Ergodic Theory: Home Assignments 7+8

Let S be a finite set, $A \in M_{S\times S}(\{0,1\})$ be a transition matrix, and (X,σ) be a TMS induced by A and S. Assume that (X,σ) is irreducible. Fix a symbol $a \in S$.

$\forall 0 \le k \le p-1$, $S_k := \{b \in S : \exists n \text{ s.t. } n = k \bmod p, a \xrightarrow{n} b\}$. We've seen in class that $X = \bigcup_{k=0}^{p-1} S_k$, and $\{S_k\}_{k=0}^{p-1}$ are pairwise disjoint. We've also seen that $\forall 0 \le k \le p-1$ $\mathcal{W}_k := \{(a_0, ..., a_{p-1}) : a_0 \in S_k, a_{p-1} \in S_{k-1 \bmod p}\}$ induces a new TMS X_k with permitted transitions $w^1 \to w^2$ iff $w^1_{p-1} \to w^2_0$, $w^1, w^2 \in \mathcal{W}_k$. $\sigma_k : X_k \to X_k$ is naturally associated with $\sigma^p : X \to X$, as seen in class.

1. $p := \gcd(\{n : b \xrightarrow{n} b\})$ is independent of the choice of $b \in S$.
2. $\forall 0 \le k \le p-1$, $\gcd(\{n : w \xrightarrow{n} w\})$ for some (any) $w \in \mathcal{W}_k$ (edges are w.r.t to σ_k).
3. $\forall 0 \le k \le p-1$, $\forall w \in \mathcal{W}_k$ $\exists N_w$ s.t. $\forall n > N_w$ $w \xrightarrow{n} w$ (edges are w.r.t to σ_k).

Please solve all three questions until Jan 5th. When solving any question, you may assume the correctness of the preceding questions. Guidance for questions 1 and 3 may be found in Omri's notes, in §1.5.4.2 Ergodicity And Mixing, within footnotes 1 and 3.

Good luck!