sigma^p(\underline{x}^*)]} \{ \underline{z} \in \Sigma_0 : z_{-p-1} = a, z_{-p}^{-2} = x_0^{p-2} ; z_{-1} = x_{p-1}, z_0^\infty = x_p^\infty \} \\
&= \lambda \mu_{[\sigma^p(\underline{x}^*)]} \{ \underline{z} \in \Sigma_0 : z_{-p}^{-1} = x_0^{p-1} ; z_0^\infty = x_p^\infty \} = \lambda \varphi^{(p)}(\underline{x}).
\end{aligned}$$

Since $\varphi^{(p)}$ is positive and measurable, $\varphi^{(p)} d\nu_0$ is a well defined σ -finite measure equivalent to ν_0 . This measure is σ -invariant, because for every positive bounded measurable function $f : \Sigma^+ \rightarrow \mathbb{R}$, the identity $L^* \nu = \lambda \nu$ implies that

$$\int f \circ \sigma \varphi^{(p)} d\nu_0 = \lambda^{-1} \int L[f \circ \sigma \cdot \varphi^{(p)}] d\nu_0 = \lambda^{-1} \int f L \varphi^{(p)} d\nu_0 = \int f \varphi^{(p)} d\nu_0.$$

We see that $\varphi^{(p)}$ is an invariant density for ν_0 . So is $h_0(\underline{x}) = \ell_{x_0}$. Since $h_0 d\nu_0$ is an ergodic probability measure, $\exists c_p > 0$ s.t. $\varphi^{(p)} = c_p h_0$ ν_0 -almost surely.

We now interpret this in terms of the two-sided shift. Suppose $\underline{x} \in \Sigma_0$, then there is a set of full measure Ω of $\underline{x} \in \Sigma_0$ such that for every p and every word (a_{-p}, \dots, a_{-1}) s.t. $t_{a_{-p}, a_{-p+1}} \cdots t_{a_{-1}, x_0} = 1$,

$$\mu_{[\underline{x}]} \left(C(a_{-p}, \dots, a_{-1}; x_0^\infty) \right) = \varphi^{(p)}(\underline{a}, x_0^\infty) = c_p h_0(\underline{a}, x_0^\infty) = c_p \ell_{a_{-p}}.$$

Keeping $(a_{-p+1}, \dots, a_{-1}, x_0^\infty)$ constant and summing the left hand side over all a_{-p} s.t. $t_{a_{-p}, a_{-p+1}} = 1$, we see that $c_p \sum_{b \in S} t_{ba_{-p+1}} \ell_b = c_{p-1} \ell_{a_{-p+1}}$, whence since $\ell A = \lambda \ell$, $c_p = \lambda^{-1} c_{p-1}$. The conclusion of all this is that for all $\underline{x} \in \Omega$,

$$\mu_{[\underline{x}]} \left(C(a_{-p}, \dots, a_{-1}; x_0^\infty) \right) = c_1 \left(\lambda^{-p} \ell_{a_{-p}} \right).$$

Comparing this with (2.2), we see that $\mu_{[\underline{x}]}$ is c_1 times the Bowen–Marcus measure we constructed in the previous step.

We now repeat this argument for the second family of measures $\nu_{[\underline{x}]}$ and obtain a second constant c'_1 and a second set Ω' of full measure of \underline{x} such that $\nu_{[\underline{x}]}$ is c'_1 times the Bowen–Marcus measure from the previous step. The lemma follows. \square

Proof of Theorem 1: Existence. The idea is to define for $t > 0$,

$$\begin{aligned}
(2.3) \quad \tau_s^t(\underline{x}) &:= \min \{ \underline{z} \sim \underline{x} : \underline{z} \succ \underline{x}, \mu_{[\underline{x}]}([\underline{x}, \underline{z}]) \geq t \}, \\
\tau_s^{-t}(\underline{x}) &:= \max \{ \underline{y} \sim \underline{x} : \underline{y} \prec \underline{x}, \mu_{[\underline{x}]}([\underline{y}, \underline{x}]) \geq t \}.
\end{aligned}$$

To see that the definition is proper, and that it defines a flow, we have to show that for every $t > 0$ there are unique $\underline{y}, \underline{z} \in \Sigma_0$ such that $\underline{y} \prec \underline{x} \prec \underline{z}$ and

$$(2.4) \quad \mu_{[\underline{x}]}([\underline{y}, \underline{x}]) = t = \mu_{[\underline{x}]}([\underline{x}, \underline{z}]).$$

Here is the proof. By (M2), there exist $\underline{y}' \prec \underline{x} \prec \underline{z}'$ s.t. $\mu_{[\underline{x}]}([\underline{x}, \underline{z}']) > t$ and $\mu_{[\underline{x}]}([\underline{y}', \underline{x}]) > t$. Set

$$\begin{aligned}\underline{y} &:= \sup\{\underline{y}' \in \Sigma_0 : \underline{y}' \prec \underline{x} \text{ and } \mu_{[\underline{x}]}([\underline{y}', \underline{x}]) > t\}, \\ \underline{z} &:= \inf\{\underline{z}' \in \Sigma_0 : \underline{z}' \succ \underline{x} \text{ and } \mu_{[\underline{x}]}([\underline{x}, \underline{z}']) > t\}.\end{aligned}$$

Since $([\underline{x}], \prec)$ is order isomorphic to $(\mathbb{R}, <)$, the infimum and supremum exist, they belong to $[\underline{x}]$, and one can find sequences $\underline{y}^{(n)} \uparrow \underline{y}$ and $\underline{z}^{(n)} \downarrow \underline{z}$ such that

$$\mu_{[\underline{x}]}([\underline{y}^{(n)}, \underline{x}]) > t \text{ and } \mu_{[\underline{x}]}([\underline{x}, \underline{z}^{(n)}]) > t.$$

It follows that $\mu_{[\underline{x}]}([\underline{y}, \underline{x}]) \geq t$ and $\mu_{[\underline{x}]}([\underline{x}, \underline{z}]) \geq t$.

Similarly, one can find sequences $\underline{x} \succ \underline{\hat{y}}^{(n)} \downarrow \underline{y}$ and $\underline{x} \prec \underline{\hat{z}}^{(n)} \uparrow \underline{z}$. These must satisfy $\mu_{[\underline{x}]}([\underline{\hat{y}}^{(n)}, \underline{x}]) \leq t$ and $\mu_{[\underline{x}]}([\underline{x}, \underline{\hat{z}}^{(n)}]) \leq t$. Consequently

$$\mu_{[\underline{x}]}([\underline{y}, \underline{x}]) \leq t \text{ and } \mu_{[\underline{x}]}([\underline{x}, \underline{z}]) \leq t.$$

Since $\mu_{[\underline{x}]}$ is non-atomic, we must have $\mu_{[\underline{x}]}([\underline{y}, \underline{x}]) = t$ and $\mu_{[\underline{x}]}([\underline{x}, \underline{z}]) = t$.

This shows that \underline{y} and \underline{z} exist. They are unique, because if there were other solutions $\underline{\hat{y}}, \underline{\hat{z}}$ to (2.4), then the intervals with endpoints $\underline{y}, \underline{\hat{y}}$ or $\underline{z}, \underline{\hat{z}}$ would be non-empty bounded intervals with zero measure, in contradiction to (M2).

We claim that the flow thus defined satisfies the statement of the theorem. All properties are clear, except for almost continuity. Fix \underline{x}, t s.t. $\tau_s^t(\underline{x})$ has no left minimal tail. Fix N . We construct $N', \epsilon > 0$ s.t. for every $\underline{x}' \in \Sigma_0$

$$(2.5) \quad \left. \begin{aligned} t' &\in (t - \epsilon, t + \epsilon) \\ x'_i &= x_i \text{ for all } |i| \leq N' \end{aligned} \right\} \Rightarrow \tau_s^{t'}(\underline{x}')_i = \tau_s^t(\underline{x})_i \text{ for all } |i| \leq N.$$

By assumption $\tau_s^t(\underline{x})$ has no minimal left tails, therefore there exists some $N_1 > N$ s.t. $\tau_s^t(\underline{x})_{-N_1} \neq P_{\min}(\tau_s^t(\underline{x})_{-N_1+1})$. Since $\tau_s^t(\underline{x}) \in \Sigma_0$, it has no maximal left tails, therefore there exists some $N_2 > N_1$ s.t. $\tau_s^t(\underline{x})_{-N_2} \neq P_{\max}(\tau_s^t(\underline{x})_{-N_2+1})$.

Let \underline{y} denote the element of Σ_0 s.t. $y_i = \tau_s^t(\underline{x})_i$ for $i \geq -N_1 + 1$ and $y_i = P_{\min}(y_{i+1})$ for $i \leq -N_1$. Note that $\underline{y} \prec \tau_s^t(\underline{x})$.

Let \underline{z} denote the element of Σ_0 s.t. $z_i = \tau_s^t(\underline{x})_i$ for $i \geq -N_2 + 1$, $z_{-N_2} = P_{\max}(z_{-N_2+1})$, and $z_i = P_{\min}(z_{i+1})$ for all $i < -N_2$. Note that $\underline{z} \succ \tau_s^t(\underline{x})$.

Now pick $\epsilon > 0$ smaller than $\mu_{[\underline{x}]}([\tau_s^t(\underline{x}), \underline{z}])$ and $\mu_{[\underline{x}]}([\underline{y}, \tau_s^t(\underline{x})])$. We claim that (2.5) holds with ϵ and $N' := N_2$. Pick some $\underline{x}' \in \Sigma_0$ such that $x'_i = x_i$ for all $|i| < N'$, and define $\underline{y}', \underline{z}'$ by

$$y'_i := \begin{cases} y_i & i \leq N' \\ \tau_s^t(\underline{x}')_i & i > N' \end{cases} \quad \text{and} \quad z'_i := \begin{cases} z_i & i \leq N' \\ \tau_s^t(\underline{x}')_i & i > N', \end{cases}$$

then $\underline{y}' \prec \tau_s^t(\underline{x}') \prec \underline{z}'$.

Since $\underline{y}' \prec \tau_s^t(\underline{x}') \prec \underline{z}'$ and $y'_i = \tau_s^t(\underline{x})_i = z'_i$ for all $|i| \leq N_1 - 1$, we also have that $\tau_s^t(\underline{x})_i = \tau_s^t(\underline{x}')_i$ for all $|i| \leq N_1 - 1$. It follows from the structure of the measure constructed in claim 1 of part 3 that

$$\mu_{[\underline{x}]}([\underline{y}', \tau_s^t(\underline{x}')]) = \mu_{[\underline{x}]}([\underline{y}, \tau_s^t(\underline{x})]) , \mu_{[\underline{x}]}([\tau_s^t(\underline{x}'), \underline{z}']) = \mu_{[\underline{x}]}([\tau_s^t(\underline{x}), \underline{z}]).$$

By the choice of ϵ , $\mu_{[\underline{x}']}((\underline{y}', \tau_s^t(\underline{x}')) > \epsilon$ and $\mu_{[\underline{x}']}([\tau_s^t(\underline{x}'), \underline{z}']) > \epsilon$. It follows that for every t' s.t. $|t - t'| < \epsilon$, $\underline{y}' \prec \tau_s^{t'}(\underline{x}') \prec \underline{z}'$. Since $y'_i = \tau_s^t(\underline{x})_i = x'_i$ for all $i > -N_1$, $\tau_s^{t'}(\underline{x}')_i = \tau_s^t(\underline{x})_i$ for all $|i| < N_1$. Since $N_1 > N$, (2.5) follows. \square

Proof of Theorem 1: Uniqueness. Suppose there are two flows τ_s, θ_s satisfying the statement of the theorem. As we saw above, τ_s and θ_s have the same (unparameterized) orbits, and they move on these orbits in the same order. Let $\mu_{[\underline{x}]}$ and $\nu_{[\underline{x}]}$ denote the time parameterization measures, given by

$$\mu_{[\underline{x}]}[\underline{x}, \tau_s^t(\underline{x})) = t, \nu_{[\underline{x}]}[\underline{x}, \theta_s^t(\underline{x})) = t.$$

We saw that they must satisfy (M1), (M2), (M3) and (M4).

Let m denote the measure of maximal entropy of $\sigma : \Sigma \rightarrow \Sigma$. By Lemma 3, there exist a positive number $c > 0$ and a set Ω s.t. $m(\Omega) = 1$ and $\mu_{[\underline{x}]} = c\nu_{[\underline{x}]}$ for all $\underline{x} \in \Omega$. Since m has global support, there exists a dense set of $\underline{x} \in \Sigma_0$ s.t. $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$ for all $t \in \mathbb{R}$. By the almost continuity property $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$ whenever $\tau_s^t(\underline{x}), \theta_s^{ct}(\underline{x})$ do not have left minimal tails.

If $\tau_s^t(\underline{x})$ or $\theta_s^{ct}(\underline{x})$ have a left minimal tail, then construct $t_n \downarrow t$ s.t. $\tau_s^{t_n}(\underline{x}), \theta_s^{ct_n}(\underline{x})$ do not have left minimal tails. All but countably many parameters are like that. Since $\tau_s^t(\underline{x}) = \inf_n \tau_s^{t_n}(\underline{x}) = \inf_n \theta_s^{ct_n}(\underline{x}) = \theta_s^{ct}(\underline{x})$, we get that $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$. \square

2.1. Ergodic properties of adic flows. The following theorem is due to Ito [I]:

Theorem 2. *Suppose (Σ, σ) is topologically mixing, then the adic flow on Σ_0 is uniquely ergodic and of zero entropy. The invariant probability measure is the measure of maximal entropy of $\sigma : \Sigma_0 \rightarrow \Sigma_0$ (Parry's measure).*

The key to the proof is that $\Lambda := \{\underline{x} \in \Sigma_0 : x_{i-1} = P_{\min}(x_i) \text{ for all } i \leq 0\}$ is a Poincaré section for the adic flow, and the Poincaré map $T : \Lambda \rightarrow \Lambda$ satisfies $T \circ \pi = \pi \circ \tau$ where $\pi : \underline{x} \mapsto x_0^\infty$ and $\tau : \Sigma_0^+ \rightarrow \Sigma_0^+$ is Vershik's adic transformation. The height function is (in the notation of Lemma 3) $H := h_0 \circ \pi$.

Since $\tau : \Sigma_0^+ \rightarrow \Sigma_0^+$ is uniquely ergodic with zero entropy [LV], $\tau_s^t : \Sigma_0 \rightarrow \Sigma_0$ is uniquely ergodic with zero entropy.

The adic flow does not have to be weak mixing. Take for example $\Sigma = \{0, 1\}^{\mathbb{Z}}$. In this case the function h_0 is constant, and τ is conjugate to a constant suspension of the adding machine. The function $\varphi(\underline{x}) = (-1)^{x_0}$ is an eigenfunction.

3. APPLICATION TO HOROCYCLE FLOWS

We recall some facts on the symbolic dynamics of *geodesic* flows on non-compact hyperbolic surfaces with finite area. These will be used at the end of the section to relate the horocycle flow to time changes of adic flows.

Suspension flows. Let $f : Y \rightarrow Y$ be an invertible map, and $\rho : Y \rightarrow \mathbb{R}^+$ a function such that $\sum_{n \geq 0} \rho \circ f^n = \sum_{n \leq 0} \rho \circ f^n = \infty$ everywhere on Y . Let $\stackrel{r}{\sim}$ denote the equivalence relation on $Y \times \mathbb{R}$ generated by the equivalences $(y, \xi) \stackrel{r}{\sim} (f(y), \xi - \rho(y))$. Let Y_r denote the set of equivalence classes of $\stackrel{r}{\sim}$, denoted by $\langle y, \xi \rangle$. Let $\pi_\rho : Y \times \mathbb{R} \rightarrow Y_r$ be the equivalence class map $\pi_\rho : (y, \rho) \mapsto \langle y, \rho \rangle$. The *suspension flow over f with roof function ρ* is $F^t : Y_r \rightarrow Y_r$ given by $F^t \langle y, \xi \rangle = \langle y, \xi + t \rangle$.

Hyperbolic surfaces. The *Poincaré disc* is $\mathbb{D} := \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$ with the Riemannian metric $2\sqrt{dx^2 + dy^2}/(1 - x^2 - y^2)$. The group of orientation preserving isometries of \mathbb{D} is $\text{Möb}(\mathbb{D})$, the group of Möbius transformations which preserve the unit disc, see e.g. [Ka].

A *hyperbolic surface* is a Riemannian surface M such that every $p \in M$ has a neighborhood which is isometric to $\{z \in \mathbb{D} : |z| < \epsilon\}$ for some $\epsilon > 0$. It is known that every complete connected orientable hyperbolic surface is isometric to an orbit space $\Gamma \backslash \mathbb{D} = \{\Gamma z := \{g(z) : g \in \Gamma\} : z \in \mathbb{D}\}$ where Γ is some discrete subgroup of $\text{Möb}(\mathbb{D})$ without elements of finite order. Henceforth we take all our hyperbolic surfaces to be complete, connected, and orientable, whence of the form above.

Cutting sequences [A],[Mo],[Ko]. Let $M = \Gamma \backslash \mathbb{D}$ be a non-compact hyperbolic surface of finite area, then Γ has a fundamental domain F satisfying:

- (1) F is a geodesic polygon all of whose vertices are on $\partial\mathbb{D}$ (“cusps”);
- (2) F has an even number of sides $a_1, \dots, a_n; \bar{a}_1, \dots, \bar{a}_n$, and there are $\varphi_{a_i} \in \Gamma$ s.t. φ_{a_i} maps a_i onto \bar{a}_i , and $\varphi_{\bar{a}_i} = \varphi_{a_i}^{-1}$ maps \bar{a}_i onto a_i .

We re-index the sides of F as s_1, \dots, s_{2n} according to the counterclockwise order of the sides as edges of F (the choice of s_1 is arbitrary). We keep the convention that \bar{s}_i is the congruent edge to s_i .

The *vertex cycle* of a vertex v of F is a sequence of minimal length $s_{i_1}s_{i_2} \cdots s_{i_p}$ s.t. v is on side s_{i_1} , $\varphi_{s_{i_1}}(v)$ is on side s_{i_2} , $(\varphi_{s_{i_2}} \circ \varphi_{s_{i_1}})(v)$ is on side s_{i_3} and so on up to step p when $(\varphi_{s_{i_p}} \circ \cdots \circ \varphi_{s_{i_1}})(v) = v$ is on side s_{i_1} .

Following [BKS] and [BS], we use the symbols s_i to label the sides of F from the inside (“interior labels”). The images of F by elements of Γ tile \mathbb{D} . Extend the labeling to the other tiles using the action of Γ . Now every edge will also have an “exterior label” (the interior label of the adjacent tile). This labeling system is the unique scheme so that (a) if the interior label of an edge is s , then its exterior label is \bar{s} , and (b) the counterclockwise order of the interior labels is the same in each tile. The second property is because Γ preserves orientation.

Let $g : T^1\mathbb{D} \rightarrow T^1\mathbb{D}$ denote the geodesic flow on the unit tangent bundle of \mathbb{D} . Let $T^1(F)$ denote collection of unit tangent vectors with base points in F . We say that a geodesic on M *tends to a cusp* at ∞ (resp. $-\infty$), if one (all) of its lifts to \mathbb{D} tends at ∞ (resp. $-\infty$) to one of the vertices of F or its image under some element of Γ .

The *cutting sequence* of $\omega \in T^1(F)$ is the sequence of the interior labels of the sides of the tiles that $\{g^t(\omega)\}_{t \in \mathbb{R}}$ intersects. This is a two sided infinite sequence as long as $g^t(\omega)$ does not tend to a cusp. We use the convention that the zeroth coordinate of the cutting sequence $(s_{i_k})_{k \in \mathbb{Z}}$ of ω is the side s_{i_0} such that ω belongs to the geodesic segment between s_{i_0} and s_{i_1} .

It is also possible to associate an infinite cutting sequence to a geodesic which emanates from or tends to a cusp, by viewing the geodesic as a limit of geodesics which do not emanate from or tend to cusps. The cutting sequences obtained this way are eventually periodic. The word which forms the period is a vertex cycle of a vertex of F which represents the cusp the geodesic tends to or comes from.

The collection of all cutting sequences is equal to the subshift of finite type

$$\Sigma := \{x \in S^{\mathbb{Z}} : x_i \neq \bar{x}_{i+1} \text{ for all } i \in \mathbb{Z}\}, S := \{s_1, \dots, s_{2n}\}.$$

One way to see this is to consider *boundary expansions* [BS]. The boundary expansion of $x \in \partial\mathbb{D}$ is the one-sided cutting sequence of (any) geodesic ray which starts at an interior point of F and terminates at x . This is a one-sided infinite sequence for all x which does not belong to the Γ -orbit of the vertices of F (a countable set of exceptions). If x does belong to the Γ -orbit of a cusp, then we can define (two) boundary sequences for it by considering the limits of boundary sequences of points which tend to x from the right or from the left. It is easy to see that the collection of all boundary expansions is equal to $\Sigma^+ := \{x \in S^{\mathbb{N} \cup \{0\}} : x_i \neq \bar{x}_{i+1} \text{ for all } i \geq 0\}$. The identification of the collection of cutting sequences with Σ easily follows.¹

A Poincaré section for the geodesic flow [S1]. Let

$$X := \{\omega \in T^1(M) : \text{the geodesic of } \omega \text{ does not tend to a cusp at } \infty \text{ or } -\infty\}.$$

If $\omega \in X$, then the lift of $g^t(\omega)$ to \mathbb{D} crosses infinitely many Γ -images of F , and the set of crossing times is discrete. Consequently, the following set is a Poincaré section for the geodesic flow on X :

$$\Omega := \{\Gamma\omega : \omega \in T^1(\partial F) \text{ s.t. } \forall \epsilon > 0 \text{ small } g^\epsilon(\omega) \in T^1(\text{int}(F))\}.$$

Let $T : \Omega \rightarrow \Omega$ denote the *section map*,

$$T(\omega) := g^{R(\omega)}(\omega), \text{ where } R(\omega) := \inf\{t > 0 : g^t(\omega) \in \Omega\}.$$

The relation to cutting sequences is as follows. Partition Ω into the sets $A_s = \{\Gamma\omega \in \Omega : \omega \text{ is based at } \Gamma s\}$ where s is one of the sides of F . For $\omega \in \Omega$, let A_{s_n} denote the unique element of this partition so that $T^n(\omega) \in A_{s_n}$. The sequence $\{A_{s_n}\}_{n \in \mathbb{Z}}$ is called the *itinerary* of ω . It is not difficult to check that $\{A_{s_n}\}_{n \in \mathbb{Z}}$ is the itinerary of $\omega \in \Omega$, iff $\{s_n\}_{n \in \mathbb{Z}}$ is the cutting sequence of the geodesic of ω , indexed in such a way that $\omega \in A_{s_0}$.

Let $\pi : \Omega \rightarrow \Sigma$ denote the map which send $\omega \in \Omega$ to the sequence $\{s_n\}_{n \in \mathbb{Z}}$ constructed above. This is an injection, and $\pi \circ T = \sigma \circ \pi$ where $\sigma : \Sigma \rightarrow \Sigma$ is the left shift. This allows us to represent the geodesic flow on X as a suspension flow $\sigma^t : \Sigma_r \rightarrow \Sigma_r$ over $\sigma : \Sigma \rightarrow \Sigma$ with roof function $r = R \circ \pi$.

Lemma 4 (Lopes & Thieullen [LT]). $r = r^+ + u - u \circ \sigma$ where

- $r^+(x) = \log |\varphi'_{x_0}[\alpha(x_1, x_2, \dots)]|$, where $\alpha(x_1, x_2, \dots)$ = unique point in $\partial\mathbb{D}$ with boundary expansion (x_1, x_2, \dots) , and φ_{x_0} is the side pairing $x_0 \rightarrow \bar{x}_0$.
- $u(x) := -\log \text{Im}[\psi_{\alpha(x)}(z(x))]$, where $z(x)$ is the base point of the unique vector in $T^1(\partial F)$ with cutting sequence x , and $\psi_{\alpha(x)} : \mathbb{D} \rightarrow \mathbb{H}$ is the Möbius transformation which maps the geodesic from 0 to $\alpha(x)$ to the geodesic from i to ∞ .

Notice that $r^+(x)$ only depends (x_0, x_1, x_2, \dots) .

It is not difficult to see that u is continuous at any cutting sequence of a geodesic which does not tend to a cusp. See lemma 2.2 in [LS] for more details.

¹The simple relationship between cutting sequences and boundary expansions is only true for geometrically finite surfaces *with cusps*. In the cocompact case, the set of cutting sequences is not necessarily a SFT [E], although it is always sofic [S2].

The strong stable foliation in symbolic coordinates [BM]. The *strong stable manifold* of $\omega \in T^1M$ is by definition

$$W^{ss}(\omega) := \{\omega' \in T^1M : d(g^t(\omega'), g^t(\omega)) \xrightarrow[t \rightarrow \infty]{} 0\}.$$

We wish to describe the lifts of these sets to the symbolic suspension space Σ_r .

We treat the stable horocycle flow; the description of the unstable flow is essentially the same. Let \mathfrak{G} denote the grand tail relation of σ :

$$\mathfrak{G} := \{(x, y) \in \Sigma \times \Sigma : \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty\},$$

where $x_k^\infty = (x_k, x_{k+1}, \dots)$. If two unit tangent vectors in X lie on the same strong stable manifold, then they have \mathfrak{G} -equivalent cutting sequences.

Suppose x, y are two \mathfrak{G} -equivalent cutting sequences. As shown in [BM], two elements $\langle x, \xi \rangle, \langle y, \eta \rangle$ in Σ_r code two line elements on the same strong stable manifold iff

$$\exists p, q \text{ s.t. } x_p^\infty = y_q^\infty \text{ and } [\xi - r_{p+n}(x)] - [\eta - r_{q+n}(y)] \xrightarrow[n \rightarrow \infty]{} 0.$$

Here and throughout, $r_m := r + r \circ \sigma + \dots + r \circ \sigma^{m-1}$.

Let $u : \Sigma \rightarrow \mathbb{R}$ from the Lopes–Thieullen Lemma. For every pair of \mathfrak{G} -equivalent x, y which are not eventually periodic with period equal to a vertex cycle,

$$\begin{aligned} r_{p+n}(x) - r_{q+n}(y) &= r_{p+n}^+(x) - r_{q+n}^+(y) + u(\sigma^{p+n}x) - u(x) - u(\sigma^{q+n}y) + u(y) \\ &\xrightarrow[n \rightarrow \infty]{} r_p^+(x) - r_q^+(y) + u(y) - u(x). \end{aligned}$$

Define for such pairs

$$\Psi(x, y) := r_p^+(x) - r_q^+(y) + u(y) - u(x), \text{ whenever } x_p^\infty = y_q^\infty$$

(with p, q minimal when x, y are eventually periodic). The calculation above shows that if $(x, y) \in \mathfrak{G}$ and $\xi - \eta = \Psi(x, y)$ then $\langle x, \xi \rangle, \langle y, \eta \rangle$ lie on the same strong stable manifold. When x, y are not eventually periodic, this is an iff.

The horocycle flow on a hyperbolic surface with cusps. We finally arrive to the description of the horocycle flow in the symbolic space Σ_r .

Recall that the elements of S (the sides of the fundamental domain F) are ordered cyclically by $s_1, s_2, \dots, s_{2n}, s_1$. For every $1 \leq i \leq 2n$, let \prec_{s_i} be the order

$$s_i \prec_{s_i} s_{i+1} \prec_{s_i} \dots \prec_{s_i} s_{2n} \prec_{s_i} s_1 \prec_{s_i} \dots \prec_{s_i} s_{i-1}.$$

Let \preceq denote the tail order $x \preceq y$ iff $x = y$ or $\exists p$ s.t. $x_{p+1}^\infty = y_{p+1}^\infty$ and $x_p \prec_{s_{p+1}} y_p$.

To see what is going on, suppose x, y are two \mathfrak{G} -equivalent cutting sequences of two directed geodesics. Lift these geodesic to \mathbb{D} in such a way that they terminate at the same point. Then $x \preceq y$ iff the geodesic with cutting sequence x is mapped by the horocycle flow to the geodesic with cutting sequence y in *positive* time.

The picture in \mathbb{D} also reveals that the eventually extremal sequences are exactly the cutting sequences of geodesics which tend to a cusp.

Let $\tau_s^t : \Sigma_0 \rightarrow \Sigma_0$ denote the adic flow induced by \preceq . Let $p : (\Sigma_0)_r \rightarrow T^1M$ denote the map which associates to $\langle x, t \rangle$ the vector $g^{t-u(x)}(\omega)$ where $\omega \in T^1M$ has cutting sequence x , g is the geodesic flow, and $u(x)$ is the function from the Lopes–Thieullen lemmas. This is one-to-one and continuous on Σ_0 (and two-to-one on Σ).

Let $\Xi(x, y) := r_p^+(x) - r_q^+(y) = \Psi(x, y) + u(x) - u(y)$, whenever $x_p^\infty = y_q^\infty$, where r^+ is given by the Lopes–Thieullen Lemma.

Let $h^t : T^1M \rightarrow T^1M$ denote the stable horocycle flow.

Theorem 3. *There exists a time change $\tau^t : \Sigma_0 \rightarrow \Sigma_0$ of the adic flow $\tau_s^t : \Sigma_0 \rightarrow \Sigma_0$ such that $p \circ H^t = h^t \circ p$, where $H^t : (\Sigma_0)_r \rightarrow (\Sigma_0)_r$ is the flow*

$$H^t \langle x, \xi \rangle = \langle \tau^{te^\xi}(x), \xi + \Xi(\tau^{te^\xi}(x), x) \rangle.$$

Proof. Since $p : (\Sigma_0)_r \rightarrow T^1M$ is one-to-one, the horocycle flow on

$$T^1M \cap \{\omega : g^s(\omega) \text{ does not tend to a cusp}\}$$

lifts to a flow H on $(\Sigma_0)_r$. We calculate it.

The orbits of the horocycle flow are the leaves of the strong stable foliation of the geodesic flow. These are

$$\text{Hor}\langle x, \xi \rangle = p\{\langle y, \xi + \Xi(y, x) \rangle : \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty\},$$

because $p(\langle x, \xi \rangle)$ and $p(\langle y, \eta \rangle)$ are in the same leaf of the strong stable foliation iff $\langle x, \xi - u(x) \rangle, \langle y, \eta - u(y) \rangle$ code vectors in the same leaf of the strong stable foliation, which by the previous section happens exactly when $\xi - \eta = \Psi(x, y) + u(x) - u(y) = \Xi(x, y)$. So the H orbit of $\langle x, \xi \rangle$ equals

$$\{\langle y, \xi + \Xi(y, x) \rangle : \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty\}.$$

We constructed \preceq in such a way that $x \preceq y$ iff the horocycle flow maps the geodesic with cutting sequence x to the geodesic with cutting sequence y at positive time. Therefore $t \mapsto \text{first coordinate of } H^t(x)$ is \preceq -increasing.

We know the orbits, and the order on the orbits. It remains to determine the time parametrization. We do this for $\text{Hor}\langle x, 0 \rangle$, and then use the commutation relation between the geodesic flow and the horocycle flow to see that the parametrization on $\text{Hor}\langle x, \xi \rangle$ is e^ξ times the parametrization on $\text{Hor}\langle x, 0 \rangle$.

By the definition of $u(x)$, $p(\langle x, 0 \rangle) \in T^1(\Gamma \setminus \mathbb{D})$ belongs to the projection of the horocycle $\text{Hor}_{\alpha(x_0^\infty)}(0)$ in $T^1\mathbb{D}$ which passes through the origin, and is tangent to ∂D at the point $\alpha(x)$ with boundary expansion x_0^∞ . So $\text{Hor}\langle x, 0 \rangle$ is the projection of a horocycle in \mathbb{D} passing through the origin.

The horocycle flow parametrizes this horocycle by the hyperbolic length measure. The parametrizing measure $\mu_{[x]}$ on $\{\langle y, \Xi(y, x) \rangle : y \in [x]\}$ is the pull back of the hyperbolic length measure.

The theorem holds with τ be the time change of the adic flow τ_s of \preceq , determined by the family of measures $\{\mu_{[x]}\}$. \square

3.1. The compact case. The results in this section were formulated for non-compact surfaces of finite area. In this case one can find a fundamental domain bounded by non-intersecting geodesics. In the compact case, the geodesic extensions of the sides of the fundamental domain have to intersect, and this leads to ambiguities in the boundary expansions of certain points. While there are ways of resolving this difficulty [BS], [S1], [S2], they have to involve non-canonical choices. Different choices lead to different boundary expansions, and as a result to different symbolic representations of the horocycle flow. Either way we choose, the order \preceq is more complicated and less canonical than in the case of cusps.

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3.3. Note added in proof. Recently J. Athreya and Y. Cheung obtained a beautiful non-symbolic representation of the horocycle flow as a suspension flow in the particular case of the modular surface [AC].

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