Ergodic Theory: Home Assignment 9

Let $(X, \mathcal{B}, T, \mu)$ be a ppt of a Lebesgue space. Define $H(T) := \{\lambda \in \mathbb{C} : \exists f \in L^2(\mu), f \circ T = \lambda f \bmod \mu\}$. This set is called *the point spcectrum of* T.

1. Show that $H(T) \subseteq \{z \in \mathbb{C} : |z| = 1\}$.
2. Show that if $f \in L^2(\mu)$ and $f \circ T = \lambda f \bmod \mu$, then $|f| \equiv$const. μ-a.e.
3. Show that if $\lambda, \sigma \in H(T)$, then also $1, \lambda\sigma, \lambda^{-1} \in H(T)$.
4. Show that if $f, g \in L^2(\mu)$ and $f \circ T = \lambda f, g \circ T = \sigma g \bmod \mu$ for $\lambda \neq \sigma$, then $\langle f, g \rangle = 0$ (the inner-product of $L^2(\mu)$ as defined in the lecture for complex functions).

Deduce that $H(T)$ is a countable subgroup of the unit circle.

Good luck!