Let $S$ be a finite set, $A \in M_{S \times S}(\{0, 1\})$ be a transition matrix, and $(X, \sigma)$ be a TMS induced by $A$ and $S$. Assume that $(X, \sigma)$ is irreducible. Fix a symbol $a \in S$.

\[
\forall 0 \leq k \leq p - 1, S_k := \{b \in S : \exists n \text{ s.t. } n = k \mod p, a \xrightarrow{n} b\}. \quad \text{We’ve seen in class that } X = \bigcup_{k=0}^{p-1} S_k, \text{ and } \{S_k\}_{k=0}^{p-1} \text{ are pairwise disjoint. We’ve also seen that } \forall 0 \leq k \leq p - 1 \text{ } W_k := \{(a_0, \ldots, a_{p-1}) : a_0 \in S_k, a_{p-1} \in S_{k-1 \mod p}\} \text{ induces a new TMS } X_k \text{ with permitted transitions } w^1 \to w^2 \text{ iff } w^1_{p-1} \to w^2_0, w^1, w^2 \in W_k. \quad \sigma_k : X_k \to X_k \text{ is naturally associated with } \sigma^p : X \to X, \text{ as seen in class.}
\]

1. \(p := \gcd(\{n : b \xrightarrow{n} b\})\) is independent of the choice of $b \in S$.

2. \(\forall 0 \leq k \leq p - 1, \gcd(\{n : w \xrightarrow{n} w\})\) for some (any) $w \in W_k$ (edges are w.r.t to $\sigma_k$).

3. \(\forall 0 \leq k \leq p - 1, \forall w \in W_k \exists N_w \text{ s.t. } \forall n > N_w w \xrightarrow{n} w\) (edges are w.r.t to $\sigma_k$).

Please solve all three questions until Jan 5th. When solving any question, you may assume the correctness of the preceding questions. Guidance for questions 1 and 3 may be found in Omri’s notes, in §1.5.4.2 Ergodicity And Mixing, within footnotes 1 and 3.

Good luck!