BERNOULLI EQUILIBRIUM STATES FOR SURFACE DIFFEOMORPHISMS

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Abstract. Suppose $f : M \to M$ is a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact smooth orientable manifold M of dimension two, and let μ_Ψ be an equilibrium measure for a Hölder continuous potential $\Psi : M \to \mathbb{R}$. We show that if μ_Ψ has positive metric entropy, then f is measure theoretically isomorphic mod μ_Ψ to the product of a Bernoulli scheme and a finite rotation.

1. Statements

Suppose $f : M \to M$ is a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact smooth orientable manifold M of dimension two. Suppose $\Psi : M \to \mathbb{R}$ is Hölder continuous. An invariant probability measure μ is called an *equilibrium measure*, if it maximizes the quantity $h_\mu(f) + \int \Psi d\mu$, where $h_\mu(f)$ is the metric entropy. Such measures always exist when f is C^∞, because in this case the function $\mu \mapsto h_\mu(f)$ is upper semi–continuous [**N**]. Let μ_Ψ be an ergodic equilibrium measure of Ψ. We prove:

Theorem 1.1. *If $h_{\mu_\Psi}(f) > 0$, then f is measure theoretically isomorphic with respect to μ_Ψ to the product of a Bernoulli scheme (see* §3.3*) and a finite rotation (a map of the form $x \mapsto (x+1) \bmod p$ on $\{0, 1, \ldots, p-1\}$).*

The proof applies to other potentials, such as $-t \log J_u$ ($t \in \mathbb{R}$) where J_u is the unstable Jacobian, see §5.2. In the particular case of the measure of maximal entropy ($\Psi \equiv 0$) we can say more, see §5.1.

The theorem is false in higher dimension: Let f denote the product of a hyperbolic toral automorphism and an irrational rotation. This C^∞ diffeomorphism has many equilibrium measures of positive entropy. But f cannot satisfy the conclusion of the theorem with respect to any of these measures, because f has the irrational rotation as a factor, and therefore none of its powers can have ergodic components with the K property.

Bowen [**B1**] and Ratner [**Ra**] proved Theorem 1.1 for uniformly hyperbolic diffeomorphisms. In the non-uniformly hyperbolic case, Pesin proved that any absolutely continuous ergodic invariant measure all of whose Lyapunov exponents are non-zero is isomorphic to the product of a Bernoulli scheme and a finite rotation [**Pe**]. By Pesin's Entropy Formula and Ruelle's Entropy Inequality, these measures are equilibrium measures of $-\log J_u$. Ledrappier extended Pesin's result to all equilibrium measures with non-zero exponents for the potential $-\log J_u$, including those which are not absolutely continuous [**L**]. These results hold in any dimension.

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The work of Pesin and Ledrappier (see also [**OW**]) uses the following property of equilibrium measures of $-\log J_u$: the conditional measures on unstable manifolds are absolutely continuous [**L**]. This is false for general Hölder potentials [**LY**].

Theorem 1.1 is proved in three steps:

(1) **Symbolic dynamics:** Any ergodic equilibrium measure on M with positive entropy is a finite-to-one Hölder factor of an ergodic equilibrium measure on a countable Markov shift (CMS).
(2) **Ornstein Theory:** Factors of equilibrium measures of Hölder potentials on topologically mixing CMS are Bernoulli.
(3) **Spectral Decomposition:** The non-mixing case.

Notation. $a = M^{\pm 1} b$ means $M^{-1} b \le a \le Mb$.

2. Step One: Symbolic dynamics

Let $\mathscr{G}$ be a directed graph with a countable collection of vertices $\mathscr{V}$ s.t. every vertex has at least one edge coming in, and at least one edge coming out. The *countable Markov shift* (CMS) associated to $\mathscr{G}$ is the set

$$\Sigma = \Sigma(\mathscr{G}) := \{(v_i)_{i\in\mathbb{Z}} \in \mathscr{V}^{\mathbb{Z}} : v_i \to v_{i+1} \text{ for all } i\}.$$

The *natural metric* $d(\underline{u}, \underline{v}) := \exp[-\min\{|i| : u_i \neq v_i\}]$ turns Σ into a complete separable metric space. Σ is compact iff $\mathscr{G}$ is finite. Σ is locally compact iff every vertex of $\mathscr{G}$ has finite degree. The *cylinder sets*

$$_m[a_m, \dots, a_n] := \{(v_i)_{i\in\mathbb{Z}} \in \Sigma : v_i = a_i \ (i = m, \dots, n)\} \tag{2.1}$$

form a basis for the topology, and they generate the Borel σ–algebra $\mathscr{B}(\Sigma)$.

The *left shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma[(v_i)_{i\in\mathbb{Z}}] = (v_{i+1})_{i\in\mathbb{Z}}$. Given $a, b \in \mathscr{V}$, write $a \xrightarrow{n} b$ when there is a path $a \to v_1 \to \cdots \to v_{n-1} \to b$ in $\mathscr{G}$. The left shift is topologically transitive iff $\forall a, b \in \mathscr{V}\ \exists n\, (a \xrightarrow{n} b)$. In this case $\gcd\{n : a \xrightarrow{n} a\}$ is the same for all $a \in \mathscr{V}$, and is called the *period* of σ. The left shift is topologically mixing iff it is topologically transitive and its period is equal to one. See [**K**].

Let $\Sigma^{\#} := \{(v_i)_{i\in\mathbb{Z}} \in \Sigma : \exists u, v \in \mathscr{V} \exists n_k, m_k \uparrow \infty \text{ s.t. } v_{-m_k} = u, v_{n_k} = v\}$. Every σ–invariant probability measure gives $\Sigma^{\#}$ full measure, because of Poincaré's Recurrence Theorem.

Suppose $f : M \to M$ is a $C^{1+\alpha}$–diffeomorphism of a compact orientable smooth manifold M s.t. $\dim M = 2$. If $h_{top}(f) = 0$, then every f–invariant measure has zero entropy by the variational principle, and Theorem 1.1 holds trivially. So we assume without loss of generality that $h_{top}(f) > 0$. Fix $0 < \chi < h_{top}(f)$.

A set $\Omega \subset M$ is called χ*–large*, if $\mu(\Omega) = 1$ for every ergodic invariant probability measure μ whose entropy is greater than χ. The following theorems are in [**S2**]:

Theorem 2.1. *There exists a locally compact countable Markov shift Σ_χ and a Hölder continuous map $\pi_\chi : \Sigma_\chi \to M$ s.t. $\pi_\chi \circ \sigma = f \circ \pi_\chi$, $\pi_\chi[\Sigma_\chi^{\#}]$ is χ–large, and s.t. every point in $\pi_\chi[\Sigma_\chi^{\#}]$ has finitely many pre-images.*

Theorem 2.2. *Denote the set of states of Σ_χ by $\mathscr{V}_\chi$. There exists a function $\varphi_\chi : \mathscr{V}_\chi \times \mathscr{V}_\chi \to \mathbb{N}$ s.t. if $x = \pi_\chi[(v_i)_{i\in\mathbb{Z}}]$ and $v_i = u$ for infinitely many negative i, and $v_i = v$ for infinitely many positive i, then $|\pi_\chi^{-1}(x)| \le \varphi_\chi(u, v)$.*

Theorem 2.3. *Every ergodic f–invariant probability measure μ on M such that $h_\mu(f) > \chi$ equals $\widehat{\mu} \circ \pi_\chi^{-1}$ for some ergodic σ–invariant probability measure $\widehat{\mu}$ on Σ_χ with the same entropy.*

We will use these results to reduce the problem of Bernoullicity for equilibrium measures for $f : M \to M$ and the potential Ψ, to the problem of Bernoullicity for equilibrium measures for $\sigma : \Sigma_\chi \to \Sigma_\chi$ and the potential $\psi := \Psi \circ \pi_\chi$.

3. Step Two: Ornstein Theory

First we describe the structure of equilibrium measures of Hölder continuous potentials on countable Markov shifts (CMS), and then we show how this structure forces, in the topologically mixing case, isomorphism to a Bernoulli scheme.

3.1. **Equilibrium measures on one–sided CMS** [**BS**]**.** Suppose $\mathscr{G}$ is countable directed graph. The *one-sided countable Markov shift* associated to $\mathscr{G}$ is

$$\Sigma^+ = \Sigma^+(\mathscr{G}) := \{(v_i)_{i\geq 0} \in \mathscr{V}^{\mathbb{N}\cup\{0\}} : v_i \to v_{i+1} \text{ for all } i\}.$$

Proceeding as in the two–sided case, we equip Σ^+ with the metric $d(\underline{u}, \underline{v}) := \exp[-\min\{i \geq 0 : u_i \neq v_i\}]$. The *cylinder sets*

$$[a_0, \dots, a_{n-1}] := \{\underline{u} \in \Sigma^+ : u_i = a_i \ (i = 0, \dots, n-1)\} \tag{3.1}$$

form a basis for the topology of Σ^+. Notice that unlike the two-sided case (2.1), there is no left subscript: the cylinder starts at the zero coordinate.

The *left shift map* $\sigma : \Sigma^+ \to \Sigma^+$ is given by $\sigma : (v_0, v_1, \dots) \mapsto (v_1, v_2, \dots)$. This map is not invertible. The natural extension of (Σ^+, σ) is conjugate to (Σ, σ).

A function $\phi : \Sigma^+ \to \mathbb{R}$ is called *weakly Hölder continuous* if there are constants $C > 0$ and $\theta \in (0, 1)$ s.t. $\mathrm{var}_n\phi < C\theta^n$ for all $n \geq 2$, where

$$\mathrm{var}_n\phi := \sup\{\phi(\underline{u}) - \phi(\underline{v}) : u_i = v_i \ (i = 0, \dots, n-1)\}.$$

The following inequality holds:

$$\mathrm{var}_{n+m}\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j\right) \leq \sum_{j=m+1}^{\infty} \mathrm{var}_j\phi. \tag{3.2}$$

If ϕ is bounded, weak Hölder continuity is the same as Hölder continuity.

The equilibrium measures for weakly Hölder potentials were described by Ruelle [**Ru**] for finite graphs and by Buzzi and the author for countable graphs [**BS**]. Make the following assumptions:

(a) $\sigma : \Sigma \to \Sigma$ is topologically mixing.
(b) ϕ is weakly Hölder continuous and $\sup\phi < \infty$. (This can be relaxed [**BS**].)
(c) $P_G(\phi) := \sup\{h_m(\sigma) + \int \phi dm\} < \infty$, where the supremum ranges over all shift invariant measures m s.t. $h_m(\sigma) + \int \phi dm \neq \infty - \infty$. The potentials we will study satisfy $P_G(\phi) \leq h_{top}(f) + \max|\Psi| < \infty$.

Define for $F : \Sigma^+ \to \mathbb{R}^+$, $(L_\phi F)(\underline{x}) = \sum_{\sigma(\underline{y})=\underline{x}} e^{\phi(\underline{y})} F(\underline{y})$ ("Ruelle's operator"). The iterates of L_ϕ are $(L_\phi^n F)(\underline{x}) = \sum_{\sigma^n(\underline{y})=\underline{x}} e^{\phi(\underline{y})+\phi(\sigma\underline{y})+\cdots+\phi(\sigma^{n-1}\underline{y})} F(\underline{y})$.

Theorem 3.1 (Buzzi & S.)**.** *Under assumptions* (a),(b),(c) $\phi : \Sigma^+ \to \mathbb{R}$ *has at most one equilibrium measure. If this measure exists, then it is equal to $hd\nu$ where*

(1) $h : \Sigma^+ \to \mathbb{R}$ *is a positive continuous function s.t.* $L_\phi h = \lambda h$*;*
(2) ν *is a Borel measure on* Σ *which is finite and positive on cylinder sets,* $L_\phi^*\nu = \lambda\nu$*, and* $\int hd\nu = 1$*;*
(3) $\lambda = \exp P_G(\phi)$ *and* $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} \nu[\underline{a}]h$ *pointwise for every cylinder* $[\underline{a}]$.

Parts (1) and (2) continue to hold if we replace (a) by topological transitivity.

Corollary 3.2. *Assume* (a),(b),(c) *and let* μ *be the equilibrium measure of* ϕ. *For every finite* $S^* \subset \mathscr{V}$ *there exists a constant* $C^* = C^*(S^*) > 1$ *s.t. for every* $m, n \geq 1$, *every* n*–cylinder* $[\underline{a}]$, *and every* m*–cylinder* $[\underline{c}]$,

(1) *if the last symbol in* $\underline{a}$ *is in* S^* *and* $[\underline{a}, \underline{c}] \neq \varnothing$, *then* $1/C^* \leq \frac{\mu[\underline{a},\underline{c}]}{\mu[\underline{a}]\mu[\underline{c}]} \leq C^*$;
(2) *if the first symbol of* $\underline{a}$ *is in* S^* *and* $[\underline{c}, \underline{a}] \neq \varnothing$, *then* $1/C^* \leq \frac{\mu[\underline{c},\underline{a}]}{\mu[\underline{a}]\mu[\underline{c}]} \leq C^*$.

Proof. We begin with a couple of observations (see [**W1**]).

Observation 1. Let $\phi^* := \phi + \log h - \log h \circ \sigma - \log \lambda$, then ϕ^* is weakly Hölder continuous, and if $L = L_{\phi^*}$ then $L^*\mu = \mu$, $L1 = 1$, and $L^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} \mu[\underline{a}]$ pointwise. Notice that ϕ^* need not be bounded.

Proof. The convergence $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} h\nu[\underline{a}]$ and (3.2) imply that $\log h$ is weakly Hölder continuous, and $\mathrm{var}_1(\log h) < \infty$. It follows that ϕ^* is weakly Hölder continuous. The identities $L^*\mu = \mu, L1 = 1$ can be verified by direct calculation. To see the convergence $L^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} h\nu[\underline{a}]$ we argue as follows. Since ϕ has an equilibrium measure, ϕ is positive recurrent, see [**BS**]. Positive recurrence is invariant under the addition of constants and coboundaries, so ϕ^* is also positive recurrent. The limit now follows from a theorem in [**S1**].

Observation 2: For any positive continuous functions $F, G : \Sigma^+ \to \mathbb{R}^+$,

$$\int F(G \circ \sigma^n) d\mu = \int (L^n F) G d\mu. \tag{3.3}$$

Proof. Integrate the identity $(L^n F)G = L^n(FG \circ \sigma^n)$ using $L^*\mu = \mu$.

Observation 3: Let $\phi_n^* := \phi^* + \phi^* \circ \sigma + \cdots + \phi^* \circ \sigma^{n-1}$, then $M := \exp(\sup_{n\geq 1} \mathrm{var}_{n+1} \phi_n^*)$ is finite, because of (3.2) and the weak Hölder continuity of ϕ^*.

We turn to the proof of the corollary. Suppose $\underline{a} = (a_0, \ldots, a_{n-1})$ and $a_{n-1} \in S^*$. By Observation 2, $\mu([\underline{a}, \underline{c}]) = \int_{[\underline{c}]} L^n 1_{[\underline{a}]} d\mu = \int e^{\phi_n^*(\underline{a},y)} 1_{[\underline{c}]}(y) d\mu(y)$.

It holds that $e^{\phi_n^*(\underline{a},y)} = M^{\pm 1} e^{\phi_n^*(\underline{a},z)}$ for all $y, z \in \sigma[a_{n-1}]$. Fixing y and averaging over $z \in \sigma[a_{n-1}]$ we obtain

$$\begin{aligned} e^{\phi_n^*(\underline{a},y)} &= M^{\pm 1} \left(\frac{1}{\mu(\sigma[a_{n-1}])} \int_{\sigma[a_{n-1}]} e^{\phi_n^*(\underline{a},z)} d\mu(z) \right) \\ &= M^{\pm 1} \left(\frac{1}{\mu(\sigma[a_{n-1}])} \int L^n 1_{[\underline{a}]} d\mu \right) = M^{\pm 1} \left(\frac{\mu[\underline{a}]}{\mu(\sigma[a_{n-1}])} \right). \end{aligned}$$

Let $C_1^* := \max\{M/\mu(\sigma[a]) : a \in S^*\}$. Since $a_{n-1} \in S^*$,

$$e^{\phi_n^*(\underline{a},y)} = (C_1^*)^{\pm 1} \mu[\underline{a}] \text{ for all } y \in \sigma[a_{n-1}].$$

Since $[\underline{a}, \underline{c}] \neq \varnothing$, $\sigma[a_{n-1}] \supseteq [\underline{c}]$, so $\mu[\underline{a}, \underline{c}] = \int e^{\phi_n^*(\underline{a},y)} 1_{[\underline{c}]}(y) d\mu(y) = (C_1^*)^{\pm 1} \mu[\underline{a}]\mu[\underline{c}]$.

Now suppose $a_0 \in S^*$ and $[\underline{c}, \underline{a}] \neq \varnothing$, where $\underline{c} = (c_0, \ldots, c_{m-1})$. As before $\mu[\underline{c}, \underline{a}] = \int_{[\underline{a}]} L^m 1_{[\underline{c}]} d\mu = \int_{[\underline{a}]} e^{\phi_m^*(\underline{c},y)} d\mu(y)$, and $e^{\phi_m^*(\underline{c},y)} = M^{\pm 1} (\frac{\mu[\underline{c}]}{\mu^+(\sigma[c_{m-1}])})$. So

$$\mu[\underline{c}, \underline{a}] = \left(\frac{M^{\pm 1}}{\mu(\sigma[c_{m-1}])} \right) \mu[\underline{a}]\mu[\underline{c}].$$

Since $[\underline{c}, \underline{a}] \neq \varnothing$, $\sigma[c_{m-1}] \supset [a_0]$, therefore the term in the brackets is in $[\frac{1}{M}, \frac{M}{\mu[a_0]}]$. If we set $C_2^* := \max\{M/\mu[a] : a \in S^*\}$, then $\mu[\underline{c}, \underline{a}] = (C_2^*)^{\pm 1} \mu[\underline{a}]\mu[\underline{c}]$.

The lemma follows with $C^* := \max\{C_1^*, C_2^*\}$. ∎

3.2. **Equilibrium measures on two–sided CMS.** We return to two sided CMS $\Sigma = \Sigma(\mathscr{G})$. A function $\psi : \Sigma \to \mathbb{R}$ is called *weakly Hölder continuous* if there are constants $C > 0$ and $0 < \theta < 1$ s.t. $\mathrm{var}_n\psi < C\theta^n$ for all $n \geq 2$, where $\mathrm{var}_n\psi := \sup\{\psi(x) - \psi(y) : x_i = y_i \ \ (i = -(n-1), \ldots, n-1)\}$.

A function $\psi : \Sigma \to \mathbb{R}$ is called *one-sided*, if $\psi(\underline{x}) = \psi(\underline{y})$ for every $\underline{x}, \underline{y} \in \Sigma$ s.t. $x_i = y_i$ for all $i \geq 0$. The following lemma was first proved (in a different setup) by Sinai. The proof given in [**B2**] for subshifts of finite type also works for CMS:

Lemma 3.3 (Sinai)**.** *If $\psi : \Sigma \to \mathbb{R}$ is weakly Hölder continuous and $var_1\psi < \infty$, then there exists a bounded Hölder continuous function φ such that $\phi := \psi + \varphi - \varphi\circ\sigma$ is weakly Hölder continuous and one–sided.*

Notice that if ψ is bounded then ϕ is bounded, and that every equilibrium measure for ψ is an equilibrium measure for ϕ and vice verse.

Since $\phi : \Sigma \to \mathbb{R}$ is one–sided, there is a function $\phi^+ : \Sigma^+ \to \mathbb{R}$ s.t. $\phi(\underline{x}) = \phi^+(x_0, x_1, \ldots)$. If $\phi : \Sigma \to \mathbb{R}$ is weakly Hölder continuous, then $\phi^+ : \Sigma^+ \to \mathbb{R}$ is weakly Hölder continuous.

Any shift invariant probability measure μ on Σ determines a shift invariant probability measure μ^+ on Σ^+ through the equations

$$\mu^+[a_0, \ldots, a_{n-1}] := \mu({}_0[a_0, \ldots, a_{n-1}])$$

(cf. (2.1) and (3.1)). The map $\mu \mapsto \mu^+$ is a bijection, and it preserves ergodicity and entropy. It follows that μ is an ergodic equilibrium measure for ϕ iff μ^+ is an ergodic equilibrium measure for ϕ^+.

Corollary 3.4. *Suppose $\sigma : \Sigma \to \Sigma$ is topologically mixing. If $\psi : \Sigma \to \mathbb{R}$ is weakly Hölder continuous, $\sup\psi < \infty$, $var_1\psi < \infty$, and $P_G(\psi) < \infty$ then ψ has at most one equilibrium measure μ. This measure is the natural extension of an equilibrium measure of a potential $\phi : \Sigma^+(\mathscr{G}) \to \mathbb{R}$ which satisfies assumptions* (a),(b),(c)*.*

3.3. **The Bernoulli property.** The *Bernoulli scheme* with probability vector $p = (p_a)_{a\in S}$ is $(S^{\mathbb{Z}}, \mathscr{B}(S^{\mathbb{Z}}), \mu_p, \sigma)$ where σ is the left shift map and μ_p is given by $\mu_p({}_m[a_m, \ldots, a_n]) = p_{a_m} \cdots p_{a_n}$. If $(\Omega, \mathscr{F}, \mu, T)$ is measure theoretically isomorphic to a Bernoulli scheme, then we say that $(\Omega, \mathscr{F}, \mu, T)$ is a *Bernoulli automorphism*, and μ has the *Bernoulli property.* In this section we prove:

Theorem 3.1. *Every equilibrium measure of a weakly Hölder continuous potential $\psi : \Sigma(\mathscr{G}) \to \mathbb{R}$ on a topologically mixing countable Markov shift s.t. $P_G(\psi) < \infty$ and $\sup\psi < \infty$ has the Bernoulli property.*

This was proved by Bowen [**B1**] in the case when $\mathscr{G}$ is finite. See [**Ra**] and [**W3**] for generalizations to larger classes of potentials.

We need some facts from Ornstein Theory. Suppose $\beta = \{P_1, \ldots, P_N\}$ is a finite measurable partition for an invertible probability preserving map $(\Omega, \mathscr{F}, \mu, T)$. For every $m, n \in \mathbb{Z}$ s.t. $m < n$, let $\beta_m^n := \bigvee_{i=m}^n T^{-i}\beta$.

Definition 3.5 (Ornstein)**.** *A finite measurable partition β is called* weak Bernoulli *if $\forall\varepsilon > 0$ $\exists k > 1$ s.t.* $\displaystyle\sum_{A\in\beta_{-n}^0}\ \sum_{B\in\beta_k^{k+n}} |\mu(A\cap B) - \mu(A)\mu(B)| < \varepsilon$ *for all $n > 0$.*

Ornstein showed that if an invertible probability preserving transformation has a generating increasing sequence of weak Bernoulli partitions, then it is measure theoretically isomorphic to a Bernoulli scheme [**O1**, **OF**].

Proof of Theorem 3.1. First we make a reduction to the case when $\mathrm{var}_1\psi < \infty$. To do this, recode $\Sigma(\mathscr{G})$ using the Markov partition of cylinders of length two and notice that var_1 of the new coding equals var_2 of the original coding. The supremum and the pressure of ψ remain finite, and the variations of ψ continue to decay exponentially.

Suppose μ is an equilibrium measure of $\psi : \Sigma(\mathscr{G}) \to \mathbb{R}$. For every $\mathscr{V}' \subset \mathscr{V}$ finite, let $\alpha(\mathscr{V}') := \big\{{}_0[v] : v \in \mathscr{V}'\big\} \cup \big\{\bigcup_{v \notin \mathscr{V}'} {}_0[v]\big\}$. We claim that $\alpha(\mathscr{V}')$ is weak Bernoulli. This implies the Bernoulli property, because of the results of Ornstein we cited above.

We saw in the previous section that the measure μ^+ on $\Sigma^+(\mathscr{G})$ given by

$$\mu^+[a_0, \dots, a_{n-1}] := \mu({}_0[a_0, \dots, a_{n-1}])$$

satisfies $L^*\mu^+ = \mu^+$ and $L^n 1_{[\underline{a}]} \xrightarrow[n\to\infty]{} \mu^+[\underline{a}]$, where $L = L_{\phi^*}$ and $\phi^* : \Sigma^+(\mathscr{G}) \to \mathbb{R}$ is weakly Hölder continuous. By (3.2),

$$\sup_{n \geq 1}(\mathrm{var}_{n+m}\phi_n^*) \xrightarrow[m\to\infty]{} 0.$$

Fix $0 < \delta_0 < 1$ so small that $1 - e^{-t} \in (\frac{1}{2}t, t)$ for all $0 < t < \delta_0$. Fix some smaller $0 < \delta < \delta_0$, to be determined later, and choose

- a finite collection S^* of states (vertices) s.t. $\mu\big(\bigcup_{a \in S^*} {}_0[a]\big) > 1 - \delta$;
- a constant $C^* = C^*(S^*) > 1$ as in Corollary 3.2;
- a natural number $m = m(\delta)$ s.t. $\sup_{n\geq 1}(\mathrm{var}_{n+m}\phi_n^*) < \delta$;
- a finite collection γ of m–cylinders ${}_0[\underline{c}]$ s.t. $\mu(\bigcup \gamma) > e^{-\delta/2(C^*)^2}$;
- points $x(\underline{c}) \in {}_0[\underline{c}] \in \gamma$;
- natural numbers $K(\underline{c}, \underline{c}')$ ($[\underline{c}], [\underline{c}'] \in \gamma$) s.t. for every $k \geq K(\underline{c}, \underline{c}')$
$$(L^k 1_{[\underline{c}]})(x(\underline{c}')) = e^{\pm\delta}\mu^+[\underline{c}]$$
(recall that $L^n 1_{[\underline{c}]} \xrightarrow[n\to\infty]{} \mu^+[\underline{c}]$);
- $K(\delta) := \max\{K(\underline{c}, \underline{c}') : [\underline{c}], [\underline{c}'] \in \gamma\} + m$.

Step 1. Let $A := {}_{-n}[a_0, \dots, a_n]$ and $B := {}_k[b_0, \dots, b_n]$ be two non–empty cylinders of length $n + 1$. If $b_0, a_n \in S^*$, then for every $k > K(\delta)$ and every $n \geq 0$,

$$|\mu(A \cap B) - \mu(A)\mu(B)| < 2\sinh(10\delta)\mu(A)\mu(B).$$

Proof. Let α_m denote the collection of all m–cylinders ${}_0[\underline{c}]$. For every $k > 2m$,

$$\begin{aligned}\mu(A \cap B) &= \sum_{{}_0[\underline{c}], {}_0[\underline{c}'] \in \alpha_m} \mu({}_{-n}[\underline{a}, \underline{c}] \cap \sigma^{-(k-m)}{}_0[\underline{c}', \underline{b}]) \\ &= \sum_{{}_0[\underline{c}], {}_0[\underline{c}'] \in \alpha_m} \mu({}_0[\underline{a}, \underline{c}] \cap \sigma^{-(k+n-m)}{}_0[\underline{c}', \underline{b}]) \quad \text{(shift invariance)} \\ &= \sum_{{}_0[\underline{c}], {}_0[\underline{c}'] \in \alpha_m} \mu^+([\underline{a}, \underline{c}] \cap \sigma^{-(k+n-m)}[\underline{c}', \underline{b}]).\end{aligned}$$

By Observation 2 in the proof of corollary 3.2,

$$\mu(A \cap B) = \sum_{{}_0[\underline{c}], {}_0[\underline{c}'] \in \gamma} \int_{[\underline{c}', \underline{b}]} (L^{k+n-m} 1_{[\underline{a}, \underline{c}]}) d\mu^+ + \sum_{\substack{{}_0[\underline{c}], {}_0[\underline{c}'] \in \alpha_m \\ {}_0[\underline{c}] \notin \gamma \text{ or } {}_0[\underline{c}'] \notin \gamma}} \int_{[\underline{c}', \underline{b}]} (L_\phi^{k+n-m} 1_{[\underline{a}, \underline{c}]}) d\mu^+.$$

We call the first sum the “main term” and the second sum the “error term”.

To estimate these sums we use the following decomposition: for every $y \in [\underline{c}', \underline{b}]$, $(L^{k+n-m}1_{[\underline{a},\underline{c}]})(y) = \sum_{\sigma^{k-m-1}z=y} e^{\phi^*_{n+1}(\underline{a},z)} e^{\phi^*_{k-m-1}(z)} 1_{[\underline{c}]}(z)$.

By the choice of m, $e^{\phi^*_{n+1}(\underline{a},z)} = e^{\pm\delta} e^{\phi^*_{n+1}(\underline{a},w)}$ for all $w, z \in [\underline{c}]$. Fixing z and averaging over $w \in [\underline{c}]$, we see that

$$
\begin{aligned}
e^{\phi^*_{n+1}(\underline{a},z)} &= e^{\pm\delta} \left(\frac{1}{\mu^+[\underline{c}]} \int_{[\underline{c}]} e^{\phi^*_{n+1}(\underline{a},w)} d\mu^+(w) \right) \\
&= e^{\pm\delta} \left(\frac{1}{\mu^+[\underline{c}]} \int (L^{n+1} 1_{[\underline{a},\underline{c}]}) d\mu^+(w) \right) = e^{\pm\delta} \left(\frac{\mu^+[\underline{a},\underline{c}]}{\mu^+[\underline{c}]} \right) \\
\therefore (L^{n+k-m} 1_{[\underline{a},\underline{c}]})(y) &= e^{\pm\delta} \left(\frac{\mu^+[\underline{a},\underline{c}]}{\mu^+[\underline{c}]} \right) (L_\phi^{k-m-1} 1_{[\underline{c}]})(y) \text{ for } y \in [\underline{c}', \underline{b}]. \qquad (3.4)
\end{aligned}
$$

Estimate of the main term: Suppose $k > K(\delta)$. If ${}_0[\underline{c}], {}_0[\underline{c}'] \in \gamma$ and $y \in [\underline{c}', \underline{b}]$, then

$$
\begin{aligned}
(L^{k-m-1} 1_{[\underline{c}]})(y) &= e^{\pm\delta} (L^{k-m-1} 1_{[\underline{c}]})(x(\underline{c}')) \quad \text{by choice of } m \text{ and since } y, x(\underline{c}') \in [\underline{c}'] \\
&= e^{\pm 2\delta} \mu^+[\underline{c}] \text{ by choice of } K(\delta).
\end{aligned}
$$

Plugging this into (3.4), we see that if $k > K(\delta)$ then $(L_\phi^{n+k-m} 1_{[\underline{a},\underline{c}]})(y) = e^{\pm 3\delta} \mu^+[\underline{a},\underline{c}]$ on $[\underline{c}', \underline{b}]$. Integrating over $[\underline{c}', \underline{b}]$, we see that for all $k > K(\delta)$ the main term equals

$$
e^{\pm 3\delta} \sum_{{}_0[\underline{c}], {}_0[\underline{c}'] \in \gamma} \mu^+[\underline{a},\underline{c}] \mu^+[\underline{c}', \underline{b}] = e^{\pm 3\delta} \left(\sum_{{}_0[\underline{c}] \in \gamma} \mu^+[\underline{a},\underline{c}] \right) \left(\sum_{{}_0[\underline{c}'] \in \gamma} \mu^+[\underline{c}', \underline{b}] \right).
$$

The first bracketed sum is bounded above by $\mu^+[\underline{a}]$. To bound it below, we use the assumption that $a_n \in S^*$ to write

$$
\begin{aligned}
\sum_{{}_0[\underline{c}] \in \gamma} \mu^+[\underline{a},\underline{c}] &= \mu^+[\underline{a}] - \sum_{{}_0[\underline{c}] \in \alpha_m \setminus \gamma, [\underline{a},\underline{c}] \neq \varnothing} \mu^+[\underline{a},\underline{c}] \\
&= \mu^+[\underline{a}] \left(1 - C^* \sum_{{}_0[\underline{c}] \in \alpha_m \setminus \gamma, [\underline{a},\underline{c}] \neq \varnothing} \mu^+[\underline{c}] \right) \\
&\geq \mu^+[\underline{a}] \left(1 - C^* \sum_{{}_0[\underline{c}] \in \alpha_m \setminus \gamma} \mu^+[\underline{c}] \right) \\
&\geq \mu^+[\underline{a}] \big(1 - C^* (1 - e^{-\delta/2(C^*)^2}) \big), \text{ by choice of } \gamma \\
&\geq e^{-\delta} \mu^+[\underline{a}], \text{ by choice of } \delta_0.
\end{aligned}
$$

So the first bracketed sum is equal to $e^{\pm\delta} \mu^+[\underline{a}]$. Similarly, the second bracketed sum is equal to $e^{\pm\delta} \mu^+[\underline{b}]$. Thus the main term is $e^{\pm 5\delta} \mu^+[\underline{a}] \mu^+[\underline{b}] = e^{\pm 5\delta} \mu(A) \mu(B)$.

Estimate of the error term: Since $a_n \in S^*$, (3.4) implies that

$$
(L^{n+k-m} 1_{[\underline{a},\underline{c}]})(y) \leq C^* e^\delta \mu^+[\underline{a}] (L^{k-m-1} 1_{[\underline{c}]})(y) \text{ on } [\underline{c}', \underline{b}].
$$

$$
\therefore \text{Error term} \leq C^* e^\delta \mu^+[\underline{a}] \sum_{\substack{{}_0[\underline{c}], {}_0[\underline{c}'] \in \alpha_m \\ {}_0[\underline{c}] \notin \gamma \text{ or } {}_0[\underline{c}'] \notin \gamma}} \int_{[\underline{c}', \underline{b}]} (L^{k-m-1} 1_{[\underline{c}]})(y) d\mu^+
$$

$$
\begin{aligned}
&= C^* e^{\delta}\mu^+[\underline{a}] \sum_{\substack{{}_0[\underline{c}],{}_0[\underline{c}']\in\alpha_m\\ {}_0[\underline{c}]\not\in\gamma \text{ or } {}_0[\underline{c}']\not\in\gamma}} \mu^+([\underline{c}]\cap\sigma^{-(k-m-1)}[\underline{c}',\underline{b}])\\
&\leq C^* e^{\delta}\mu^+[\underline{a}]\Bigg(\sum_{{}_0[\underline{c}]\in\alpha_m\setminus\gamma}\mu^+([\underline{c}]\cap\sigma^{-(k-1)}[\underline{b}]) + \sum_{{}_0[\underline{c}']\in\alpha_m\setminus\gamma}\mu^+(\sigma^{-(k-m-1)}[\underline{c}',\underline{b}])\Bigg)\\
&= C^* e^{\delta}\mu^+[\underline{a}]\Bigg(\sum_{\substack{{}_0[\underline{d}]\in\alpha_{k-1}\\ {}_0[d_0,\ldots,d_{m-1}]\not\in\gamma}}\mu^+[\underline{d},\underline{b}] + \sum_{{}_0[\underline{c}']\in\alpha_m\setminus\gamma}\mu^+[\underline{c}',\underline{b}]\Bigg)\\
&\leq (C^*)^2 e^{\delta}\mu^+[\underline{a}]\Bigg(\sum_{\substack{{}_0[\underline{d}]\in\alpha_{k-1}\\ {}_0[d_0,\ldots,d_{m-1}]\not\in\gamma}}\mu^+[\underline{d}]\mu^+[\underline{b}] + \sum_{{}_0[\underline{c}']\in\alpha_m\setminus\gamma}\mu^+[\underline{c}']\mu^+[\underline{b}]\Bigg)\quad (\because b_0\in S^*)\\
&\leq (C^*)^2 e^{\delta}\mu^+[\underline{a}]\mu^+[\underline{b}]\cdot 2\mu[(\cup\gamma)^c] \leq 2e^{\delta}\delta\mu^+[\underline{a}]\mu^+[\underline{b}] < 5\delta\mu^+[\underline{a}]\mu^+[\underline{b}].
\end{aligned}
$$

We get that the error term is less than $5\delta\mu(A)\mu(B)$.

We see that for all $k > K(\delta)$, $\mu(A\cap B) = (e^{\pm 5\delta}\pm 5\delta)\mu(A)\mu(B)$, whence

$$|\mu(A\cap B)-\mu(A)\mu(B)| \leq \mu(A)\mu(B)\max\{e^{5\delta}+5\delta-1, 1-e^{-5\delta}+5\delta\}.$$

It follows that $|\mu(A\cap B)-\mu(A)\mu(B)|\leq 2\sinh(10\delta)\mu(A)\mu(B)$.

Step 2. For every $k > K(\delta)$, for every $n\geq 0$,

$$\sum_{A\in\alpha_{-n}^0, B\in\alpha_k^{k+n}} |\mu(A\cap B)-\mu(A)\mu(B)| < 2\sinh(10\delta)+4\delta.$$

Proof. Write $A = {}_{-n}[a_0,\ldots,a_n]$ and $B = {}_k[b_0,\ldots,b_n]$. We break the sum into

1. the sum over A, B s.t. $a_n, b_0\in S^*$;
2. the sum over A, B s.t. $a_n\not\in S^*$;
3. the sum over A, B s.t. $a_n\in S^*$ and $b_0\not\in S^*$.

The first sum is less than $2\sinh(10\delta)$. The second and third sums are bounded by $2\mu[(\bigcup_{a\in S^*}{}_0[a])^c] < 2(1-e^{-\delta}) < 2\delta$.

Step 3. $\alpha(\mathscr{V}')$ has the weak Bernoulli property for every finite $\mathscr{V}'\subset\mathscr{V}$.

Proof. Choose δ so small that $2\sinh(10\delta)+4\delta<\varepsilon$ and take $K = K(\delta)$ as above, then $\sum_{A\in\alpha_{-n}^0, B\in\alpha_k^{k+n}}|\mu(A\cap B)-\mu(A)\mu(B)|<\varepsilon$ for all $n\geq 0$. Since the partitions $\alpha(\mathscr{V}')_{-n}^0$ and $\alpha(\mathscr{V}')_k^{k+n}$ are coarser than α_{-n}^0 and α_k^{k+n}, the weak Bernoulli property for $\alpha(\mathscr{V}')$ follows by the triangle inequality. □

4. Step 3: The Non-Mixing Case

Lemma 4.1 (Adler, Shields, and Smorodinsky)**.** *Let $(X,\mathscr{B},\mu,T)$ be an ergodic invertible probability preserving transformation with a measurable set X_0 of positive measure such that*

1. $T^p(X_0) = X_0 \bmod \mu$;
2. $X_0, T(X_0),\ldots,T^{p-1}(X_0)$ *are pairwise disjoint* $\bmod \mu$;
3. $T^p : X_0\to X_0$ *equipped with* $\mu(\cdot|X_0)$ *is a Bernoulli automorphism.*

Then $(X,\mathscr{B},\mu,T)$ is measure theoretically isomorphic to the product of a Bernoulli scheme and a finite rotation.

Proof (see [**ASS**]*).* Let $X_i := T^i(X)$ $(i = 0, \dots, p-1)$. Since T is ergodic and measure preserving, $\mu(X_i) = \frac{1}{p}$ for all p. Also, $T^p(X_i) = X_i \bmod \mu$ for all i. Since T is invertible, $T^p : X_i \to X_i$ equipped with $\mu_i := \mu(\cdot|X_i)$ is isomorphic to $T^p : X_0 \to X_0$. It follows that $h_{\mu_i}(T^p)$ are all equal. Since $\mu = \frac{1}{p}(\mu_0 + \cdots + \mu_{p-1})$ and since $\mu \mapsto h_\mu(T^p)$ is affine, $h_{\mu_i}(T^p|_{X_i}) = h_\mu(T^p) = p h_\mu(T)$ for every i.

Let $(\Sigma, \mathscr{F}, m, S)$ denote a Bernoulli scheme s.t. $h_m(S) = h_\mu(T)$. The map $S^p : \Sigma \to \Sigma$ is isomorphic to a Bernoulli scheme with entropy $p h_\mu(T)$. It follows that S^p is isomorphic to $T^p : X_0 \to X_0$. Let $\vartheta : X_0 \to \Sigma$ be an isomorphism map: $\vartheta \circ T^p = S^p \circ \vartheta$. Define:

- $F_p := \{0, 1, \dots, p-1\}$
- $R : F_p \to F_p$, $R(x) = x + 1 \pmod p$
- $\Pi : X \to \Sigma \times F_p$, $\Pi(x) = (S^i[\vartheta(y)], i)$ for the unique $(y, i) \in X_0 \times F_p$ s.t. $x = T^i(y)$ (this makes sense on a set of full measure).

Π is an isomorphism from $(X, \mathscr{B}, \mu, T)$ to $(\Sigma \times F_p, \mathscr{F} \otimes 2^{F_0}, m \times c, S \times R)$, where c is $\frac{1}{p}\times$the counting measure on F_p:

(1) Π is invertible: The inverse function is $(z, i) \mapsto T^i(\vartheta^{-1}[S^{-i}(z)])$.
(2) $\Pi \circ T = (S \times R) \circ \Pi$: Suppose $x \in X$, and write $x = T^i(y)$ with $(y, i) \in X_0 \times F_p$. If $i < p-1$, then $T(x) = T^{i+1}(y)$ with $(y, i+1) \in X_0 \times F_p$, so $\Pi[T(x)] = (S^{i+1}[\vartheta(y)], i+1) = (S \times R)(S^i[\vartheta(y)], i)) = (S \times R)[\Pi(x)]$. If $i = p-1$, then $T(x) = T[T^{p-1}(y)]$ and $(T^p(y), 0) \in X_0 \times F_p$. Since $\vartheta \circ T^p = S^p \circ \vartheta$ on X_0, $\Pi[T(x)] = (\vartheta(T^p y), 0) = (S^p[\vartheta(y)], R(p-1)) = (S \times R)(S^i[\vartheta(y)], i) = (S \times R)[\Pi(x)]$. In all cases, $\Pi \circ T = (S \times R) \circ \Pi$.
(3) $\mu \circ \Pi^{-1} = m \times c$: For every Borel set $E \subset \Sigma$ and $i \in F_p$,

$$\begin{aligned}(\mu \circ \Pi^{-1})(E \times \{i\}) &= \mu[\vartheta^{-1} S^{-i}(E)] = \mu(X_0)\mu(\vartheta^{-1} S^{-i}(E)|X_0) \\ &= \mu(X_0)(\mu_0 \circ \vartheta^{-1})(S^{-i}E) = \frac{1}{p} m(S^{-i}E) = \frac{1}{p} m(E) = (m \times c)(E \times \{i\}).\end{aligned}$$

It follows that Π is a measure theoretic isomorphism. $\square$

Proof of Theorem 1.1. Suppose μ is an equilibrium measure with positive entropy for f and the Hölder potential $\Psi : M \to \mathbb{R}$. Fix some $0 < \chi < h_\mu(f)$. By Theorems 2.1 and 2.3, there exists a countable Markov shift $\sigma : \Sigma \to \Sigma$, a Hölder continuous map $\pi : \Sigma \to M$, and a shift invariant ergodic probability measure $\widehat{\mu}$ on Σ s.t. $\widehat{\mu} \circ \pi^{-1} = \mu$ and $h_{\widehat{\mu}}(\sigma) = h_\mu(f)$. In particular, if $\psi := \Psi \circ \pi$, then $h_{\widehat{\mu}}(\sigma) + \int \psi d\widehat{\mu} = h_\mu(f) + \int \Psi d\mu$.

For any other ergodic shift invariant probability measure $\widehat{m}$, there is a set of full measure $\widehat{\Sigma} \subset \Sigma$ s.t. $\pi : \widehat{\Sigma} \to M$ is finite-to-one (Theorem 2.2). Therefore the f–invariant measure $m := \widehat{m} \circ \pi^{-1}$ has the same entropy as $\widehat{m}$, whence

$$h_{\widehat{m}}(\sigma) + \int \psi d\widehat{m} = h_m(f) + \int \Psi dm \le h_\mu(f) + \int \Psi d\mu = h_{\widehat{\mu}}(\sigma) + \int \psi d\widehat{\mu}.$$

It follows that $\widehat{\mu}$ is an equilibrium measure for $\sigma : \Sigma \to \Sigma$ and ψ.

We wish to apply Theorem 3.1. The potential ψ is Hölder continuous, bounded, and $P_G(\psi) = h_\mu(f) + \int \Psi d\mu < \infty$. But $\sigma : \Sigma \to \Sigma$ may not be topologically mixing. To deal with this difficulty we appeal to the spectral decomposition theorem.

Since $\widehat{\mu}$ is ergodic, it is carried by a topologically transitive $\Sigma' = \Sigma(\mathscr{G}')$ where $\mathscr{G}'$ is a subgraph of $\mathscr{G}$. Let p denote the period of Σ' (see §2). The Spectral

Decomposition Theorem for CMS [**K**, Remark 7.1.35] states that

$$\Sigma' = \Sigma'_0 \uplus \Sigma'_1 \uplus \cdots \uplus \Sigma'_{p-1}$$

where every Σ'_i is a union of states of Σ, $\sigma(\Sigma'_i) = \Sigma'_{(i+1)\mathrm{mod} p}$, and $\sigma^p : \Sigma'_i \to \Sigma'_i$ is topologically mixing. Each $\sigma^p : \Sigma'_i \to \Sigma'_i$ is topologically conjugate to the CMS $\Sigma(\mathscr{G}'_i)$ where $\mathscr{G}'_i$ is the directed graph with

- vertices $(v_0, v_1, \ldots, v_{p-1})$ where $v_0 \to \cdots \to v_{p-1}$ is a path in $\mathscr{G}'$ which starts at one of the states in Σ'_i,
- and edges $(v_0, \ldots, v_{p-1}) \to (w_0, \ldots, w_{p-1})$ iff $v_{p-1} = w_0$.

Let $\widehat{\mu}_i := \widehat{\mu}(\cdot|\Sigma'_i)$. It is not difficult to see that $\widehat{\mu}_i$ is an equilibrium measure for $\sigma^p : \Sigma'_i \to \Sigma'_i$ with respect to the potential $\psi_p := \psi + \psi \circ \sigma + \cdots + \psi \circ \sigma^{p-1}$. It is also not difficult to see that ψ_p can be identified with a bounded Hölder continuous potential ψ^i_p on $\Sigma(\mathscr{G}'_i)$ and that $P_G(\psi^i_p) = pP_G(\psi) < \infty$.

By Theorem 3.1, $\sigma^p : \Sigma'_i \to \Sigma'_i$ equipped with $\widehat{\mu}_i$ is isomorphic to a Bernoulli scheme.

Let $X_i := \pi(\Sigma'_i)$. Since $\pi \circ \sigma = f \circ \pi$, $f(X_i) = X_{(i+1)\mathrm{mod}\ p}$. Each X_i is f^p–invariant, and $f^p : X_i \to X_i$ equipped with $\mu_i := \mu(\cdot|X_i)$ is a factor of $\sigma^p : \Sigma'_i \to \Sigma'_i$. By Ornstein's Theorem [**O1**], factors of Bernoulli automorphisms are Bernoulli automorphisms. So $f^p : X_i \to X_i$ are Bernoulli automorphisms.

In particular, $f^p : X_i \to X_i$ are ergodic. Since $X_i \cap X_j$ is f^p–invariant, either $X_i = X_j$ or $X_i \cap X_j = \varnothing \mod \mu$. So there exists $q|p$ s.t. $M = X_0 \uplus \cdots \uplus X_{q-1} \mod \mu$. Since $q|p$, $f(X_i) = X_{(i+1)\mathrm{mod}\ q}$, and $f^q : X_0 \to X_0$ is a root of $f^p : X_0 \to X_0$. Since f^p is Bernoulli, f^q is Bernoulli [**O3**]. By Lemma 4.1, $(M, \mathscr{B}(M), \mu, f)$ is isomorphic to the product of a Bernoulli scheme and a finite rotation. ☐

5. Concluding remarks

We discuss some additional consequences of the proof we presented in the previous sections. In what follows $f : M \to M$ is a $C^{1+\alpha}$ surface diffeomorphism on a compact smooth orientable surface. We assume throughout that the topological entropy of f is positive.

5.1. The measure of maximal entropy is virtually Markov.

Equilibrium measures for $\Psi \equiv 0$ are called *measures of maximal entropy* for obvious reasons.

A famous theorem of Adler & Weiss [**AW**] says that an ergodic measure of maximal entropy $\mu_{\max}$ for a hyperbolic toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ can be coded as finite state Markov chain. More precisely, there exists a subshift of finite type $\sigma : \Sigma \to \Sigma$ and a Hölder continuous map $\pi : \Sigma \to \mathbb{T}^2$ such that (a) $\pi \circ \sigma = f \circ \pi$; (b) $\mu_{\max} = \widehat{\mu}_{\max} \circ \pi^{-1}$ where $\widehat{\mu}_{\max}$ is an ergodic Markov measure on Σ; and (c) π is a measure theoretic isomorphism.

This was extended by Bowen [**B2**] to all Axiom A diffeomorphisms, using Parry's characterization of the measure of maximal entropy for a subshift of finite type [**Pa**]. Bowen's result holds in any dimension.

In dimension two, we have the following generalization to general $C^{1+\alpha}$ surface diffeomorphisms with positive topological entropy:

Theorem 5.1. *Suppose $\mu_{\max}$ is an ergodic measure of maximal entropy for f, then there exists a topologically transitive CMS $\sigma : \Sigma \to \Sigma$ and a Hölder continuous map $\pi : \Sigma \to M$ s.t.* (a) *$\pi \circ \sigma = f \circ \pi$;* (b) *$\mu_{\max} = \widehat{\mu}_{\max} \circ \pi^{-1}$ where $\widehat{\mu}_{\max}$ is an ergodic Markov measure on Σ; and* (c) *$\exists \Sigma' \subset \Sigma$ of full measure s.t. $\pi|_{\Sigma'}$ is n–to–one.*

Proof. The arguments in the previous section show that $\mu_{\max} = \widehat{\mu}_{\max} \circ \pi^{-1}$ where $\widehat{\mu}_{\max}$ is an ergodic measure of maximal entropy on some topologically transitive countable Markov shift $\Sigma(\mathscr{G})$ and $\pi : \Sigma(\mathscr{G}) \to M$ is Hölder continuous map s.t. $\pi \circ \sigma = f \circ \pi$ and such that π is finite-to-one on a set of full $\widehat{\mu}_{\max}$–measure. Since $x \mapsto |\pi^{-1}(x)|$ is f–invariant, π is n–to–one on a set of full measure for some $n \in \mathbb{N}$.

Gurevich's Theorem [**G**] says that $\widehat{\mu}_{\max}$ is a Markov measure. Ergodicity forces the support of $\widehat{\mu}_{\max}$ to be a topologically transitive sub–CMS of $\Sigma(\mathscr{G})$. □

The example mentioned in the introduction shows that the theorem is false in dimension larger than two.

5.2. **Equilibrium measures for $-t \log J_u$.** Theorem 1.1 was stated for equilibrium measures μ of Hölder continuous functions $\Psi : M \to \mathbb{R}$, but the proof works equally well for any function Ψ s.t. $\psi := \Psi \circ \pi_\chi$ is a bounded Hölder continuous function on Σ_χ. Here χ is any positive number strictly smaller than $h_\mu(f)$, and $\pi_\chi : \Sigma_\chi \to M$ is the Markov extension described in §2.

We discuss a particular example which appears naturally in hyperbolic dynamics (see e.g. [**BP**], [**L**],[**B1**]).

Let M' denote the set of $x \in M$ s.t. T_xM splits into the direct sum of two one–dimensional spaces $E^s(x)$ and $E^u(x)$ so that $\limsup\limits_{n\to\infty} \frac{1}{n} \log \|df_x^n \underline{v}\|_{f^n(x)} < 0$ for all $\underline{v} \in E^s(x) \setminus \{\underline{0}\}$, and $\limsup\limits_{n\to\infty} \frac{1}{n} \log \|df_x^{-n} \underline{v}\|_{f^{-n}(x)} < 0$ for all $\underline{v} \in E^u(x) \setminus \{\underline{0}\}$. It is well–known that if the spaces $E^s(x)$, $E^u(x)$ exist, then they are unique, and $df_x[E^u(x)] = E^u(f(x))$, $df_x[E^s(x)] = E^s(f(x))$.

Definition 5.1. *The* unstable Jacobian *is* $J_u(x) := |\det(df_x|_{E^u(x)})|$ $(x \in M')$.

Equivalently, $J_u(x)$ is the unique positive number s.t. $\|df_x(\underline{v})\|_{f(x)} = J_u(x)\|\underline{v}\|_x$ for all $\underline{v} \in E^u(x)$.

Notice that $J_u(x)$ is only defined on M'. Oseledets' Theorem and Ruelle's Entropy Inequality guarantee that $\mu(M \setminus M') = 0$ for every f–ergodic invariant measure with positive entropy.

The maps $x \mapsto E^u(x)$, $x \mapsto E^s(x)$ are in general not smooth. Brin's Theorem states that these maps are Hölder continuous on Pesin sets [**BP**, §5.3]. Therefore $J_u(x)$ is Hölder continuous on Pesin sets. We have no reason to expect $J_u(x)$ to extend to a Hölder continuous function on M.

Luckily, the following holds [**S2**, Proposition 12.2.1]: For the Markov extension $\pi_\chi : \Sigma_\chi \to M$, $E^u(\pi(\underline{u})), E^s(\pi(\underline{u}))$ are well–defined for every $\underline{u} \in \Sigma$, and the maps $\underline{u} \mapsto E^u(\underline{u}), \underline{u} \mapsto E^s(\underline{u})$ are Hölder continuous on Σ_χ. As a result $J_u \circ \pi$ is a globally defined bounded Hölder continuous function on Σ_χ.

Since f is a diffeomorphism, $\log(J_u \circ \pi)$ is also globally defined, bounded and Hölder continuous.

Theorem 5.2. *Suppose μ maximizes $h_\mu(f) - t\int(\log J_u)d\mu$ among all ergodic invariant probability measures carried by M'. If $h_\mu(f) > 0$, then f is measure theoretically isomorphic w.r.t. μ a Bernoulli scheme times a finite rotation.*

The case $t = 1$ follows from the work of Ledrappier [**L**], see also Pesin [**Pe**].

5.3. **How many ergodic equilibrium measures with positive entropy?**

Theorem 5.3. *A Hölder continuous potential on M has at most countably many ergodic equilibrium measures with positive entropy.*

Proof. Fix $\Psi : M \to \mathbb{R}$ Hölder continuous (more generally a function such that ψ defined below is Hölder continuous).

Given $0 < \chi < h_{top}(f)$, we show that Ψ has at most countably many ergodic equilibrium measures μ s.t. $h_\mu(f) > \chi$.

Let $\pi_\chi : \Sigma_\chi \to M$ denote the Markov extension described in §2, and let $\mathscr{G}$ denote the directed graph s.t. $\Sigma_\chi = \Sigma(\mathscr{G})$. We saw in the proof of Theorem 1.1 that every ergodic equilibrium measure μ for Ψ s.t. $h_\mu(f) > \chi$ is the projection of some ergodic equilibrium measure for $\psi := \Psi \circ \pi_\chi : \Sigma(\mathscr{G}) \to \mathbb{R}$. So it is enough to show that ψ has at most countably many ergodic equilibrium measures.

Every ergodic equilibrium measure μ on $\Sigma(\mathscr{G})$ is carried by $\Sigma(\mathscr{H})$ where (i) $\mathscr{H}$ is a subgraph of $\mathscr{G}$, (ii) $\sigma : \Sigma(\mathscr{H}) \to \Sigma(\mathscr{H})$ is topologically transitive, and (iii) $\Sigma(\mathscr{H})$ carries an equilibrium measure for $\psi : \Sigma(\mathscr{G}) \to \mathbb{R}$. Simply take the subgraph with vertices a s.t. $\mu({}_0[a]) \neq 0$ and edges $a \to b$ s.t. $\mu({}_0[a,b]) \neq 0$.

For every subgraph $\mathscr{H}$ satisfying (i),(ii), and (iii) there is exactly one equilibrium measure for ψ on $\Sigma(\mathscr{H})$. The support of this measure is $\Sigma(\mathscr{H})$, see Corollary 3.4 and Theorem 3.1.

So every ergodic equilibrium measure sits on $\Sigma(\mathscr{H})$ where $\mathscr{H}$ satisfies (i), (ii), and (iii), and every such $\Sigma(\mathscr{H})$ carries exactly one measure like that. As a result, it is enough to show that $\mathscr{G}$ contains at most countably many subgraphs $\mathscr{H}$ satisfying (i), (ii), and (iii).

We do this by showing that any two different subgraphs $\mathscr{H}_1$, $\mathscr{H}_2$ like that have disjoint sets of vertices. Assume by contradiction that $\mathscr{H}_1, \mathscr{H}_2$ share a vertex. Then $\mathscr{H} := \mathscr{H}_1 \cup \mathscr{H}_2$ satisfies (i), (ii), and (iii). By the discussion above, $\Sigma(\mathscr{H})$ carries at most one equilibrium measure for ψ. But it carries at least two such measures: one with support $\Sigma(\mathscr{H}_1)$ and one with support $\Sigma(\mathscr{H}_2)$. This contradiction shows that $\mathscr{H}_1$ and $\mathscr{H}_2$ cannot have common vertices. □

The case $\Psi = -\log J_u$ is due to Ledrappier [**L**] and Pesin [**Pe**]. The case $\Psi \equiv 0$ was done at [**S2**]. Buzzi [**Bu**] had shown that the measure of maximal entropy of a piecewise affine surface homeomorphism has finitely many ergodic components, and has conjectured that a similar result holds for C^∞ surface diffeomorphisms with positive topological entropy.

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