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Lecture Notes on Ergodic Theory

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If you find additional errors please let me know! O.S.

Contents

Chapter 1
Basic definitions and constructions

1.1 What is ergodic theory and how it came about

Dynamical systems and ergodic theory. Ergodic theory is a part of the theory of dynamical systems. At its simplest form, a *dynamical system* is a function T defined on a set X. The *iterates* of the map are defined by induction $T^0 := id$, $T^n := T \circ T^{n-1}$, and the aim of the theory is to describe the behavior of $T^n(x)$ as $n \to \infty$.

More generally one may consider the *action of a semi-group* of transformations, namely a family of maps $T_g : X \to X$ $(g \in G)$ satisfying $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$. In the particular case $G = \mathbb{R}^+$ or $G = \mathbb{R}$ we have a family of maps T_t such that $T_t \circ T_s = T_{t+s}$, and we speak of a *semi-flow* or a *flow*.

The original motivation was classical mechanics. There X is the set of all possible states of given dynamical system (sometimes called *configuration space*) or *phase space*), and $T : X \to X$ is the *law of motion* which prescribes that if the system is at state x now, then it will evolve to state $T(x)$ after one unit of time. The *orbit* $\{T^n(x)\}_{n \in \mathbb{Z}}$ is simply a record of the time evolution of the system, and the understanding the behavior of $T^n(x)$ as $n \to \infty$ is the same as understanding the behavior of the system at the far future. Flows T_t arise when one insists on studying continuous, rather than discrete time. More complicated group actions, e.g. $\mathbb{Z}^d$–actions, arise in material science. There $x \in X$ codes the configuration of a d–dimensional lattice (e.g. a crystal), and $\{T_{\underline{v}} : \underline{v} \in \mathbb{Z}^d\}$ are the symmetries of the lattice.

The theory of dynamical systems splits into subfields which differ by the structure which one imposes on X and T:

1. *Differentiable dynamics* deals with actions by differentiable maps on smooth manifolds;
2. *Topological dynamics* deals with actions of continuous maps on topological spaces, usually compact metric spaces;
3. *Ergodic theory* deals with measure preserving actions of measurable maps on a measure space, usually assumed to be finite.

It may seem strange to assume so little on X and T. The discovery that such meagre assumptions yield non trivial information is due to Poincaré, who should be considered the progenitor of the field.

Poincaré's Recurrence Theorem and the birth of ergodic theory. Imagine a box filled with gas, made of N identical molecules. Classical mechanics says that if we know the positions $\underline{q}_i = (q_i^1, q_i^2, q_i^3)$ and momenta $\underline{p}_i = (p_i^1, p_i^2, p_i^3)$ of the i-th molecule for all $i = 1, \ldots, N$, then we can determine the positions and momenta of each particle at time t by solving Hamilton's equations

$$\begin{aligned} \dot{p}_i^j(t) &= -\partial H / \partial q_i^j \\ \dot{q}_i^j(t) &= \partial H / \partial p_i^j. \end{aligned} \tag{1.1}$$

$H = H(\underline{q}_1, \ldots, \underline{q}_N; \underline{p}_1, \ldots, \underline{p}_N)$, the *Hamiltonian*, is the total energy of the system.

It is natural to call $(\underline{q}, \underline{p}) := (\underline{q}_1, \ldots, \underline{q}_N; \underline{p}_1, \ldots, \underline{p}_N)$ the *state* of the system. Let X denote the collection of all possible states. If we assume (as we may) that the total energy is bounded above, then for many reasonable choices of H this is a open bounded subset of $\mathbb{R}^{6N}$. Let

$$T_t : (\underline{q}, \underline{p}) \mapsto (\underline{q}(t), \underline{p}(t))$$

denote the map which gives solution of (1.1) with initial condition $(\underline{q}(0), \underline{p}(0))$. If H is sufficiently regular, then (1.1) had a unique solution for all t and every initial condition. The uniqueness of the solution implies that T_t is a flow. The law of conservation of energy implies that $\underline{x} \in X \Rightarrow T_t(\underline{x}) \in X$ for all t.

Question: Suppose the system starts at a certain state $(\underline{q}(0), \underline{p}(0))$, will it eventually return to a state close to $(\underline{q}(0), \underline{p}(0))$?

For general H, the question seems intractable because (1.1) is strongly coupled system of an enormous number of equations ($N \sim 10^{24}$). Poincaré's startling discovery is that the question is trivial, if viewed from the right perspective. To understand his solution, we need to recall a classical fact, known as *Liouville's theorem*: The Lebesgue measure m on X satisfies $m(T_t E) = m(E)$ for all t and all measurable $E \subset X$ (problem 1.1).

Here is Poincaré's solution. Define $T := T_1$, and observe that $T^n = T_n$. Fix $\varepsilon > 0$ and consider the set W of all states $\underline{x} = (\underline{q}, \underline{p})$ such that $d(\underline{x}, T^n(\underline{x})) > \varepsilon$ for all $n \geq 1$ (here d is the Euclidean distance). Divide W into finitely many disjoint pieces W_i of diameter less than ε.

For each fixed i, the sets $T^{-n}(W_i)$ $(n \geq 1)$ are pairwise disjoint: Otherwise $T^{-n}(W_i) \cap T^{-(n+k)}(W_i) \neq \varnothing$, so $W_i \cap T^{-k}(W_i) \neq \varnothing$, and there exists $\underline{x} \in W_i \cap T^{-k}(W_i)$. But this leads to a contradiction:

1. $\underline{x} \in T^{-k}(W_i)$ implies that $T^k(\underline{x}) \in W_i$ whence $d(\underline{x}, T^k \underline{x}) \leq \operatorname{diam}(W_i) < \varepsilon$, whereas
2. $\underline{x} \in W_i \subset W$ implies that $d(\underline{x}, T^k \underline{x}) > \varepsilon$ by the definition of W.

So $T^{-n}(W_i)$ $(n \geq 1)$ are pairwise disjoint.

Since $\{T^{-n}W_i\}_{n\geq 1}$ are pairwise disjoint, $m(X) \geq \sum_{k\geq 1} m(T^{-k}W_i)$. But $T^{-k}(W_i)$ all have the same measure (Liouville theorem), and $m(X) < \infty$, so we must have $m(W_i) = 0$. Summing over i we get that $m(W) = 0$. In summary, a.e. $\underline{x}$ has the property that $d(T^n(\underline{x}), \underline{x}) < \varepsilon$ for some $n \geq 1$. Considering the countable collection $\varepsilon = 1/n$, we obtain the following result:

Poincaré's Recurrence Theorem: For almost every $\underline{x} = (\underline{q}(0), \underline{p}(0))$, if the system is at state $\underline{x}$ at time zero, then it will return arbitrarily close to this state infinitely many times in the arbitrarily far future.

Poincaré's Recurrence Theorem is a tour de force, because it turns a problem which looks intractable to a triviality by simply looking at it from a different angle. The only thing the solution requires is the existence of a finite measure on X such that $m(T^{-1}E) = m(E)$ for all measurable sets E. This startling realization raises the following mathematical question: What other dynamical information can one extract from the existence of a measure m such that $m = m \circ T^{-1}$? Of particular interest was the justification of the following "assumption" made by Boltzmann in his work on statistical mechanics:

The Ergodic Hypothesis: For certain invariant measures μ, many functions $f : X \to \mathbb{R}$, and many states $\underline{x} = (\underline{q}, \underline{p})$, the time average $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T f(T_t(\underline{x}))dt$ exists, and equals the space average $\frac{1}{\mu(X)}\int f d\mu$.

(This is not Boltzmann's original formulation.) The ergodic hypothesis is a quantitative version of Poincaré's recurrence theorem: If f is the indicator of the ε–ball around a state $\underline{x}$, then the time average of f is the frequency of times when $T_t(\underline{x})$ is ε–away from $\underline{x}$, and the ergodic hypothesis is a statement on its value.

1.2 The abstract setup of ergodic theory

The proof of Poincaré's Recurrence Theorem suggests the study of the following setup.

Definition 1.1. A *measure space* is a triplet $(X, \mathscr{B}, \mu)$ where

1. X is a set, sometime called the *space*.
2. $\mathscr{B}$ is a σ*–algebra*: a collection of subsets of X which contains the empty set, and which is closed under complements, countable unions and countable intersections. The elements of $\mathscr{B}$ are called *measurable sets*.
3. $\mu : \mathscr{B} \to [0, \infty]$, called the *measure*, is a σ*–additive function*: if $E_1, E_2, \ldots \in \mathscr{B}$ are pairwise disjoint, then $\mu(\biguplus_i E_i) = \sum_i \mu(E_i)$.

If $\mu(X) = 1$ then we say that μ is a *probability measure* and $(X, \mathscr{B}, \mu)$ is a *probability space*.

In order to avoid measure theoretic pathologies, we will always assume that $(X, \mathscr{B}, \mu)$ is the completion (see problem 1.2) of a *standard measure space*: a measure space $(X,\mathscr{B}',\mu')$, where X is a complete and separable metric space and $\mathscr{B}'$ is its Borel σ–algebra.

It can be shown that such spaces are *Lebesgue spaces*: They are isomorphic to the union of a compact interval equipped with the Lebesgue's σ–algebra and Lebesgue's measure, and a finite or countable or empty collection of points with positive measure. See the appendix for details.

Definition 1.2. A *measure preserving transformation* (mpt) is a quartet $(X,\mathscr{B},\mu,T)$ where $(X,\mathscr{B},\mu)$ is a measure space, and

1. T is measurable: $E\in\mathscr{B}\Rightarrow T^{-1}E\in\mathscr{B}$;
2. m is T–invariant: $\mu(T^{-1}E)=\mu(E)$ for all $E\in\mathscr{B}$.

A *probability preserving transformation* (ppt) is a mpt on a probability space.

This is the minimal setup needed to prove (problem 1.3):

Theorem 1.1 (Poincaré's Recurrence Theorem). *Suppose $(X,\mathscr{B},\mu,T)$ is a p.p.t. If E is a measurable set, then for almost every $x\in E$ there is a sequence $n_k\to\infty$ such that $T^{n_k}(x)\in E$.*

Poincaré's theorem is not true for general infinite measure preserving transformations, as the example $T(x)=x+1$ on $\mathbb{Z}$ demonstrates.

Having defined the objects of the theory, we proceed to declare when do we consider two objects to be isomorphic:

Definition 1.3. Two m.p.t. $(X_i,\mathscr{B}_i,\mu_i,T_i)$ are called *measure theoretically isomorphic*, if there exists a measurable map $\pi: X_1\to X_2$ such that

1. there are $X_i'\in\mathscr{B}_i$ such that $m_i(X_i\setminus X_i')=0$ and such that $\pi: X_1'\to X_2'$ is invertible with measurable inverse;
2. for every $E\in\mathscr{B}_2$, $\pi^{-1}(E)\in\mathscr{B}_1$ and $m_1(\pi^{-1}E)=m_2(E)$;
3. $T_2\circ\pi=\pi\circ T_1$ on X_1.

One of the main aims of ergodic theorists is to develop tools for deciding whether two mpt's are isomorphic.

1.3 The probabilistic point of view.

Much of the power and usefulness of ergodic theory is due to the following probabilistic interpretation of the abstract set up discussed above. Suppose $(X,\mathscr{B},\mu,T)$ is a ppt.

1. We imagine X to be a *sample space*: the collection of all possible outcomes ω of a random experiment.
2. We interpret $\mathscr{B}$ as the collection of all *measurable events*: all sets $E\subset X$ such that we have enough information to answer the question "is $\omega\in E$?."
3. We use μ to define the *probability law*: $\Pr[\omega\in E]:=\mu(E)$;

4. Measurable functions $f : X \to \mathbb{R}$ are *random variables* $f(\omega)$;
5. The sequence $X_n := f \circ T^n$ $(n \geq 1)$ is a *stochastic process*, whose distribution is given by the formula

$$\Pr[X_{i_1} \in E_{i_1}, \dots, X_{i_k} \in E_{i_k}] := \mu\left(\bigcap_{j=1}^{k} \{\omega \in X : f(T^{i_j}\omega) \in E_{i_j}\}\right).$$

The invariance of μ guarantees that such stochastic processes are always *stationary*: $\Pr[X_{i_1+m} \in E_{i_1+m}, \dots, X_{i_k} \in E_{i_k+m}] = \Pr[X_{i_1} \in E_{i_1}, \dots, X_{i_k} \in E_{i_k+m}]$ for all m.

The point is that we can ask what are the properties of the stochastic processes $\{f \circ T^n\}_{n \geq 1}$ arising out of the ppt $(X, \mathscr{B}, \mu, T)$, and bring in tools and intuition from probability theory to the study of dynamical systems.

Note that we have found a way of studying stochastic phenomena in a context which is, a priori, completely deterministic (if we know the state of the system at time zero is x, then we know with full certainty that its state at time n is $T^n(x)$). The modern treatment of the question "how come a deterministic system can behave randomly" is based on this idea.

1.4 Ergodicity and mixing

Suppose $(X, \mathscr{B}, \mu, T)$ is a mpt. A measurable set $E \in \mathscr{B}$ is called *invariant*, if $T^{-1}(E) = E$. Evidently, in this case T can be split into two measure preserving transformations $T|_E : E \to E$ and $T|_{E^c} : E^c \to E^c$, which can be analyzed separately.

Definition 1.4. A mpt $(X, \mathscr{B}, \mu, T)$ is called *ergodic*, if every invariant set E satisfies $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. We say μ is an ergodic measure.

Proposition 1.1. *Suppose $(X, \mathscr{B}, \mu, T)$ is a mpt on a complete measure space, then the following are equivalent:*

1. μ is ergodic;
2. if $E \in \mathscr{B}$ and $\mu(T^{-1}E \triangle E) = 0$, then $\mu(E) = 0$ or $\mu(X \setminus E) = 0$;
3. if $f : X \to \mathbb{R}$ is measurable and $f \circ T = f$ a.e., then there is $c \in \mathbb{R}$ s.t. $f = c$ a.e.

Proof. Suppose μ is ergodic, and E is measurable s.t. $\mu(E \triangle T^{-1}E) = 0$. We construct a measurable set E_0 such that $T^{-1}E_0 = E_0$ and $\mu(E_0 \triangle E) = 0$. By ergodicity $\mu(E_0) = 0$ or $\mu(X \setminus E_0) = 0$. Since $\mu(E \triangle E_0) = 0$ implies that $\mu(E) = \mu(E_0)$ and $\mu(X \setminus E) = \mu(X \setminus E_0)$ we get that either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

The set E_0 we use is $E_0 := \{x \in X : T^k(x) \in E$ infinitely often.$\}$. It is obvious that this set is measurable and invariant. To estimate $\mu(E_0 \triangle E)$ note that

(a) if $x \in E_0 \setminus E$, then there exists some k s.t. $x \in T^{-k}(E) \setminus E$;
(b) if $x \in E \setminus E_0$, then there exists some k s.t. $x \notin T^{-k}(E)$, whence $x \in E \setminus T^{-k}(E)$.

Thus $E_0 \triangle E \subset \bigcup_{k\geq 1} E \triangle T^{-k}(E)$.

We now use the following "triangle inequality" :

$$\mu(A_1 \triangle A_3) \leq \mu(A_1 \triangle A_2) + \mu(A_2 \triangle A_3) \quad (A_i \in \mathscr{B})$$

(This is because $\mu(A_i \triangle A_i) = \|1_{A_i} - 1_{A_j}\|_1$ where 1_{A_i} is the *indicator function* of A_i, equal to one on A_i and to zero outside A_i). The triangle inequality implies that

$$\mu(E_0 \triangle E) \leq \sum_{k=1}^{\infty} \mu(E \triangle T^{-k}E) \leq \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \mu(T^{-i}E \triangle T^{-(i+1)}E)$$
$$= \sum_{k=1}^{\infty} k\mu(E \triangle T^{-1}E) \quad (\because \mu \circ T^{-1} = \mu).$$

Since $\mu(E \triangle T^{-1}E) = 0$, $\mu(E_0 \triangle E) = 0$ and we have shown that (1) implies (2).

Next assume (2). and let f be a measurable function s.t. $f \circ T = f$ almost everywhere. For every t, $[f > t] \triangle T^{-1}[f > t] \subset [f \neq f \circ T]$, so

$$\mu([f > t] \triangle T^{-1}[f > t]) = 0.$$

By assumption, this implies that either $\mu[f > t] = 0$ or $\mu[f \leq t] = 0$. In other words, either $f > t$ a.e., or $f \leq t$ a.e. Define $c := \sup\{t : f > t \text{ a.e.}\}$, then $f = c$ almost everywhere, proving (3). The implication (3)⇒(1) is obvious: take $f = 1_E$. □

An immediate corollary is that ergodicity is an invariant of measure theoretic isomorphism: If two mpt are isomorphic, then the ergodicity of one implies the ergodicity of the other.

The next definition is motivated by the probabilistic notion of *independence*. Suppose $(X, \mathscr{B}, \mu)$ is a *probability* space. We think of elements of $\mathscr{B}$ as of "events", we interpret measurable functions $f : X \to \mathbb{R}$ as "random variables", and we view μ as a "probability law" $\mu(E) = \mathbb{P}[x \in E]$. Two events $E, F \in \mathscr{B}$ are called *independent*, if $\mu(E \cap F) = \mu(E)\mu(F)$ (because in the case $\mu(E), \mu(F) \neq 0$ this is equivalent to saying that $\mu(E|F) = \mu(E), \mu(F|E) = \mu(F)$).

Definition 1.5. A <u>probability</u> preserving transformation $(X, \mathscr{B}, \mu, T)$ is called *mixing* (or *strongly mixing*), if for all $E, F \in \mathscr{B}$, $\mu(E \cap T^{-k}F) \xrightarrow[k\to\infty]{} \mu(E)\mu(F)$.

In other words, $T^{-k}(F)$ is "asymptotically independent" of E. It is easy to see that strong mixing is an invariant of measure theoretic isomorphism.

It can be shown that the sets E, F in the definition of mixing can be taken to be equal (problem 1.12).

Proposition 1.2. *Strong mixing implies ergodicity.*

Proof. Suppose E is invariant, then $\mu(E) = \mu(E \cap T^{-n}E) \xrightarrow[n\to\infty]{} \mu(E)^2$, whence $\mu(E)^2 = \mu(E)$. It follows that $\mu(E) = 0$ or $\mu(E) = 1 = \mu(X)$. □

Just like ergodicity, strong mixing can be defined in terms of functions. Before we state the condition, we recall a relevant notion from statistics. The *correlation coefficient* of non-constant $f,g \in L^2(\mu)$ is defined to be

$$\rho(f,g) := \frac{\int fg\,d\mu - \int f d\mu \cdot \int g d\mu}{\|f - \int f d\mu\|_2 \, \|g - \int g d\mu\|_2}.$$

The numerator is equal to

$$\mathrm{Cov}(f,g) := \int \Big[\Big(f - \int f\Big)\Big(g - \int g\Big)\Big] d\mu,$$

called the *covariance* of f,g. This works as follows: If f,g are weakly correlated then they will not always deviate from their means in the same direction, leading to many cancelations in the integral, and a small net result. If f,g are strongly correlated, there will be less cancelations, and a larger absolute value for the net result. Positive covariance signifies that the deviations from the mean are often in the same direction, and negative covariance indicates that that they are often in opposite directions. The denominator in the definition of ρ is not important. It is there to force $\rho(f,g)$ to have values in $[-1,1]$ (Cauchy-Schwarz).

Proposition 1.3. *A ppt $(X,\mathscr{B},\mu,T)$ is strongly mixing iff for every $f,g \in L^2$, $\int fg\circ T^n d\mu \xrightarrow[n\to\infty]{} \int f d\mu \int g d\mu$, equivalently* $\mathrm{Cov}(f,g\circ T^n) \xrightarrow[n\to\infty]{} 0$.

Proof. We need the following trivial observations:

1. Since $\mu\circ T^{-1} = \mu$, $\|f\circ T\|_p = \|f\|_p$ for all $f \in L^p$ and $1 \le p \le \infty$;
2. $\mathrm{Cov}(f,g)$ is bilinear in f,g;
3. $|\mathrm{Cov}(f,g)| \le 4\|f - \int f\|_2 \|g - \int g\|_2$.

The first two statements are left as an exercise. For the third we use the Cauchy-Schwarz inequality: $|\mathrm{Cov}(f,g)| \le \|f - \int f\|_2\|g - \int g\|_2 \le (\|f\|_2 + \|f\|_1)(\|g\|_2 + \|g\|_1) \overset{!}{\le} (2\|f\|_2)(2\|g\|_2)$, where $\overset{!}{\le}$ is because $\|\varphi\|_1 = \|\varphi\cdot 1\|_1 \le \|\varphi\|_2\|1\|_2 = \|\varphi\|_2$.

Now for the proof of the proposition. The condition that $\mathrm{Cov}(f,g\circ T^n) \xrightarrow[n\to\infty]{} 0$ for all $f,g \in L^2$ implies mixing by substituting $f = 1_E$, $g = 1_F$. For the other direction, assume that μ is mixing, and let f,g be two elements of L^2. If f,g are indicators of measurable sets, then $\mathrm{Cov}(f,g\circ T^n) \to 0$ by mixing. If f,g are finite linear combinations of indicators, $\mathrm{Cov}(f,g\circ T^n) \to 0$ because of the bilinearity of the covariance. For general $f,g \in L^2$, we can find for every $\varepsilon > 0$ finite linear combinations of indicators $f_\varepsilon, g_\varepsilon$ s.t. $\|f - f_\varepsilon\|_2, \|g - g_\varepsilon\|_2 < \varepsilon$. By the observations above,

$$\begin{aligned}|\mathrm{Cov}(f,g\circ T^n)| &\le |\mathrm{Cov}(f - f_\varepsilon, g\circ T^n)| + |\mathrm{Cov}(f_\varepsilon, g_\varepsilon\circ T^n)| + |\mathrm{Cov}(f_\varepsilon,(g-g_\varepsilon)\circ T^n)|\\ &\le 4\varepsilon\|g\|_2 + o(1) + 4(\|f\|_2 + \varepsilon)\varepsilon, \text{ as } n\to\infty.\end{aligned}$$

It follows that $\limsup|\mathrm{Cov}(f,g\circ T^n)| \le 4\varepsilon(\|f\|_2 + \|g\|_2 + \varepsilon)$. Since ε is arbitrary, the limsup, whence the limit itself, is equal to zero. ☐

1.5 Examples

1.5.1 Circle rotations

Let $\mathbb{T} := [0,1)$ equipped with the Lebesgue measure m, and define for $\alpha \in [0,1)$ $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(x) = x + \alpha \mod 1$. R_α is called a *circle rotation*, because the map $\pi(x) = \exp[2\pi i x]$ is an isomorphism between R_α and the rotation by the angle $2\pi\alpha$ on the unit circle S^1.

Proposition 1.4.

1. *R_α is measure preserving for every α;*
2. *R_α is ergodic iff $\alpha \notin \mathbb{Q}$;*
3. *R_α is never strongly mixing.*

Proof. A direct calculation shows that the Lebesgue measure m satisfies $m(R_\alpha^{-1}I) = m(I)$ for all intervals $I \subset [0,1)$. Thus the collection $\mathscr{M} := \{E \in \mathscr{B} : m(R_\alpha^{-1}E) = m(E)\}$ contains the algebra of finite disjoint unions of intervals. It is easy to check $\mathscr{M}$ is a monotone class, so by the monotone class theorem (see appendix) $\mathscr{M}$ contains all Borel sets. It clearly contains all null sets. Therefore it contains all Lebesgue measurable sets. Thus $\mathscr{M} = \mathscr{B}$ and (1) is proved.

We prove (2). Suppose first that $\alpha = p/q$ for $p,q \in \mathbb{N}$. Then $R_\alpha^q = id$. Fix some $x \in [0,1)$, and pick ε so small that the ε–neighborhoods of $x + k\alpha$ for $k = 0,\dots,q-1$ are disjoint. The union of these neighborhoods is an invariant set of positive measure, and if ε is sufficiently small then it is not equal to $\mathbb{T}$. Thus R_α is not ergodic.

Next assume that $\alpha \notin \mathbb{Q}$. Suppose E is an invariant set, and set $f = 1_E$. Expand f to a Fourier series:

$$f(t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n t} \quad (\text{ convergence in } L^2).$$

The invariance of E dictates $f = f \circ R_\alpha$. The Fourier expansion of $f \circ R_\alpha$ is

$$(f \circ R_\alpha)(t) = f(t + \alpha \mod 1) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\alpha} \widehat{f}(n) e^{2\pi i n t}.$$

Equating coefficients, we see that $\widehat{f}(n) = \widehat{f}(n)\exp[2\pi i n\alpha]$. Thus either $\widehat{f}(n) = 0$ or $\exp[2\pi i n\alpha] = 1$. Since $\alpha \notin \mathbb{Q}$, $\widehat{f}(n) = 0$ for all $n \neq 0$. We obtain that $f = \widehat{f}(0)$ a.e., whence $1_E = m(E)$ almost everywhere. This can only happen if $m(E) = 0$ or $m(E) = 1$, proving the ergodicity of m.

To show that m is not mixing, we consider the function $f(x) = \exp[2\pi i x]$. This function satisfies $f \circ R_\alpha = \lambda f$ with $\lambda = \exp[2\pi i\alpha]$ (such a function is called an *eigenfunction*). For every α there is a sequence $n_k \to \infty$ s.t. $n_k\alpha \mod 1 \to 0$ (Dirichlet theorem), thus $\|f \circ R_\alpha^{n_k} - f\|_2 = |\lambda^{n_k} - 1| \xrightarrow[k\to\infty]{} 0$. It follows that $F := \mathrm{Re}(f) = \cos(2\pi x)$ satisfies $\|F \circ R_\alpha^{n_k} - F\|_2 \xrightarrow[k\to\infty]{} 0$, whence $\int F \circ R_\alpha^{n_k} F dm \xrightarrow[k\to\infty]{} \int F^2 dm \neq (\int F)^2$, and m is not mixing. ☐

1.5.2 The angle doubling map

Again, we work with $\mathbb{T} := [0,1]$ equipped with the Lebesgue measure m, and define $T : \mathbb{T} \to \mathbb{T}$ by $T(x) = 2x \mod 1$. T is called the *angle doubling map*, because the map $\pi(x) := \exp[2\pi i x]$ is an isomorphism between T and the map $e^{i\theta} \mapsto e^{2i\theta}$ on S^1.

Proposition 1.5. *The angle doubling map is probability preserving, and strong mixing, whence ergodic.*

Proof. It is convenient to work with binary expansions $x = (0.d_1d_2d_3\ldots)$, $(d_i = 0,1)$, because with this representation $T(0.d_1d_2\ldots) = (0.d_2d_3\ldots)$. For every finite n–word of zeroes and ones $(d_1,\ldots,d_n)$, define the sets (called "cylinders")

$$[d_1,\ldots,d_n] := \{x \in [0,1) : x = (0.d_1\cdots d_n\varepsilon_1\varepsilon_2\ldots),\ \text{for some } \varepsilon_i \in \{0,1\}\}.$$

This is a (dyadic) interval, of length $1/2^n$.

It is clear that $T^{-1}[d_1,\ldots,d_n] = [*,d_1,\ldots,d_n]$ where $*$ stands for "0 or 1". Thus, $m(T^{-1}[\underline{d}]) = m[0,\underline{d}] + m[1,\underline{d}] = 2\cdot 2^{-(n+1)} = 2^{-n} = m[\underline{d}]$. We see that $\mathscr{M} := \{E \in \mathscr{B} : m(T^{-1}E) = m(E)\}$ contains the algebra of finite disjoint unions of cylinders. Since $\mathscr{M}$ is obviously a monotone class, and since the cylinders generate the Borel σ–algebra (prove!), we get that $\mathscr{M} = \mathscr{B}$, whence T is measure preserving.

We prove that T is mixing. Suppose f,g are indicators of cylinders: $f = 1_{[a_1,\ldots,a_n]}$, $g = 1_{[b_1,\ldots,b_m]}$. Then for all $k > n$,

$$\int f\cdot g\circ T^k dm = m[\underline{a},\underbrace{*\cdots*}_{k-n},\underline{b}] = m[\underline{a}]m[\underline{b}].$$

Thus $\mathrm{Cov}(f, g\circ T^k) \xrightarrow[k\to\infty]{} 0$ for all indicators of cylinders. Every L^2–function can be approximated in L^2 by a finite linear combination of indicators of cylinders (prove!). One can therefore proceed as in the proof of proposition 1.3 to show that $\mathrm{Cov}(f, g\circ T^k) \xrightarrow[k\to\infty]{} 0$ for all L^2 functions. ☐

1.5.3 Bernoulli Schemes

Let S be a finite set, called the *alphabet*, and let $X := S^{\mathbb{N}}$ be the set of all one–sided infinite sequences of elements of S. Impose the following metric on X:

$$d((x_n)_{n\geq 0},(y_n)_{n\geq 0}) := 2^{-\min\{k: x_k\neq y_k\}}. \tag{1.2}$$

The resulting topology is generated by the collection of *cylinders*:

$$[a_0,\ldots,a_{n-1}] := \{\underline{x} \in X : x_i = a_i \ \ (0 \leq i \leq n-1)\}.$$

It can also be characterized as being the product topology on $S^{\mathbb{N}}$, when S is given the discrete topology. In particular this topology is compact.

The *left shift* is the transformation $T:(x_0,x_1,x_2,\dots)\mapsto(x_1,x_2,\dots)$. The left shift is continuous.

Next fix a vector $\underline{p}=(p_a)_{a\in S}$ of positive numbers whose sum is equal to one.

Definition 1.6. The *Bernoulli measure* corresponding to $\underline{p}$ is the unique measure on the Borel σ–algebra of X such that $\mu[a_0,\dots,a_{n-1}]=p_{a_0}\cdots p_{a_{n-1}}$ for all cylinders.

It is useful to recall why such a measure exists. Here is a review of the necessary tools from measure theory.

Definition 1.7. Let X be a non-empty set.

1. A *semi-algebra* on X is a collection $\mathscr{S}$ os subsets of X such that
   a. $\varnothing,X\in\mathscr{S}$
   b. $\mathscr{S}$ is closed under intersections, and
   c. for every $A\in\mathscr{S}$, $X\setminus A$ is a finite disjoint union of elements from $\mathscr{S}$.
2. The σ-algebra *generated* by $\mathscr{S}$ is the smallest σ-algebra of subsets of X which contains $\mathscr{S}$. Equivalently, this is the σ-algebra equal to the intersection of all σ-algebras which contain $\mathscr{S}$ (e.g. the power set of X).
3. A function $\mu:\mathscr{S}\to[0,\infty]$ is called *σ-finite* if X is a countable disjoint union of elements $A_i\in\mathscr{S}$ such that $\mu(A_i)<\infty$ for all i. This always happens if $\mu(X)<\infty$.
4. A function $\mu:\mathscr{S}\to[0,\infty]$ is called *σ-additive on $\mathscr{S}$* if for all pairwise disjoint countable collection of $A_i\in\mathscr{S}$, if $\biguplus A_i\in\mathscr{S}$ then $\mu(\biguplus A_i)=\sum\mu(A_i)$.

Theorem 1.2 (Carathéodory's Extension Theorem). *Let X be a set and $\mathscr{S}$ a semi-algebra of subsets of X. Every σ–additive σ-finite $\mu:\mathscr{S}\to[0,\infty]$ has a unique extension to a σ–additive σ-finite function on the σ–algebra generated by $\mathscr{S}$.*

In our case $X=\{(x_0,x_1,\dots):x_i\in S\text{ for all }i\}$, $\mathscr{S}=\{\varnothing,X\}\cup\{\text{cylinders}\}$, and $\mu:\mathscr{S}\to[0,\infty]$ is defined by $\mu(\varnothing):=0$, $\mu(X):=1$, $\mu[a_0,\dots,a_{n-1}]:=p_{a_0}\cdots p_{a_{n-1}}$.

$\mathscr{S}$ is a semi-algebra, because the intersection of cylinders is empty or a cylinder, and because the complement of a cylinder $[a_0,\dots,a_{n-1}]$ is the finite disjoint union of cylinders $[b_0,\dots,b_{n-1}]$ such that for some i, $b_i\neq a_i$.

It is also clear that $\mu:\mathscr{S}\to[0,\infty]$ is σ-finite. Indeed it is finite ($\mu(X)<\infty$)! It remain to check that μ is σ-additive on $\mathscr{S}$.

Suppose $[\underline{a}]$ is a countable disjoint union of cylinders $[\underline{b}^j]$. Each cylinder is open and compact (prove!), so such unions are necessarily finite. Let N be the maximal length of the cylinder $[\underline{b}^j]$. Since $[\underline{b}^j]\subseteq[\underline{a}]$, we can write $[\underline{b}^j]=[\underline{a},\underline{\beta}^j]=\biguplus_{|\underline{c}|=N-|\underline{b}^j|}[\underline{a},\underline{\beta}^j,\underline{c}]$, and a direct calculation shows that

$$\sum_{|\underline{c}|=N-|\underline{b}^j|}\mu[\underline{a},\underline{\beta}^j,\underline{c}]=\mu[\underline{a},\underline{\beta}^j]\left(\sum_c p_c\right)^{N-|\underline{b}^j|}=\mu[\underline{a},\underline{\beta}^j]\equiv\mu[\underline{b}^j].$$

Summing over j, we get that $\sum_j\mu[\underline{b}^j]=\sum_j\sum_{|\underline{c}|=N-|\underline{b}^j|}\mu[\underline{a},\underline{\beta}^j,\underline{c}]$.

Now $[\underline{a}] = \biguplus_j [\underline{b}^j] = \biguplus_j \biguplus_{|\underline{c}|=N-|\underline{b}^j|} [\underline{a}, \underline{\beta}^j, \underline{c}]$, so the collection of $(\underline{\beta}^j, \underline{c})$ is equal to the collection of all possible words $\underline{w}$ of length $N - |\underline{a}|$ (otherwise the right hand side misses some sequences). Thus

$$\sum_j \mu[\underline{b}^j] = \sum_{|\underline{w}|=N-|\underline{a}|} \mu[\underline{a}, \underline{w}] = \mu[\underline{a}] \left(\sum_c p_c\right)^{N-|\underline{a}|} = \mu[\underline{a}],$$

proving the σ–additivity of μ on $\mathscr{S}$.

It now follows from the Carathéodory's Extension Theorem that $\mu : \mathscr{S} \to [0, \infty]$ has a unique extension to a probability measure on the smallest σ-algebra containing the cylinders.

Call this σ-algebra $\mathscr{B}$. Since the cylinders generate the topology of X, every open set is a union of cylinders. This union is countable, because there are only countably many cylinders. So every open set belongs to $\mathscr{B}$. It follows that $\mathscr{B}$ contains the Borel σ-algebra of X (which equals by definition to the smallest σ-algebra containing all open sets). So μ extends uniquely to a non-negative σ-additive function on the Borel σ-algebra of X. Definition probability measure. Definition 1.6 is justified.

Proposition 1.6. *Suppose $X = \{0,1\}^{\mathbb{N}}$, μ is the $(\frac{1}{2}, \frac{1}{2})$–Bernoulli measure, and σ is the left shift, then $(X, \mathscr{B}(X), \mu, T)$ is measure theoretically isomorphic to the angle doubling map.*

Proof. The isomorphism is $\pi(x_0, x_1, \ldots,) := \sum 2^{-n} x_n$. This is a bijection between

$$X' := \{\underline{x} \in \{0,1\}^{\mathbb{N}} : \nexists n \text{ s.t. } x_m = 1 \text{ for all } m \geq n\}$$

and $[0,1)$ (prove that $\mu(X') = 1$), and it is clear that $\pi \circ \sigma = T \circ \pi$. Since the image of a cylinder of length n is a dyadic interval of length 2^{-n}, π preserves the measures of cylinders. The collection of measurable sets which are mapped by π to sets of the same measure is a σ–algebra. Since this σ–algebra contains all the cylinders and all the null sets, it contains all measurable sets. $\square$

Proposition 1.7. *Every Bernoulli scheme is mixing, whence ergodic.*

The proof is the same as in the case of the angle doubling map. Alternatively, it follows from the mixing of the angle doubling map and the fact that the two are isomorphic.

1.5.4 Finite Markov Chains

We saw that the angle doubling map is isomorphic to a dynamical system acting as the left shift on a space of sequences (a Bernoulli scheme). Such representations appear frequently in the theory of dynamical systems, but more often than not, the space of sequences is slightly more complicated than the set of all sequences.

1.5.4.1 Subshifts of finite type

Let S be a finite set, and $A = (t_{ij})_{S\times S}$ a matrix of zeroes and ones without columns or rows made entirely of zeroes.

Definition 1.8. The *subshift of finite type (SFT)* with *alphabet* S and *transition matrix* A is the set $\Sigma_A^+ = \{\underline{x} = (x_0, x_1, \ldots) \in S^{\mathbb{N}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}$, together with the metric $d(\underline{x}, \underline{y}) := 2^{-\min\{k: x_k \neq y_k\}}$ and the action $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$.

This is a compact metric space, and the left shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is continuous. We think of Σ_A^+ as of the space of all infinite paths on a directed graph with vertices S and edge $a \to b$ connecting $a, b \in S$ such that $t_{ab} = 1$.

Let Σ_A^+ be a SFT with set of states S, $|S| < \infty$, and transition matrix $A = (A_{ab})_{S\times S}$.

- A *stochastic matrix* is a matrix $P = (p_{ab})_{a,b\in S}$ with non-negative entries, such that $\sum_b p_{ab} = 1$ for all a, i.e. $P1 = 1$. The matrix is called *compatible* with A, if $A_{ab} = 0 \Rightarrow p_{ab} = 0$.
- A *probability vector* is a vector $\underline{p} = \langle p_a : a \in S\rangle$ of non-negative entries, s.t. $\sum p_a = 1$
- A *stationary probability vector* is a probability vector $\underline{p} = \langle p_a : a \in S\rangle$ s.t. $\underline{p}P = \underline{p}$: $\sum_a p_a p_{ab} = p_b$.

Given a probability vector $\underline{p}$ and a stochastic matrix P compatible with A, one can define a *Markov measure* μ on Σ_A^+ (or Σ_A) by

$$\mu[a_0, \ldots, a_{n-1}] := p_{a_0} p_{a_0 a_1} \cdots p_{a_{n-2} a_{n-1}},$$

where $[a_0, \ldots, a_{n-1}] = \{\underline{x} \in \Sigma_A^+ : x_i = a_i \ (i = 0, \ldots, n-1)\}$. The stochasticity of P guarantees that this measure is finitely additive on the algebra of cylinders, and σ–subadditivity can be checked as for Bernoulli measures using compactness. Thus this gives a Borel probability measure on Σ_A^+.

Proposition 1.8. *μ is shift invariant iff $\underline{p}$ is stationary w.r.t. P. Any stochastic matrix has a stationary probability vector.*

Proof. To see the first half of the statement, we note that μ is shift invariant iff $\mu[*, \underline{b}] = \mu[\underline{b}]$ for all $[\underline{b}]$, which is equivalent to

$$\sum_a p_a p_{ab_0} p_{b_0 b_1} \cdots p_{b_{n-2} b_{n-1}} = p_{b_0} p_{b_0 b_1} \cdots p_{b_{n-2} b_{n-1}}.$$

Canceling the identical terms on both sides gives $\sum_a p_a p_{ab_0} = p_{b_0}$. Thus μ is shift invariant iff $\underline{p}$ is P–stationary.

We now show that every stochastic matrix has a stationary probability vector. Consider the right action of P on the simplex Δ of probability vectors in $\mathbb{R}^N$:

$$\Delta := \{(x_1, \ldots, x_N) : x_i \geq 0, \sum x_i = 1\}, \ T(\underline{x}) = \underline{x}P.$$

We have $T(\Delta) \subseteq \Delta$, since $\sum_a (T\underline{x})_a = \sum_a \sum_b x_b p_{ba} = \sum_b x_b \sum_a p_{ba} = \sum_b x_b = 1$. Recall *Brouwer's fixed point theorem*: A continuous mapping of a compact convex subset of $\mathbb{R}^d$ into itself has a fixed point. Applying this to $T:\Delta\to\Delta$ we find $\underline{x}\in\Delta$ such that $\underline{x}P=\underline{x}$. This is the stationary probability vector. $\square$

Thus every stochastic matrix determines (at least one) shift invariant measure. Such measures are called *Markov measures* We ask when is this measure ergodic, and when is it mixing.

1.5.4.2 Ergodicity and mixing of Markov measures

There are obvious obstructions to ergodicity and mixing. To state them concisely, we introduce some terminology. Suppose $P=(p_{ab})_{S\times S}$ is a stochastic matrix. We say that *a connects* to b in n steps, and write $a\xrightarrow{n}b$, if there is a path of length $n+1$ $(a,\xi_1,\dots,\xi_{n-1},b)\in S^{n+1}$ s.t. $p_{a\xi_1}p_{\xi_1\xi_2}\cdot\dots\cdot p_{\xi_{n-1}b}>0$ (see problem 1.5).

Definition 1.9. A stochastic matrix $P=(p_{ab})_{a,b\in S}$ is called *irreducible*, if for every $a,b\in S$ there exists an n s.t. $a\xrightarrow{n}b$.

Lemma 1.1. *If A is irreducible, then $p:=\gcd\{n:a\xrightarrow{n}a\}$ is independent of a. (*$\gcd$ *=greatest common divisor).*

Proof. Let $p_a:=\gcd\{n:a\xrightarrow{n}a\}$, $p_b:=\gcd\{n:b\xrightarrow{n}b\}$, and $\Lambda_b:=\{n:b\xrightarrow{n}b\}$. Then $\Lambda_b+\Lambda_b\subset\Lambda_b$, and therefore $\Lambda_b-\Lambda_b$ is a subgroup of $\mathbb{Z}$. Necessarily $\Lambda_b-\Lambda_b=p_b\mathbb{Z}$. By irreducibility, there are α,β s.t. $a\xrightarrow{\alpha}b\xrightarrow{\beta}a$. So for all $n\in\Lambda_b$, $a\xrightarrow{\alpha}b\xrightarrow{n}b\xrightarrow{\beta}a$, whence $p_a|\gcd(\alpha+\beta+\Lambda_b)$. This clearly implies that p_a divides every number in $\Lambda_b-\Lambda_b=p_b\mathbb{Z}$. So $p_a|p_b$. Similarly one shows that $p_b|p_a$, whence $p_a=p_b$. $\square$

Definition 1.10. The *period* of an irreducible stochastic matrix P is the number $p:=\gcd\{n:a\xrightarrow{n}a\}$ (this is independent of a by the lemma). An irreducible stochastic matrix is called *aperiodic* if its period is equal to one.

For example, the SFT with transition matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ is irreducible with period two.

If P is not irreducible, then any Markov measure of P and a *strictly positive* stationary probability vector of P is non-ergodic. To see why, pick $a,b\in S$ s.t. a does not connect to b in any number of steps. The set

$$E:=\{\underline{x}\in\Sigma_A^+:x_i\neq b\text{ for all }i\text{ sufficiently large}\}$$

is a shift invariant set which contains $[a]$, but which is disjoint from $[b]$. So E is an invariant set with such that $0<p_a\le\mu(E)\le1-p_b<1$. So μ is not ergodic.

If P is irreducible, but not aperiodic, then any Markov measure on Σ_A^+ is non-mixing, because the function $f:=1_{[a]}$ satisfies $ff\circ T^n\equiv0$ for all n not divisible by the period. This means that $\int ff\circ T^nd\mu$ is equal to zero on a subsequence, and therefore cannot converge to $\mu[a]^2$.

We claim that these are the only possible obstructions to ergodicity and mixing. The proof is based on the following fundamental fact, whose proof will be given at the next section.

Theorem 1.3 (Ergodic Theorem for Markov Chains). *Suppose P is a stochastic matrix, and write $P^n = (p_{ab}^{(n)})$, then P has a stationary probability vector $\underline{p}$, and*

1. *if P is irreducible, then $\frac{1}{n}\sum_{k=1}^{n} p_{ab}^{(k)} \xrightarrow[n\to\infty]{} p_b \quad (a,b \in S)$;*
2. *if P is irreducible and aperiodic, then $p_{ab}^{(k)} \xrightarrow[n\to\infty]{} p_b \quad (a,b \in S)$.*

Corollary 1.1. *A shift invariant Markov measure on a SFT Σ_A^+ is ergodic iff its stochastic matrix is irreducible, and mixing iff its stochastic matrix is irreducible and aperiodic.*

Proof. Let μ be a Markov measure with stochastic matrix P and stationary probability vector $\underline{p}$. For all cylinders $[\underline{a}] = [a_0, \ldots, a_{n-1}]$ and $[\underline{b}] = [b_0, \ldots, b_{m-1}]$,

$$\mu([\underline{a}] \cap \sigma^{-k}[\underline{b}]) = \mu\left(\biguplus_{\underline{\xi}\in\mathscr{W}_{k-n}} [\underline{a},\underline{\xi},\underline{b}]\right), \ \ \mathscr{W}_\ell := \{\underline{\xi} = (\xi_1,\ldots,\xi_\ell) : [\underline{a},\underline{\xi},\underline{b}] \neq \varnothing\}$$

$$= \mu[\underline{a}] \cdot \sum_{\underline{\xi}\in\mathscr{W}_{k-n}} p_{a_{n-1}\xi_1}\cdots p_{\xi_{k-n}b_0} \cdot \frac{\mu[\underline{b}]}{p_{b_0}} = \mu[\underline{a}]\mu[\underline{b}] \cdot \frac{p_{a_{n-1}b_0}^{(k-n)}}{p_{b_0}}.$$

If P is irreducible, then by theorem 1.3, $\frac{1}{n}\sum_{k=0}^{n-1}\mu([\underline{a}]\cap\sigma^{-k}[\underline{b}]) \xrightarrow[n\to\infty]{} \mu[\underline{a}]\mu[\underline{b}]$.

We claim that this implies ergodicity. Suppose E is an invariant set, and fix $\varepsilon > 0$, arbitrarily small. There are cylinders $A_1, \ldots, A_N \in \mathscr{S}$ s.t. $\mu\left(E \triangle \biguplus_{i=1}^N A_i\right) < \varepsilon$.[1] Thus

$$\mu(E) = \mu(E\cap\sigma^{-k}E) = \sum_{i=1}^{N}\mu(A_i\cap\sigma^{-k}E) \pm \varepsilon = \sum_{i,j=1}^{N}\mu(A_i\cap\sigma^{-k}A_j) \pm 2\varepsilon.$$

Averaging over k, and passing to the limit, we get

$$\mu(E) = \sum_{i,j=1}^{N}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mu(A_i\cap\sigma^{-k}A_j) \pm 2\varepsilon = \sum_{i,j=1}^{N}\mu(A_i)\mu(A_j) \pm 2\varepsilon$$

$$= \left(\sum_{i=1}^{N}\mu(A_i)\right)^2 \pm 2\varepsilon = [\mu(E)\pm\varepsilon]^2 \pm 2\varepsilon.$$

Passing to the limit $\varepsilon \to 0^+$, we obtain $\mu(E) = \mu(E)^2$, whence $\mu(E) = 0$ or 1.

Now assume that that P is irreducible and aperiodic. The ergodic theorem for Markov chains says that $\mu([\underline{a}]\cap\sigma^{-k}[\underline{b}]) \xrightarrow[k\to\infty]{} \mu[\underline{a}]\mu[\underline{b}]$. Since any measurable sets E, F can approximated by finite disjoint unions of cylinders, an argument similar to the previous one shows that $\mu(E\cap\sigma^{-k}F) \xrightarrow[k\to\infty]{} \mu(E)\mu(F)$ for all $E, F \in \mathscr{B}$. This is is mixing. ☐

[1] Proof: The collection of sets E satisfying this approximation property is a σ–algebra which contains all cylinders, therefore it is equal to $\mathscr{B}$.

Remark 1. The ergodic theorem for Markov chains can be visualized as follows. Imagine that we distribute mass on the states of S according to a probability distribution $\underline{q} = (q_a)_{a\in S}$. Now shift mass from one state to another using the rule that a p_{ab}–fraction of the mass at a is moved to b. The new mass distribution is $\underline{q}P$ (check). After n steps, the mass distribution is $\underline{q}P^n$. The previous theorem says that, in the aperiodic case, the mass distribution converges to the stationary distribution —- the equilibrium state. It can be shown that the rate of convergence is exponential (problem 1.7).

Remark 2: The ergodic theorem for Markov chains has an important generalization to all matrices with non-negative entries, see problem 1.6.

1.5.4.3 Proof of the Ergodic Theorem for Markov chains

Suppose first that P is an irreducible and aperiodic stochastic matrix. This implies that there is some power m such that all the entries of P^m are strictly positive.[2]

Let $N := |S|$ denote the number of states, and consider the set of all probability vectors $\Delta := \{(x_1,\dots,x_N) : x_i \geq 0, \sum x_i = 1\}$. Since P is stochastic, the map $T(\underline{x}) = \underline{x}P$ maps Δ continuously into itself. By the Brouwer Fixed Point Theorem, there is a probability vector $\underline{p}$ s.t. $\underline{p}P = \underline{p}$ (this is the stationary probability vector).

The irreducibility of P implies that all the coordinates of $\underline{p}$ are strictly positive. Indeed, $\sum p_k = 1$ so at least one coordinate p_i is positive. For any other coordinate p_j there is by irreducibility a path $(i, \xi_1, \dots, \xi_{n-1}, j)$ such that

$$p_{i\xi_1} p_{\xi_1\xi_2} \cdots p_{\xi_{n-2}\xi_{n-1}} p_{\xi_{n-1}j} > 0.$$

So $p_j = (\underline{p}P^n)_j = \sum_k p_k p_{kj}^{(n)} \geq p_i p_{i\xi_1} p_{\xi_1\xi_2} \cdots p_{\xi_{n-2}\xi_{n-1}} p_{\xi_{n-1}j} > 0$.

So $\underline{p}$ lies in the interior of Δ, and the set $C := \Delta - \underline{p}$ is a compact convex neighborhood of the origin such that $T(C) \subset C$, $T^m(C) \subset int[C]$. (We mean the relative interior in the $(N-1)$-dimensional body C, not the ambient space $\mathbb{R}^N$.)

Now consider $L := \mathrm{span}(C)$ (an $N-1$–dimensional space). This is an invariant space for T, whence for P^t (the transpose of P). We claim that all the eigenvalues of $P^t|_L$ have absolute value less than 1:

1. *Eigenvalues of modulus larger than one are impossible*, because P is stochastic, so $\|\underline{v}P\|_1 \leq \|\underline{v}\|_1$, so the spectral radius of P^t cannot be more than 1.
2. *Roots of unity are impossible*, because in this case for some k, P^{km} has a real eigenvector $\underline{v}$ with eigenvalue one. There is no loss of generality in assuming that $\underline{v} \in \partial C$, otherwise multiply $\underline{v}$ by a suitable scalar. But P^{km} cannot have fixed points on ∂C, because $P^{km}(C) \subset int[C]$
3. *Eigenvalues $e^{i\theta}$ with $\theta \notin 2\pi\mathbb{Q}$ are impossible*, because if $e^{i\theta}$ is an eigenvalue then there are two real eigenvectors $\underline{u}, \underline{v} \in \partial C$ such that the action of P on $\mathrm{span}\{\underline{u}, \underline{v}\}$

[2] Begin by proving that if A is irreducible and aperiodic, then for every a there is an N_a s.t. $a \xrightarrow{n} a$ for all $n > N_a$. Use this to show that for all a, b there is an N_{ab} s.t. $a \xrightarrow{n} b$ for all $n > N_{ab}$. Take $m = \max\{N_{ab}\}$.

is conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, an irrational rotation. This means $\exists k_n \to \infty$ s.t. $\underline{u}P^{mk_n} \to \underline{u} \in \partial C$. But this cannot be the case because $P^m[C] \subset int[C]$, and by compactness, this cannot intersect ∂C.

In summary the spectral radius of $P^t|_L$ is less than one.

But $\mathbb{R}^N = L \oplus \text{span}\{\underline{p}\}$. If we decompose a general vector $\underline{v} = \underline{q} + t\underline{p}$ with $\underline{q} \in L$, then the above implies that $\underline{v}P^n = t\underline{p} + O(\|P^n|_L\|)\|\underline{q}\| \xrightarrow[n\to\infty]{} t\underline{p}$. It follows that $p_{ab}^{(n)} \xrightarrow[n\to\infty]{} p_b$ for all a,b.

This is almost the ergodic theorem for irreducible aperiodic Markov chains, the only thing which remains is to show that P has a unique stationary vector. Suppose $\underline{q}$ is another probability vector s.t. $\underline{q}P = \underline{q}$. We can write $p_{ab}^{(n)} \to p_b$ in matrix form as follows:

$$P^n \xrightarrow[n\to\infty]{} Q, \text{ where } Q = (q_{ab})_{S\times S} \text{ and } q_{ab} = p_b.$$

This means that $\underline{q} = \underline{q}P^n \to \underline{q}Q$, whence $\underline{q}Q = \underline{q}$, so $q_a = \sum_b q_b q_{ba} = \sum_b q_b p_a = p_a$.

Consider now the periodic irreducible case. Let A be the transition matrix associated to P (with entries $t_{ab} = 1$ when $p_{ab} > 0$ and $t_{ab} = 0$ otherwise), and let p denote the period of P. Fix once and for all a state v. Working with the SFT Σ_A^+, we let

$$S_k := \{b \in S : v \xrightarrow{n} b \text{ for some } n = k \mod p\} \ (k = 0, \dots, p-1).$$

S_k are pairwise disjoint, because if $b \in S_{k_1} \cap S_{k_2}$, then $\exists \alpha_i = k_i \mod p$ and $\exists \beta$ s.t. $v \xrightarrow{\alpha_i} b \xrightarrow{\beta} v$ for $i = 1,2$. By the definition of the period, $p|\alpha_i + \beta$ for $i = 1,2$, whence $k_1 = \alpha_1 = -\beta = \alpha_2 = k_2 \mod p$.

It is also clear that every path of length ℓ which starts at S_k, ends at $S_{k'}$ where $k' = k + \ell \mod p$. In particular, every path of length p which starts at S_k ends at S_k. This means that if $p_{ab}^{(p)} > 0$, then a,b belong to the same S_k.

It follows that P^p is conjugate, via a coordinate permutation, to a block matrix with blocks $(p_{ab}^{(p)})_{S_k\times S_k}$. Each of the blocks is stochastic, irreducible, and aperiodic. Let $\underline{\pi}^{(k)}$ denote the stationary probability vectors of the blocks.

By the first part of the proof, $p_{ab}^{(\ell p)} \xrightarrow[\ell\to\infty]{} \pi_b^{(k)}$ for all a,b in the same S_k, and $p_{ab}^{(n)} = 0$ for $n \neq 0 \mod p$. More generally, if $a \in S_{k_1}$ and $b \in S_{k_2}$, then

$$\begin{aligned}
\lim_{\ell\to\infty} p_{ab}^{(\ell p + k_2 - k_1)} &= \lim_{\ell\to\infty} \sum_{\xi \in S_{k_2}} p_{a\xi}^{(k_2-k_1+p)} p_{\xi b}^{((\ell-1)p)} \\
&= \sum_{\xi \in S_{k_2}} p_{a\xi}^{(k_2-k_1+p)} \pi_b^{(k_2)} \quad \text{(by the above)} \\
&= \pi_b^{(k_2)} \sum_{\xi \in S} p_{a\xi}^{(k_2-k_1+p)} = \pi_b^{(k_2)}. \qquad (\because p_{a\xi}^{(k_2-k_1+p)} = 0 \text{ when } \xi \notin S_{k_2})
\end{aligned}$$

Similarly, $p_{ab}^{(\ell p+\alpha)} = 0$ when $\alpha \neq k_2 - k_1 \mod p$. Thus

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}p_{ab}^{(k)}=\frac{1}{p}\pi_b^{(k)} \text{ for the unique } k \text{ s.t. } S_k\ni b$$

The limiting vector $\underline{\pi}$ is a probability vector, because $\sum_{k=1}^{p}\sum_{b\in S_k}\frac{1}{p}\pi_b^{(k)}=1$.

We claim that $\underline{\pi}$ is the unique stationary probability vector of P. The limit theorem for $p_{ab}^{(n)}$ can be written in the form $\frac{1}{n}\sum_{k=0}^{n-1}P^k\to Q$ where $Q=(q_{ab})_{S\times S}$ and $q_{ab}=\pi_b$. As before this implies that $\underline{\pi}P=\underline{\pi}$ and that any probability vector $\underline{q}$ such that $\underline{q}P=\underline{q}$, we also have $\underline{q}Q=\underline{q}$, whence $\underline{q}=\underline{p}$. $\square$

1.5.5 The geodesic flow on a hyperbolic surface

The *hyperbolic plane* is the surface $\mathbb{H}:=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$ equipped with the Riemannian metric $ds=|dz|/\mathrm{Im}(z)$, which gives it constant curvature (-1).

It is known that the orientation preserving *isometries* (i.e. distance preserving maps) are the Möbius transformations which preserve $\mathbb{H}$. They form the group

$$\begin{aligned}\text{Möb}(\mathbb{H})&=\left\{z\mapsto\frac{az+b}{cz+d}:a,b,c,d\in\mathbb{R},ad-bc=1\right\}\\&\simeq\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}:a,b,c,d\in\mathbb{R},ad-bc=1\right\}\Big/\{\pm id\}=:\mathrm{PSL}(2,\mathbb{R}).\end{aligned}$$

(We quotient by $\{\pm id\}$ to identify $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\left(\begin{smallmatrix}-a&-b\\-c&-d\end{smallmatrix}\right)$ which represent the same Möbius transformation.)

The *geodesics* (i.e. length minimizing curves) on the hyperbolic plane are vertical half–lines, or circle arcs which meet $\partial\mathbb{H}$ at right angles. Here is why: Suppose $\omega\in TM$ is a unit tangent vector which points directly up, then it is not difficult to see that the geodesic at direction ω is a vertical line. For general unit tangent vectors ω, find an element $\varphi\in$ Möb$(\mathbb{H})$ which rotates them so that $d\varphi(\omega)$ points up. The geodesic in direction ω is the φ–preimage of the geodesic in direction $d\varphi(\omega)$ (a vertical half–line). Since Möbius transformations map lines to lines or circles in a conformal way, the geodesic of ω is a circle meeting $\partial\mathbb{H}$ at right angles.

The *geodesic flow* of $\mathbb{H}$ is the flow g^t on the *unit tangent bundle* of $\mathbb{H}$,

$$T^1\mathbb{H}:=\{\text{tangent vectors with length one}\}$$

which moves ω along the geodesic it determines at unit speed.

To describe this flow it useful to find a convenient parametrization for $T^1\mathbb{H}$. Fix $\omega_0\in T^1\mathbb{H}$ (e.g. the unit vector based at i and pointing up). For every ω, there is a unique $\varphi_\omega\in$ Möb$(\mathbb{H})$ such that $\omega=d\varphi_\omega(\omega_0)$, thus we can identify

$$T^1\mathbb{H}\simeq\text{Möb}(\mathbb{H})\simeq\mathrm{PSL}(2,\mathbb{R}).$$

It can be shown that in this coordinate system the geodesic flow takes the form

$$g^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

To verify this, it is enough to calculate the geodesic flow on $\omega_0 \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Next we describe the Riemannian volume measure on $T^1\mathbb{H}$ (up to normalization). Such a measure must be invariant under the action of all isometries. In our coordinate system, the isometry $\varphi(z) = (az+b)/(cz+d)$ acts by

$$\varphi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Since $\mathrm{PSL}(2,\mathbb{R})$ is a locally compact topological group, there is only one Borel measure on $\mathrm{PSL}(2,\mathbb{R})$ (up to normalization), which is left invariant by all left translations on the group: the Haar measure of $\mathrm{PSL}(2,\mathbb{R})$. Thus the Riemannian volume measure is a left Haar measure of $\mathrm{PSL}(2,\mathbb{R})$, and this determines it up to normalization.

It is a convenient feature of $\mathrm{PSL}(2,\mathbb{R})$ that its left Haar measure is also invariant under right translations. It follows that the geodesic flow preserves the volume measure on $T^1\mathbb{H}$. But this measure is infinite, and it is not ergodic (prove!).

To obtain ergodic flows, we need to pass to compact quotients of $\mathbb{H}$. These are called *hyperbolic surfaces*.

A hyperbolic surface is a Riemannian surface M such that every point in M has a neighborhood V which is isometric to an open subset of $\mathbb{H}$. Recall that a Riemannian surface is called *complete*, if every geodesic can be extended indefinitely in both directions.

Theorem 1.4 (Killing–Hopf Theorem). *Every complete connected hyperbolic surface is isometric to* ${}_{\Gamma}\backslash^{\mathbb{H}} := \{\Gamma z : z \in \mathbb{H}\}$*, where*

1. *Γ is a subgroup of Möb$(\mathbb{H})$ and $\Gamma z := \{g(z) : g \in \Gamma\}$.*
2. *Every point $z \in \mathbb{H}$ is in the interior of some open set $U \subset PSL(2,\mathbb{R})$ s.t. $\{g(U) : g \in \Gamma\}$ are pairwise disjoint. So $\Gamma z' \mapsto$ the unique point in $\Gamma z' \cap U$ is a bijection.*
3. *the Riemannian structure on $\{\Gamma z' : z' \in U\}$ is the one induced by the Riemannian structure on U.*

If we identify Γ with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, then we get the identification $T^1({}_{\Gamma}\backslash^{\mathbb{H}}) \simeq {}_{\Gamma}\backslash^{\mathrm{PSL}(2,\mathbb{R})}$. It is clear that the Haar measure on $\mathrm{PSL}(2,\mathbb{R})$ induces a unique locally finite measure on ${}_{\Gamma}\backslash^{\mathrm{PSL}(2,\mathbb{R})}$, and that the geodesic flow on $T^1({}_{\Gamma}\backslash^{\mathbb{H}})$ takes the form

$$g^t(\Gamma\omega) = \Gamma\omega \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

and preserves this measure.

Definition 1.11. A measure preserving *flow* $g^t : X \to X$ is called *ergodic*, if every measurable set E such that $g^{-t}(E) = E$ for all t satisfies $m(E) = 0$ or $m(E^c) = 0$.

Theorem 1.5. *If $\Gamma\backslash^{\mathbb{H}}$ is compact, then the geodesic flow on $T^1(\Gamma\backslash^{\mathbb{H}})$ is ergodic.*

Proof. If $\Gamma\backslash^{\mathbb{H}}$ is compact, then $\Gamma\backslash^{\mathbb{H}}$ has finite volume. In this case, ergodicity is equivalent to the following property: Every square integrable function f such that for a.e. x, $f(g^t x) = f(x)$ for all $t \in \mathbb{R}$, is equal a.e. to a constant function (prove!).

Consider the following flows:

$$h^t_{st}(\Gamma\omega) = \Gamma\omega \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$h^t_{un}(\Gamma\omega) = \Gamma\omega \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

If we can show that any geodesic invariant function f is also invariant under these flows then we are done, because it is known that

$$\left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle = PSL(2,\mathbb{R})$$

(prove!), and any $\mathrm{PSL}(2,\mathbb{R})$–invariant function on $\Gamma\backslash^{\mathrm{PSL}(2,\mathbb{R})}$ is constant.

Since our measure is induced by the the Haar measure, the flows h^t_{un}, h^t_{st} are measure preserving. A matrix calculation shows:

$$g^s h^t_{st} g^{-s} = h^{te^{-s}}_{st} \xrightarrow[s\to\infty]{} id$$

$$g^{-s} h^t_{un} g^s = h^{te^s}_{un} \xrightarrow[s\to-\infty]{} id$$

Step 1. If $f \in L^2$, then $f \circ h^{te^{-s}} \xrightarrow[s\to\infty]{L^2} f$.

Proof. Approximate by continuous functions of compact support, and observe that h^t is an isometry of L^2.

Step 2. If $f \in L^2$ and $f \circ g^s = f$, then $f \circ h^t_{un} = f$ and $f \circ h^t_{st} = f$.

Proof. $\|f \circ h^t_{st} - f\|_2 = \|f \circ g^s \circ h^t_{st} - f\|_2 = \|f \circ g^s \circ h^t_{st} \circ g^{-s} - f\|_2 \to 0.$

Thus f is h^t_{st}–invariant. A similar calculation shows that it is h^t_{un}–invariant, and we are done. $\square$

1.6 Basic constructions

In this section we discuss several standard methods for creating new measure preserving transformations from old ones. These constructions appear quite frequently in applications.

Products

Recall that the product of two measure spaces $(X_i, \mathscr{B}_i, \mu_i)$ $(i = 1,2)$ is the measure space $(X_1 \times X_2, \mathscr{B}_1 \otimes \mathscr{B}_2, \mu_1 \times \mu_2)$ where $\mathscr{B}_1 \times \mathscr{B}_2$ is the smallest σ–algebra which contains all sets of the form $B_1 \times B_2$ where $B_i \in \mathscr{B}_i$, and $\mu_1 \times \mu_2$ is the unique measure such that $(\mu_1 \times \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$.

This construction captures the idea of *independence* from probability theory: if $(X_i, \mathscr{B}_i, \mu_i)$ are the probability models of two random experiments, and these experiments are "independent", then $(X_1 \times X_2, \mathscr{B}_1 \otimes \mathscr{B}_2, \mu_1 \times \mu_2)$ is the probability model of the pair of the experiments, because for every $E_1 \in \mathscr{B}_1, E_2 \in \mathscr{B}_2$,

$$\begin{aligned} F_1 &:= E_1 \times X_2 \text{ is the event "in experiment 1, } E_1 \text{ happened"} \\ F_2 &:= X_1 \times E_2 \text{ is the event "in experiment 2, } E_2 \text{ happened"} \\ F_1 \cap F_2 &= E_1 \times E_2 \end{aligned}$$

and $F_1 \cap F_2 = E_1 \times E_2$; so $(\mu_1 \times \mu_2)(F_1 \cap F_2) = (\mu_1 \times \mu_2)(F_1)(\mu_1 \times \mu_2)(F_2)$, showing that the events F_1, F_2 are independent.

Definition 1.12. The *product* of two measure preserving systems $(X_i, \mathscr{B}_i, \mu_i, T_i)$ $(i = 1,2)$ is the measure preserving system $(X_1 \times X_2, \mathscr{B}_1 \otimes \mathscr{B}_2, \mu_1 \times \mu_2, T_1 \times T_2)$, where $(T_1 \times T_2)(x_1, x_2) = (T_1 x_1, T_2 x_2)$.

(Check that S is measure preserving.)

Proposition 1.9. *The product of two ergodic mpt is not always ergodic. The product of two mixing mpt is always mixing.*

Proof. The product of two (ergodic) irrational rotations $S := R_\alpha \times R_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$, $S(x,y) = (x + \alpha, y + \alpha) \mod 1$ is not ergodic: $F(x,y) = y - x \mod 1$ is a non-constant invariant function. (See problem 1.8.)

The product of two mixing mpt is however mixing. To see this set $\mu = \mu_1 \times \mu_2$, $S = T_1 \times T_2$, and $\mathscr{S} := \{A \times B : A \in \mathscr{B}_1, B \in \mathscr{B}_2\}$. For any $E_1 := A_1 \times B_1, E_2 := A_2 \times B_2 \in \mathscr{S}$,

$$\begin{aligned} \mu(E_1 \cap S^{-n} E_2) &= \mu\big((A_1 \times B_1) \cap (T_1 \times T_2)^{-n}(A_2 \times B_2)\big) \\ &= \mu\big((A_1 \cap T^{-n} A_2) \cap (B_1 \cap T^{-n} B_2)\big) \\ &= \mu_1(A_1 \cap T^{-n} A_2)\mu_2(B_1 \cap T^{-n} B_2) \\ &\xrightarrow[n\to\infty]{} \mu_1(A_1)\mu_2(B_1)\mu_1(A_2)\mu_2(B_2) = \mu(A_1 \times B_1)\mu(A_2 \times B_2). \end{aligned}$$

Since $\mathscr{S}$ is a semi-algebra which generates $\mathscr{B}_1 \otimes \mathscr{B}_2$, any element of $\mathscr{B}_1 \otimes \mathscr{B}_2$ can be approximated by a finite disjoint elements of $\mathscr{S}$, and a routine approximation argument shows that $\mu(E \cap S^{-n} F) \to \mu(E)\mu(F)$ for all $E, F \in \mathscr{B}$. $\square$

1.6.1 Skew-products

We start with an example. Let μ be the $(\frac{1}{2},\frac{1}{2})$–Bernoulli measure on the two shift $\Sigma_2^+ := \{0,1\}^{\mathbb{N}}$. Let $f:\Sigma_2^+ \to \mathbb{Z}$ be the function $f(x_0,x_1,\ldots) = (-1)^{x_0}$. Consider the transformation

$$T_f : \Sigma_2^+ \times \mathbb{Z} \to \Sigma_2^+ \times \mathbb{Z} \ , \ T_f(\underline{x},k) = (\sigma(\underline{x}), k + f(x)),$$

where $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is the left shift. This system preserves the (infinite) measure $\mu \times m_{\mathbb{Z}}$ where $m_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$. The n-th iterate is

$$T_f^n(\underline{x},k) = (\sigma^n \underline{x}, k + X_0 + \cdots + X_{n-1}), \text{ where } X_i := (-1)^{x_i}.$$

What we see in the second coordinate is the symmetric random walk on $\mathbb{Z}$, started at k, because (1) the steps X_i take the values ± 1, and (2) $\{X_i\}$ are independent because of the choice of μ. We say that the second coordinate is a "random walk on $\mathbb{Z}$ driven by the noise process $(\Sigma_2^+, \mathscr{B}, \mu, \sigma)$."

Here is a variation on this example. Suppose T_0, T_1 are two measure preserving transformations of the same measure space $(Y, \mathscr{C}, \nu)$. Consider the transformation $(X \times Y, \mathscr{B} \otimes \mathscr{C}, \mu \times \nu, T_f)$, where

$$T_f(x,y) = (Tx, T_{f(x)} y).$$

The n–th iterate takes the form $T_f^n(x,y) = (\sigma^n x, T_{x_{n-1}} \cdots T_{x_0} y)$. The second coordinate looks like the random concatenation of elements of $\{T_0, T_1\}$. We say that T_f is a "random dynamical system driven by the noise process $(X, \mathscr{B}, \mu, T)$."

These examples suggest the following abstract constructions.

Suppose $(X, \mathscr{B}, \mu, T)$ is a mpt, and G is a locally compact Polish[3] topological group, equipped with a left invariant Haar measure m_G. Suppose $f : X \to G$ is measurable.

Definition 1.13. The *skew–product* with cocycle f over the basis $(X, \mathscr{B}, \mu, T)$ is the mpt $(X \times G, \mathscr{B} \otimes \mathscr{B}(G), \mu \times m_G, T_f)$, where $T_f : X \times G \to X \times G$ is the transformation $T_f(x,g) = (Tx, g \cdot f(x))$.

(Check, using Fubini's theorem, that this is a mpt.) The n–th iterate $T_f^n(x,g) = (T^{n-1}x, g \cdot f(x) \cdot f(Tx) \cdots f(T^{n-1}x))$, is a "random walk on G driven by the noise process $(X, \mathscr{B}, \mu, T)$."

Now imagine that the group G is acting in a measure preserving way on some space $(Y, \mathscr{C}, \nu)$. This means that there are measurable maps $T_g : Y \to Y$ such that $\nu \circ T_g^{-1} = \nu$, $T_{g_1 g_2} = T_{g_1} T_{g_2}$, and $(g,y) \mapsto T_g(y)$ is a measurable from $X \times G$ to Y.

Definition 1.14. The *random dynamical system* on $(Y, \mathscr{C}, \nu)$ with action $\{T_g : g \in G\}$, cocycle $f : X \to G$, and noise process $(X, \mathscr{B}, \mu, T)$, is the system $(X \times Y, \mathscr{B} \otimes \mathscr{C}, \mu \times \nu, T_f)$ given by $T_f(x,y) = (Tx, T_{f(x)} y)$.

[3] "Polish"=has a topology which makes it a complete separable metric space.

(Check using Fubini's theorem that this is measure preserving.) Here the n-th iterate is $T_f^n(x,y) = (T^n x, T_{f(T^n x)} \cdots T_{f(Tx)} T_{f(x)} y)$.

It is obvious that if a skew–product (or a random dynamical system) is ergodic or mixing, then its base is ergodic or mixing. The converse is not always true. The ergodic properties of a skew–product depend in a subtle way on the interaction between the base and the cocycle.

Here are two important obstructions to ergodicity and mixing for skew–products. In what follows G is a polish group and $\widehat{G}$ is its group of characters,

$$\widehat{G} := \{\gamma : G \to S^1 : \gamma \text{ is a continuous homomorphism}\}.$$

Definition 1.15. Suppose $(X, \mathscr{B}, \mu, T)$ is a ppt and $f : X \to G$ is Borel.

1. f is called *arithmetic* w.r.t. μ, if $\exists h : X \to S^1$ measurable, and $\gamma \in \widehat{G}$ non-constant, such that $\gamma \circ f = h/h \circ T$ a.e.
2. f is called *periodic* w.r.t. μ, if $\exists h : X \to S^1$ measurable, $|\lambda| = 1$, and $\gamma \in \widehat{G}$ non-constant, such that $\gamma \circ f = \lambda h/h \circ T$ a.e.

Proposition 1.10. *Let $(X, \mathscr{B}, \mu, T)$ be a ppt, $f : X \to G$ Borel, and $(X \times G, \mathscr{B} \otimes \mathscr{B}(G), \mu \times m_G, T_f)$ the corresponding skew–product. If f is arithmetic, then T_f is not ergodic, and if f is periodic, then T_f is not mixing.*

Proof. Suppose f is arithmetic. The function $F(x,y) := h(x)\gamma(y)$ satisfies

$$F(Tx, yf(x)) = h(Tx)\gamma(y)\gamma(f(x)) = h(Tx)\gamma(y)h(x)/h(Tx) = F(x,y),$$

and we have a non-constant invariant function. So arithmeticity⇒non-ergodicity. Similarly, if f is periodic, then $F(x,y) = h(x)\gamma(y)$ satisfies $F(Tx, yf(x)) = \lambda F(x,y)$, whence $F \circ T_f = \lambda F$. Pick $n_k \to \infty$ s.t. $\lambda^{n_k} \to 1$, then $\mathrm{Cov}(F, F \circ T_f^{n_k}) \xrightarrow[k\to\infty]{} \int F^2 - (\int F)^2$. Since $F \neq \int F$ a.e., the limit is non-zero and we get a contradiction to mixing. So periodicity⇒non-mixing. □

1.6.2 Factors

When we construct skew-products over a base, we enrich the space. A factor is a constructions which depletes the space.

Definition 1.16. A mpt transformation $(X, \mathscr{B}, \mu, T)$ is called a *(measure theoretic) factor* of a mpt transformation $(Y, \mathscr{C}, \nu, S)$, if there are sets of full measure $X' \subset X, Y' \subset Y$ such that $T(X') \subset X'$, $S(Y') \subset Y'$, and a measurable *onto* map $\pi : Y' \to X'$ such that $\nu \circ \pi^{-1} = \mu$ and $\pi \circ S = T \circ \pi$ on Y'. We call π the *factor map*.

$$\begin{array}{ccc} Y' & \xrightarrow{S} & Y' \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{T} & X' \end{array}$$

If $(X,\mathscr{B},\mu,T)$ is a factor of $(Y,\mathscr{C},\nu,S)$, then it is customary to call $(Y,\mathscr{C},\nu,S)$ an *extension* of $(X,\mathscr{B},\mu,T)$ and π the *factor map*.

There are three principle examples:

1. Any measure theoretic isomorphism between two mpt is a factor map between them. But some factor maps are not isomorphisms because they are not injective.
2. A skew product $T_f : X\times Y \to X\times Y$ is an extension of its base $T : X \to X$. The factor map is $\pi : X\times G \to X$, $\pi(x,y) = x$.
3. Suppose $(X,\mathscr{B},\mu,T)$ is an mpt and T is measurable w.r.t. a smaller σ–algebra $\mathscr{C}\subset\mathscr{B}$ (i.e. $T^{-1}\mathscr{C}\subset\mathscr{C}$), then $(X,\mathscr{C},\mu,T)$ is a factor of $(X,\mathscr{B},\mu,T)$. The factor map is the identity.

We dwell a bit more on the third example. In probability theory, σ–algebras model *information*: a set E is "measurable", if we can answer the question "is ω in E?" using the information available to use. For example, if a real number $x\in\mathbb{R}$ is unknown, but we can "measure" $|x|$, then the information we have on x is modeled by the σ–algebra $\{E\subset\mathbb{R} : E = -E\}$, because we can determined whether $x\in E$ only for symmetric sets E. By decreasing the σ–algebra, we are forgetting some information. For example if instead of knowing $|x|$, we only know whether $0\leq|x|\leq 1$ or not, then our σ–algebra is the finite σ–algebra $\{\varnothing,\mathbb{R},[-1,1],\mathbb{R}\setminus[-1,1]\}$.

Here is a typical example. Suppose we have a dynamical system $(X,\mathscr{B},\mu,T)$, and we cannot "measure" x, but we can "measure" $f(x)$ for some measurable $f : X\to\mathbb{R}$. Then the information we have by observing the dynamical system is encoded in the smallest σ–algebra $\mathscr{C}\subset\mathscr{B}$ with respect to which $f\circ T^n$ are all measurable.[4] The dynamical properties we feel in this case are those of the factor $(X,\mathscr{C},\mu,T)$ and not of the $(X,\mathscr{B},\mu,T)$. For example, it could be the case that $\mu(E\cap T^{-n}F)\to\mu(E)\mu(F)$ for all $E,F\in\mathscr{C}$ but not for all $E,F\in\mathscr{B}$ — and then we will observe "mixing" simply because our information is not sufficient to observe the non-mixing in the system.

1.6.3 The natural extension

An *invertible* mpt is a mpt $(X,\mathscr{B},\mu,T)$ such that for some invariant set $X'\subset X$ of full measure, $T : X'\to X'$ is invertible, and $T,T^{-1} : X'\to X'$ are measurable. Invertible mpt are more convenient to handle than general mpt because they have the following properties: If $(X,\mathscr{B},\mu)$ is a complete measure space, then the forward image of a measurable set is measurable, and for any countable collection of measurable sets A_i, $T(\bigcap A_i) = \bigcap T(A_i)$ up to a set of measure zero. This is not true in general for non-invertible maps. Luckily, every "reasonable" non-invertible mpt is the factor of an invertible mpt. Here is the construction.

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[4] Such a σ–algebra exists: take the intersection of all sub-σ–algebras which make $f\circ T^n$ all measurable, and note that this intersection is not empty because it contains $\mathscr{B}$.

Definition 1.17. Suppose $(X,\mathscr{B},\mu,T)$ is a ppt defined on a Lebesgue measure space, and assume $T(X)=X$. The *natural extension* of $(X,\mathscr{B},\mu,T)$ is the system $(\widetilde{X},\widetilde{\mathscr{B}},\widetilde{\mu},\widetilde{T})$, where

1. $\widetilde{X}:=\{\underline{x}=(\dots,x_{-1},x_0,x_1,x_2,\dots):x_i\in X, T(x_i)=x_{i+1}$ for all $i\}$;
2. $\widetilde{\mathscr{B}}$ is the smallest σ-algebra which contains all sets of the form $\{\underline{x}\in\widetilde{X}:x_i\in E\}$ with $i\le 0$ and $E\in\mathscr{B}$ (see below);
3. $\widetilde{\mu}$ is the unique probability measure on $\widetilde{\mathscr{B}}$ such that $\mu\{\underline{x}\in\widetilde{X}:x_i\in E_i\}=\mu(E_i)$ for all $i\le 0$ and $E_i\in T^{-i}\mathscr{B}$;
4. $\widetilde{T}$ is the left shift.

Lemma 1.2. *The measure $\widetilde{\mu}$ exists and is unique.*

Proof (the proof can be omitted on first reading). Let $\mathscr{S}$ denote the collection of all sets of the form $[E_{-n},\dots,E_0]:=\{\underline{x}\in\widetilde{X}:x_{-i}\in E_{-i}\ \ (i=0,\dots,n)\}$ where $n\ge 0$ and $E_{-n},\dots,E_{-1},E_0\in\mathscr{B}$. We call the elements of $\mathscr{S}$ *cylinders*.

It is easy to see that $\mathscr{S}$ is a semi-algebra. Our plan is to define $\widetilde{\mu}$ on $\mathscr{S}$ and then apply Carathéodory's extension theorem. To do this we first observe the following important identity:

$$[E_{-n},\dots,E_0]=\{\underline{x}\in\widetilde{X}:x_{-n}\in\bigcap_{i=0}^{n}T^{-(n-i)}E_{-i}\}. \tag{1.3}$$

The inclusion $\subseteq$ is because for every $\underline{x}\in[E_{-n},\dots,E_0]$, $x_{-n}\equiv T^{-(n-i)}x_{-i}\in T^{-(n-i)}E_{-i}$ by the definition of $\widetilde{X}$; and $\supseteq$ is because if $x_{-n}\in T^{-(n-i)}E_{-i}$ then $x_{-i}\equiv T^{(n-i)}(x_{-n})\in E_{-i}$ for $i=0,\dots,n$. Motivated by this identity we define $\widetilde{\mu}:\mathscr{S}\to[0,\infty]$ by

$$\widetilde{\mu}[E_{-n},\dots,E_0]:=\mu\left(\bigcap_{i=0}^{n}T^{-(n-i)}E_{-i}\right).$$

Since $\widetilde{\mu}(\widetilde{X})=\mu(X)=1$, $\widetilde{\mu}$ is σ-finite on $\widetilde{X}$. We will check that $\widetilde{\mu}$ is σ-additive on $\mathscr{S}$ and deduce the lemma from Carathéodory's extension theorem. We begin with simpler finite statements.

STEP 1. *Suppose $C_1,\dots,C_\alpha$ are pairwise disjoint cylinders and $D_1,\dots,D_\beta$ are pairwise disjoint cylinders. If $\biguplus_{i=1}^{\alpha}C_i=\biguplus_{i=1}^{\beta}D_i$, then $\sum\widetilde{\mu}(C_i)=\sum\widetilde{\mu}(D_i)$.*

Proof. Notice that $[E_{-n},\dots,E_0]\equiv[X,\dots,X,E_{-n},\dots,E_0]$ and (by the T-invariance of μ) $\widetilde{\mu}[E_{-n},\dots,E_0]=\widetilde{\mu}[X,\dots,X,E_{-n},\dots,E_0]$, no matter how many X's we add to the left. So there is no loss of generality in assuming that C_i,D_i all have the same length: $C_i=[C^{(i)}_{-n},\dots,C^{(i)}_0]$, $D_j=[D^{(i)}_{-n},\dots,D^{(i)}_0]$.

By (1.3), since C_i are pairwise disjoint, $\bigcap_{k=0}^{n}T^{-(n-k)}C^{(i)}_{-k}$ are pairwise disjoint. Similarly, $\bigcap_{k=0}^{n}T^{-(n-k)}D^{(i)}_{-k}$ are pairwise disjoint. Since $\biguplus_{i=1}^{\alpha}C_i=\biguplus_{i=1}^{\beta}D_i$, the identity (1.3) also implies that $\biguplus_{i=1}^{\alpha}\bigcap_{k=0}^{n}T^{-(n-k)}C^{(i)}_{-k}=\biguplus_{j=1}^{\beta}\bigcap_{k=0}^{n}T^{-(n-k)}D^{(i)}_{-k}$. So

$$\sum_{i=1}^{\alpha} \widetilde{\mu}(C_i) = \sum_{i=1}^{\alpha} \mu \left(\bigcap_{k=0}^{n} T^{-(n-k)} C_{-k}^{(i)} \right) = \sum_{j=1}^{\beta} \mu \left(\bigcap_{k=0}^{n} T^{-(n-k)} D_{-k}^{(j)} \right) = \sum_{j=1}^{\alpha} \widetilde{\mu}(D_j),$$

which proves our claim.

STEP 2. *Suppose $C, C_1, \dots, C_n$ are cylinders. If $C \subset \biguplus_{i=1}^{n} C_i$, then $\widetilde{\mu}(C) \le \sum_{i=1}^{n} \widetilde{\mu}(C_i)$.*

Proof. Notice that $\bigcup_{i=1}^{n} C_i = \biguplus_{i=1}^{n} (C_i \cap \bigcap_{j=1}^{i-1} C_j^c)$. Using the fact that $\mathscr{S}$ is a semi-algebra, it is not difficult to see that $C_i \cap \bigcap_{j=1}^{i-1} C_j^c$ is a finite pairwise disjoint union of cylinders: $C_i \cap \bigcap_{j=1}^{i-1} C_j^c = \biguplus_{k=1}^{n_i} C_{ik}$. So

$$\bigcup_{i=1}^{n} C_i = \biguplus_{i=1}^{n} \biguplus_{k=1}^{n_i} C_{ik} \text{ where } \biguplus_{k=1}^{n_i} C_{ik} \subset C_i.$$

Look at the inclusion $C \subset \bigcup_{i=1}^{n} C_i \equiv \biguplus_{i=1}^{n} \biguplus_{k=1}^{n_i} C_{ik}$. The set difference of the two sides of the equation is a finite pairwise disjoint union of cylinders because $\mathscr{S}$ is a semi-algebra. So by step 1 $\widetilde{\mu}(C) \le \sum_{i=1}^{n} \left(\sum_{k=1}^{n_i} \widetilde{\mu}(C_{ik}) \right)$.

Similarly, $\biguplus_{k=1}^{n_i} C_{ik} \subset C_i$ and the set difference of the two sides of the inclusion is a finite pairwise disjoint union of cylinders. So by step 1, $\sum_{j=1}^{n_i} \widetilde{\mu}(C_{ik}) \le \widetilde{\mu}(C_i)$. In summary $\widetilde{\mu}(C) \le \sum \widetilde{\mu}(C_i)$.

We are finally ready to prove σ-additivity. Suppose $C_k = [E_{-n_k}^{(k)}, \dots, E_0^{(k)}]$ are a countable collection of pairwise disjoint cylinders such that $\biguplus_{k=1}^{\infty} C_k = C = [E_{-n}, \dots, E_0]$. Our aim is to show that $\mu(C) = \sum \mu(C_k)$.

STEP 3. $\mu(C) \ge \sum \mu(C_k)$

Proof. Without loss of generality $n_k \ge n$ for all k, otherwise we can replace C_k by the equal cylinder $[X, \dots, X, E_{-n_k}^{(k)}, \dots, E_0^{(k)}]$ with X repeated $n - n_k - 1$ times.

For every K, let $N_K := \max\{n_k : 1 \le k \le K\}$, then $N_K \ge n$. Define $E_{-m} := X$ for $m > n$ and $E_{-m}^{(k)} := X$ for $m > n_k$. If $m > N_K$ and $1 \le k \le K$ then $C = [E_{-m}, \dots, E_0]$ and $C_k = [E_{-m}^{(k)}, \dots, E_0^{(k)}]$. So $[E_{-m}, \dots, E_0] \supseteq \biguplus_{k=1}^{K} [E_{-m}^{(k)}, \dots, E_0^{(k)}]$.

It now follows from (1.3) that $\bigcap_{i=0}^{m} T^{-(m-i)} E_{-i} \supseteq \biguplus_{k=1}^{K} \bigcap_{i=0}^{m} T^{-(m-i)} E_{-i}^{(k)}$, whence $\mu\left(\bigcap_{i=0}^{m} T^{-(m-i)} E_{-i}\right) \ge \mu\left(\biguplus_{k=1}^{K} \bigcap_{i=0}^{m} T^{-(m-i)} E_{-i}^{(k)}\right)$.

Since μ is σ-additive on $\mathscr{B}$ and $\mu \circ T^{-1} = \mu$, the left-hand-side equals $\widetilde{\mu}(C)$ and the right-hand-side equals $\sum_{k=1}^{K} \widetilde{\mu}(C_k)$.

STEP 2. $\mu(C) \le \sum \mu(C_k)$.

Proof. It is here that we use the assumption that $(X, \mathscr{B}, \mu)$ is a Lebesgue measure space. The assumption allows us to assume without loss of generality that X is a compact metric space and $\mathscr{B}$ is the completion of the Borel σ-algebra (because any union of an interval and a countable collection of atoms can be isomorphically embedded in such a space). In this case μ is *regular*: for every $E \in \mathscr{B}$ and for every $\varepsilon > 0$ there is an open set U and a compact set F such that $F \subset E \subset U$ and $\mu(E \setminus F), \mu(U \setminus E) < \varepsilon$.

In particular, given $\varepsilon > 0$ there is no problem in finding a compact set $F \subset \bigcap_{i=0}^{n} T^{-(n-i)} E_{-i}$ and open sets $U_k \supset \bigcap_{i=0}^{n_k} T^{-(n_k-i)} E_{-i}^{(k)}$ open so that

$$\mu(F) \geq \widetilde{\mu}(C) - \varepsilon \text{ and } \mu(U_k) \leq \widetilde{\mu}(C_k) + \frac{\varepsilon}{2^k}.$$

By (1.3), $[F, \underbrace{X, \ldots, X}_{n}] \subset C = \biguplus_{k=1}^{\infty} C_k \subset \bigcup_{k=1}^{\infty} [U_k, \underbrace{X, \ldots, X}_{n_k}]$, and with respect to the product topology on $\widetilde{X}$, $[F, X, \ldots, X]$ is compact and $[U_k, X, \ldots, X]$ are open. So there is a finite N s.t. $[F, \underbrace{X, \ldots, X}_{n}] \subset \bigcup_{k=1}^{N} [U_k, \underbrace{X, \ldots, X}_{n_k}]$. By step 2

$$\widetilde{\mu}[F, \underbrace{X, \ldots, X}_{n}] \leq \sum_{k=1}^{N} \widetilde{\mu}[U_k, \underbrace{X, \ldots, X}_{n_k}]$$

So $\mu(F) \leq \sum_{k=1}^{N} \mu(U_k)$, whence $\widetilde{\mu}(C) \leq \sum_{k=1}^{N} \widetilde{\mu}(C_k) + 2\varepsilon \leq \sum_{k=1}^{\infty} \widetilde{\mu}(C_k) + 2\varepsilon$.

The step follows by taking $\varepsilon \to 0$. ☐

Theorem 1.6. *The natural extension of $(X, \mathscr{B}, \mu, T)$ is an invertible extension of $(X, \mathscr{B}, \mu, T)$, and is the factor of any other invertible extension of $(X, \mathscr{B}, \mu, T)$. $\widetilde{T}$ is ergodic iff T is ergodic, and $\widetilde{T}$ is mixing iff T is mixing.*

Proof. It is clear that the natural extension is an invertible ppt. Let $\pi : \widetilde{X} \to X$ denote the map $\pi(\underline{x}) = x_0$, then π is measurable and $\pi \circ \widetilde{T} = T \circ \pi$. Since $TX = X$, every point has a pre-image, and so π is onto. Finally, for every $E \in \mathscr{B}$,

$$\widetilde{\mu}(\pi^{-1}(E)) = \widetilde{\mu}(\{\widetilde{x} \in \widetilde{X} : x_0 \in E\}) = \mu(E)$$

by construction. So $(\widetilde{X}, \widetilde{\mathscr{B}}, \widetilde{\mu}, \widetilde{T})$ is an invertible extension of $(X, \mathscr{B}, \mu, T)$.

Suppose $(Y, \mathscr{C}, \nu, S)$ is another invertible extension, and let $\pi_Y : Y \to X$ be the factor map (defined a.e. on Y). We show that $(Y, \mathscr{C}, \nu, S)$ extends $(\widetilde{X}, \widetilde{\mathscr{B}}, \widetilde{\mu}, \widetilde{T})$.

Let $(\widetilde{Y}, \widetilde{\mathscr{C}}, \widetilde{\nu}, \widetilde{S})$ be the natural extension of $(Y, \mathscr{C}, \nu, S)$. It is isomorphic to $(Y, \mathscr{C}, \nu, S)$, with the isomorphism given by $\vartheta(y) = (y_k)_{k \in \mathbb{Z}}$, $y_k := S^k(y)$. Thus it is enough to show that $(\widetilde{X}, \widetilde{\mathscr{B}}, \widetilde{\mu}, \widetilde{T})$ is a factor of $(\widetilde{Y}, \widetilde{\mathscr{C}}, \widetilde{\nu}, \widetilde{T})$. Here is the factor map: $\theta : (y_k)_{k \in \mathbb{Z}} \mapsto (\pi_Y(y_k))_{k \in \mathbb{Z}}$.

If $\widetilde{T}$ is ergodic, then T is ergodic, because every T-invariant set E lifts to a $\widetilde{T}$-invariant set $\widetilde{E} := \pi^{-1}(E)$. The ergodicity of $\widetilde{T}$ implies that $\widetilde{\mu}(\widetilde{E}) = 0$ or 1, whence $\mu(E) = \widetilde{\mu}(\pi^{-1}(E)) = \widetilde{\mu}(\widetilde{E}) = 0$ or 1.

To see the converse (T is ergodic $\Rightarrow \widetilde{T}$ is ergodic) we make use of the following observation:

CLAIM: *Let $\widetilde{\mathscr{B}}_n := \left\{ \{\widetilde{x} \in \widetilde{X} : x_{-n} \in E\} : E \in \mathscr{B} \right\}$, then*

1. *$\widetilde{\mathscr{B}}_n$ are σ-algebras*
2. *$\widetilde{\mathscr{B}}_1 \subset \widetilde{\mathscr{B}}_2 \subset \cdots$ and $\bigcup_{n \geq 0} \widetilde{\mathscr{B}}_n$ generate $\widetilde{\mathscr{B}}$*

3. *$\widetilde{T}^{-1}(\widetilde{\mathscr{B}}_n) \subset \widetilde{\mathscr{B}}_n$ and $(\widetilde{X}, \widetilde{\mathscr{B}}_n, \widetilde{\mu}, \widetilde{T})$ is a ppt*
4. *if T is ergodic then $(\widetilde{X}, \widetilde{\mathscr{B}}_n, \widetilde{\mu}, \widetilde{T})$ is ergodic.*

Proof. We leave the first three items as exercises to the reader. To see the last item suppose T is ergodic and $\widetilde{E} \in \widetilde{\mathscr{B}}_n$ is $\widetilde{T}$-invariant. By the definition of $\widetilde{\mathscr{B}}_n$, $\widetilde{E} = \{\widetilde{x} : \widetilde{x}_{-n} \in E\}$ with $E \in \mathsf{B}$, and it is not difficult to see that E must be T-invariant. Since T is ergodic, $\mu(E) = 0$ or 1. So $\widetilde{\mu}(\widetilde{E}) = \mu(E) = 0$ or 1. So $(\widetilde{X}, \widetilde{\mathscr{B}}_n, \widetilde{\mu}, \widetilde{T})$ is ergodic.

We can now prove the ergodicity of $(\widetilde{X}, \widetilde{\mathscr{B}}, \widetilde{\mu}, \widetilde{T})$ as follows. Suppose $\widetilde{f}$ is absolutely integrable and $\widetilde{T}$-invariant, and let

$$\widetilde{f}_n := \mathbb{E}(\widetilde{f} | \widetilde{\mathscr{B}}_n)$$

(readers who are not familiar with conditional expectations can find their definition in section 2.3.1).

We claim that $\widetilde{f}_n \circ \widetilde{T} = \widetilde{f}_n$. This is because for every bounded $\widetilde{\mathscr{B}}_n$–measurable test function $\widetilde{\varphi}$,

$$\int [\widetilde{\varphi} \cdot \widetilde{f}_n \circ \widetilde{T}^{-1}] d\widetilde{\mu} \overset{(1)}{=} \int (\widetilde{\varphi} \cdot \widetilde{f}_n \circ \widetilde{T}^{-1}) \circ \widetilde{T} d\widetilde{\mu} = \int \widetilde{\varphi} \circ \widetilde{T} \cdot \widetilde{f}_n d\widetilde{\mu} \overset{(2)}{=} \int \widetilde{\varphi} \circ \widetilde{T} \cdot \widetilde{f} d\widetilde{\mu}$$
$$\overset{(3)}{=} \int (\widetilde{\varphi} \circ \widetilde{T} \cdot \widetilde{f} \circ \widetilde{T}) d\widetilde{\mu} \overset{(4)}{=} \int \widetilde{\varphi} \widetilde{f} d\widetilde{\mu} \overset{(5)}{=} \int \widetilde{\varphi} \widetilde{f}_n d\widetilde{\mu}.$$

Justifications: (1) is because $\widetilde{\mu} \circ \widetilde{T}^{-1} = \widetilde{\mu}$; (2) is because $\widetilde{f}_n = \mathbb{E}(\widetilde{f} | \widetilde{\mathscr{B}}_n)$ and $\widetilde{\varphi} \circ \widetilde{T}$ is $\widetilde{\mathscr{B}}_n$–measurable by part (3) of the claim; (3) is because $\widetilde{f} \circ \widetilde{T} = \widetilde{f}$; (4) is because $\widetilde{\mu} \circ \widetilde{T}^{-1} = \widetilde{\mu}$; and (5) is by the definition of the conditional expectation.

In summary $\int [\widetilde{\varphi} \cdot \widetilde{f}_n \circ \widetilde{T}^{-1}] d\widetilde{\mu} = \int \widetilde{\varphi} \widetilde{f}_n d\widetilde{\mu}$ for all bounded $\widetilde{\mathscr{B}}_n$–measurable functions, whence $\widetilde{f}_n \circ \widetilde{T} = \widetilde{f}_n$ a.e. as claimed.

We saw above that $(\widetilde{X}, \widetilde{\mathscr{B}}_n, \widetilde{\mu}, \widetilde{T})$ is ergodic. Therefore, the invariance $\widetilde{\mathscr{B}}_n$–measurability of $\widetilde{f}_n$ imply that $\widetilde{f}_n = \int \widetilde{f} d\widetilde{\mu}$ almost everywhere. By the martingale convergence theorem,

$$\widetilde{f}_n \xrightarrow[n \to \infty]{} \widetilde{f} \text{ almost everywhere.}$$

So $\widetilde{f}$ is constant almost everywhere. This shows that every $\widetilde{T}$-invariant integrable function is constant a.e., so $\widetilde{T}$ is ergodic.

Next we consider the mixing of T, $\widetilde{T}$. $\widetilde{T}$ mixing $\Rightarrow$ T mixing because for every $A, B \in \mathscr{B}$, $\mu(A \cap T^{-n}B) = \widetilde{\mu}(\pi^{-1}A \cap T^{-n}\pi^{-1}B) \xrightarrow[n \to \infty]{} \widetilde{\mu}(\pi^{-1}A)\widetilde{\mu}(\pi^{-1}B) \equiv \mu(A)\mu(B)$. We show the other direction. Suppose T is mixing and consider $\widetilde{A}, \widetilde{B} \in \mathscr{B}_n$ with $\mathscr{B}_n$ defined as before. Then $\widetilde{A} = \{\widetilde{x} : x_{-n} \in A\}$ and $\widetilde{B} = \{\widetilde{x} : x_{-n} \in B\}$ with $A, B \in \mathscr{B}$ and using the identity $(\widetilde{T}\widetilde{x})_i = T(\widetilde{x}_i)$ it is easy to see that

$$\widetilde{A} \cap \widetilde{T}^{-k}\widetilde{B} = \{\underline{x} \in \widetilde{X} : \widetilde{x}_{-n} \in A \cap T^{-k}B\}.$$

So $\widetilde{\mu}(\widetilde{A} \cap \widetilde{T}^{-k}\widetilde{B}) = \mu(A \cap T^{-k}B) \xrightarrow[k \to \infty]{} \mu(A)\mu(B) \equiv \widetilde{\mu}(\widetilde{A})\widetilde{\mu}(\widetilde{B})$.

This proves mixing for $\widetilde{A},\widetilde{B}\in\mathscr{B}_n$. To get mixing for general $\widetilde{A},\widetilde{B}\in\mathscr{B}$ we note $\bigcup\widetilde{\mathscr{B}}_n$ generates $\widetilde{\mathscr{B}}$ so for any $\widetilde{E}$ and for every $\varepsilon>0$ there are n and $\widetilde{E}'\in\widetilde{\mathscr{B}}_n$ such that $\widetilde{\mu}(\widetilde{E}\triangle\widetilde{E}')<\varepsilon$ (because the collection of sets $\widetilde{E}$ with such n and $\widetilde{E}'$ forms a σ-algebra which contains $\bigcup\widetilde{\mathscr{B}}_n$). A standard approximation argument now shows that $\widetilde{\mu}(\widetilde{A}\cap\widetilde{T}^{-k}\widetilde{B})\xrightarrow[k\to\infty]{}\widetilde{\mu}(\widetilde{A})\widetilde{\mu}(\widetilde{B})$ for all $\widetilde{A},\widetilde{B}\in\widetilde{\mathscr{B}}$. $\square$

1.6.4 Induced transformations

Suppose $(X,\mathscr{B},\mu,T)$ is a *probability* preserving transformation, and let $A\in\mathscr{B}$ be a set of positive measure. By Poincaré's Recurrence Theorem, for a.e. $x\in A$ there is some $n\geq 1$ such that $T^n(x)\in A$. Define

$$\varphi_A(x):=\min\{n\geq 1: T^n x\in A\},$$

with the minimum of the empty set being understood as infinity. Note that $\varphi_A<\infty$ a.e. on A, hence $A_0:=\{x\in A:\varphi_A(x)<\infty\}$ is equal to A up to a set of measure zero.

Definition 1.18. The *induced transformation* on A is $(A_0,\mathscr{B}(A),\mu_A,T_A)$, where $A_0:=\{x\in A:\varphi_A(x)<\infty\}$, $\mathscr{B}(A):=\{E\cap A_0:E\in\mathscr{B}\}$, μ_A is the measure $\mu_A(E):=\mu(E|A)=\mu(E\cap A)/\mu(A)$, and $T_A:A_0\to A_0$ is $T_A(x)=T^{\varphi_A(x)}(x)$.

Theorem 1.7. *Suppose $(X,\mathscr{B},\mu,T)$ is a ppt, and $A\in\mathscr{B}$ has positive finite measure.*

1. *$\mu_A\circ T_A^{-1}=\mu_A$;*
2. *if T is ergodic, then T_A is ergodic (but the mixing of $T\not\Rightarrow$ the mixing of T_A);*
3. ***Kac Formula:** If μ is ergodic, then $\int f d\mu=\int_A\sum_{k=0}^{\varphi_A-1}f\circ T^k d\mu$ for every $f\in L^1(X)$. In particular $\int_A\varphi_A d\mu_A=1/\mu(A)$.*

Proof. Given $E\subset A$ measurable, $\mu(E)=\underbrace{\mu(T^{-1}E\cap A)}_{\mu(T_A^{-1}E\cap[\varphi_A=1])}+\mu(T^{-1}E\cap A^c)=$

$$=\underbrace{\mu(T^{-1}E\cap A)}_{\mu(T_A^{-1}E\cap[\varphi_A=1])}+\underbrace{\mu(T^{-2}E\cap T^{-1}A^c\cap A)}_{\mu(T_A^{-1}E\cap[\varphi_A=2])}+\mu(T^{-2}E\cap T^{-1}A^c\cap A^c)$$
$$=\cdots=\sum_{j=1}^{N}\mu(T_A^{-1}E\cap[\varphi_A=j])+\mu(T^{-N}E\cap\bigcap_{j=0}^{N-1}T^{-j}A^c).$$

Passing to the limit as $N\to\infty$, we see that $\mu(E)\geq\mu_A(T_A^{-1}E)$. Working with $A\setminus E$, and using the assumption that $\mu(X)<\infty$, we get that $\mu(A)-\mu(E)\geq\mu(A)-\mu(T_A^{-1}E)$ whence $\mu(E)=\mu(T_A^{-1}E)$. Since μ_A is proportional to μ on $\mathscr{B}(A)$, we get $\mu_A=\mu_A\circ T_A^{-1}$.

We assume that T is ergodic, and prove that T_A is ergodic. The set

$$\Omega:=\{x:T^n(x)\in A\text{ for infinitely many }n\geq 0\}$$

is a T–invariant set of non-zero measure (bounded below by $\mu(A)$), so it must has full measure. Thus a.e. $x \in X$ has some $n \geq 0$ s.t. $T^n(x) \in A$, and

$$r_A(x) := \min\{n \geq 0 : T^n x \in A\} < \infty \text{ a.e. in } X.$$

Suppose $f : A_0 \to \mathbb{R}$ is a T_A–invariant L^2–function. Define

$$F(x) := f(T^{r_A(x)} x).$$

This makes sense a.e. in X, because $r_A < \infty$ almost everywhere. This function is T–invariant, because either $x, Tx \in A$ and then $F(Tx) = f(Tx) = f(T_A x) = f(x) = F(x)$ or one of x, Tx is outside A and then $F(Tx) = f(T^{r_A(Tx)} Tx) = f(T^{r_A(x)} x) = F(x)$. Since T is ergodic, F is constant a.e. on X, and therefore $f = F|_A$ is constant a.e. on A. Thus the ergodicity of T implies the ergodicity of T_A.

Here is an example showing that the mixing of T does not imply the mixing of T_A. Let Σ^+ be a SFT with states $\{a, 1, 2, b\}$ and allowed transitions

$$a \to 1; 1 \to 1, b; b \to 2; 2 \to a.$$

Let $A = \{\underline{x} : x_0 = a, b\}$. Any shift invariant Markov measure μ on Σ^+ is mixing, because Σ^+ is irreducible and aperiodic ($1 \to 1$). But T_A is not mixing, because $T_A[a] = [b]$ and $T_A[b] = [a]$, so $[a] \cap T_A^{-n}[a] = \varnothing$ for all n odd.

Next we prove the Kac formula. Suppose first that $f \in L^\infty(X, \mathscr{B}, \mu)$ and $f \geq 0$.

$$\begin{aligned}
\int f d\mu &= \int_A f d\mu + \int f \cdot 1_{A^c} d\mu = \int_A f d\mu + \int f \circ T \cdot 1_{T^{-1}A^c} d\mu \\
&= \int_A f d\mu + \int f \circ T \cdot 1_{T^{-1}A^c \cap A} d\mu + \int f \circ T \cdot 1_{T^{-1}A^c \cap A^c} d\mu \\
&= \int_A f d\mu + \int_A f \circ T \cdot 1_{[\varphi_A > 1]} d\mu + \int f \circ T^2 \cdot 1_{T^{-2}A^c \cap T^{-1}A^c} d\mu \\
&= \cdots = \int_A \sum_{j=0}^{N-1} f \circ T^j \cdot 1_{[\varphi_A > j]} d\mu + \int f \circ T^N \cdot 1_{\bigcap_{j=1}^N T^{-j}A^c} d\mu.
\end{aligned}$$

The first term tends, as $N \to \infty$, to

$$\int_A \sum_{j=0}^{\infty} f \circ T^j \sum_{i=j+1}^{\infty} 1_{[\varphi_A = i]} d\mu \equiv \int_A \sum_{j=0}^{\varphi_A - 1} f \circ T^j d\mu.$$

The second term is bounded by $\|f\|_\infty \mu\{x : T^j(x) \notin A \text{ for all } k \leq N\}$. This bound tends to zero, because $\mu\{x : T^j(x) \notin A \text{ for all } k\} = 0$ because T is ergodic and recurrent (fill in the details). This proves the Kac formula for all L^∞ functions.

Every non-negative L^1–function is the increasing limit of L^∞ functions. By the monotone convergence theorem, the Kac formula must hold for all non-negative L^1–function. Every L^1–function is the difference of two non-negative L^1–functions ($f = f \cdot 1_{[f>0]} - |f| \cdot 1_{[f<0]}$). It follows that the Kac formula holds for all $f \in L^1$. $\square$

1.6.5 Suspensions and Kakutani skyscrapers

The operation of inducing can be "inverted", as follows. Let $(X,\mathscr{B},\mu,T)$ be a ppt, and $r:X\to\mathbb{N}$ an integrable measurable function.

Definition 1.19. The *Kakutani skyscraper* with *base* $(X,\mathscr{B},\mu,T)$ and *height function r* is the system $(X_r),\mathscr{B}(X_r),\nu,S)$, where

1. $X_r := \{(x,n) : x\in X, 0\le n\le r(x)-1\}$;
2. $\mathscr{B}(X_r)=\{E\in\mathscr{B}(X)\otimes\mathscr{B}(\mathbb{N}) : E\subseteq X_r\}$, where $\mathscr{B}(\mathbb{N})=2^{\mathbb{N}}$;
3. ν is the unique measure such that $\nu(B\times\{k\})=\mu(B)/\int r d\mu$;
4. S is defined by $S(x,n)=(x,n+1)$, when $n<r(x)-1$, and $S(x,n)=(Tx,0)$, when $n=r(x)-1$.

(Check that this is a ppt.)

We think of X_r as a skyscraper made of stories $\{(x,k) : r(x)>k\}$; the orbits of S climb up the skyscraper until the reach the top floor possible, and then move to the ground floor according to T.

If we induce a Kakutani skyscraper on $\{(x,0) : x\in X\}$, we get a system which is isomorphic to $(X,\mathscr{B},\mu,T)$.

Proposition 1.11. *A Kakutani skyscraper over an ergodic base is ergodic, but there are non-mixing skyscrapers over mixing bases.*

The proof is left as an exercise.

There is a straightforward important continuous–time version of this construction: Suppose $(X,\mathscr{B},\mu,T)$ is a ppt, and $r:X\to\mathbb{R}^+$ is a measurable function such that $\inf r>0$.

Definition 1.20. The *suspension semi-flow* with *base* $(X,\mathscr{B},\mu,T)$ and *height function r* is the semi-flow $(X_r,\mathscr{B}(X_r),\nu,T_s)$, where

1. $X_r := \{(x,t)\in X\times\mathbb{R} : 0\le t<r(x)\}$;
2. $\mathscr{B}(X_r)=\{E\in\mathscr{B}(X)\otimes\mathscr{B}(\mathbb{R}) : E\subseteq X_r\}$;
3. ν is the measure such that $\int_{X_r} f d\nu=\int_X\int_0^{r(x)} f(x,t)dtd\mu(x)/\int_X r d\mu$;
4. $T_s(x,t)=(T^n x, t+s-\sum\limits_{k=0}^{n-1} r(T^k x))$, where n is s.t. $0\le t+s-\sum\limits_{k=0}^{n-1} r(T^k x)<r(T^n x)$.

(Check that this is a measure preserving semi-flow.)

Suspension flows appear in applications in the following way. Imagine a flow T_t on a manifold X. It is often possible to find a submanifold $S\subset X$ such that (almost) every orbit of the flow intersects S transversally infinitely many times. Such a submanifold is called a *Poincaré section*. If it exists, then one can define a map $T_S : S\to S$ which maps $x\in S$ into $T_t x$ with $t:=\min\{s>0 : T_s(x)\in S\}$. This map is called the *Section map*. The flow itself is isomorphic to a suspension flow over its Poincaré section.

Problems

1.1. Proof of Liouville's theorem in section 1.1
(a) Write $\underline{x} := (\underline{q}, \underline{p})$ and $\underline{y} := T_t(\underline{q}, \underline{p})$. Use Hamilton's equations to show that the Jacobian matrix of $\underline{y} = \underline{y}(\underline{x})$ satisfies $\frac{\partial \underline{y}}{\partial \underline{x}} = I + tA + O(t^2)$ as $t \to 0$, where $\mathrm{tr}(A) = 0$.
(b) Show that for every matrix A, $\det(I + tA + O(t^2)) = 1 + t\,\mathrm{tr}(A) + O(t^2)$ as $t \to 0$.
(c) Prove that the Jacobian of T_t is equal to one for all t. Deduce Liouville's theorem.

1.2. The completion of a measure space. Suppose $(X, \mathscr{B}, \mu)$ is a measure space. A set $N \subset X$ is called a *null set*, if there is a measurable set $E \supseteq N$ such that $\mu(E) = 0$. A measure space is called *complete*, if every null set is measurable. Every measure space can be completed, and this exercise shows how to do this.
(a) Let $\mathscr{B}_0$ denote the the collection of all sets of the form $E \cup N$ where $E \in \mathscr{B}$ and N is a null set. Show that $\mathscr{B}_0$ is a σ–algebra.
(b) Show that μ has a unique extension to a σ–additive measure on $\mathscr{B}_0$.

1.3. Prove Poincaré's Recurrence Theorem for a general probability preserving transformation (theorem 1.1).

1.4. Fill in the details in the proof above that the Markov measure corresponding to a stationary probability vector and a stochastic matrix exists, and is shift invariant measure.

1.5. Suppose Σ_A^+ is a SFT with stochastic matrix P. Let $A = (t_{ab})_{S \times S}$ denote the matrix of zeroes and ones where $t_{ab} = 1$ if $p_{ab} > 0$ and $t_{ab} = 0$ otherwise. Write $A^n = (t_{ab}^{(n)})$. Prove that $t_{ab}^{(n)}$ is the number of paths of length n starting at a and ending at b. In particular: $a \xrightarrow{n} b \Leftrightarrow t_{ab}^{(n)} > 0$.

1.6. The Perron–Frobenius Theorem[5]: *Suppose $A = (a_{ij})$ is a matrix all of whose entries are non-negative, and let $B := (b_{ij})$ be the matrix $b_{ij} = 1$ if $a_{ij} > 0$ and $b_{ij} = 0$ if $a_{ij} = 0$. Assume that B is irreducible, then A has a positive eigenvalue λ with the following properties:*

(i) There are positive vectors $\underline{r}$ and $\underline{\ell}$ s.t. $\underline{\ell}A = \lambda\underline{\ell}$, $A\underline{r} = \lambda\underline{r}$.
(ii) The eigenvalue λ is simple.
(iii) The spectrum of $\lambda^{-1}A$ consists of 1, several (possibly zero) roots of unity, and a finite subset of the open unit disc. In this case the limit $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-k}A^k$ exists.
(iv) If B is irreducible and aperiodic, then the spectrum of $\lambda^{-1}A$ consists of 1 and a finite subset of the open unit disc. In this case the limit $\lim\limits_{n\to\infty} \lambda^{-n}A^n$ exists.

1. Prove the Perron–Frobenius theorem in case A is stochastic, first in the aperiodic case, then in the general case.
2. Now consider the case of a non-negative matrix:

[5] Perron first proved this in the aperiodic case. Frobenius later treated the periodic irreducible case.

a. Use a fixed point theorem to show that $\lambda, \underline{\ell}, \underline{r}$ exist;
b. Set $\underline{1} := (1, \dots, 1)$ and let V be the diagonal matrix such that $V\underline{1} = \underline{r}$. Prove that $\lambda^{-1}V^{-1}AV$ is stochastic.
c. Prove the Perron-Frobenius theorem.

1.7. Suppose $P = (p_{ab}^{(n)})_{S\times S}$ is an irreducible aperiodic stochastic matrix. Use the spectral description of P obtained in problem 1.6 to show that $p_{ab}^{(n)} \to p_b$ exponentially fast.

1.8. Show that the product of n irrational rotations $R_{\alpha_1}, \dots, R_{\alpha_n}$ is ergodic iff $(\alpha_1, \dots, \alpha_n)$ are independent over the irrationals.

1.9. Suppose $g^t : X \to X$ is a measure preserving flow. The *time one map* of the flow is the measure preserving map $g^1 : X \to X$. Give an example of an ergodic flow whose time one map is not ergodic.

1.10. The adding machine
Let $X = \{0,1\}^{\mathbb{N}}$ equipped with the σ–algebra $\mathscr{B}$ generated by the cylinders, and the Bernoulli $(\frac{1}{2}, \frac{1}{2})$–measure μ. The *adding machine* is the ppt $(X, \mathscr{B}, \mu, T)$ defined by the rule $T(1^n 0*) = (0^n 1*)$, $T(1^\infty) = 0^\infty$. Prove that the adding machine is invertible and probability preserving. Show that $T(\underline{x}) = \underline{x} \oplus (10^\infty)$ where $\oplus$ is "addition with carry to the right".

1.11. Prove proposition 1.11.

1.12. Show that a ppt $(X, \mathscr{B}, \mu, T)$ is mixing whenever $\mu(A \cap T^{-n}A) \xrightarrow[n\to\infty]{} \mu(A)^2$ for all $A \in \mathscr{B}$. *Guidence:*

1. $\int 1_A f \circ T^n d\mu \xrightarrow[n\to\infty]{} \mu(A) \int f d\mu$ for all $f \in \overline{\mathrm{span}}_{L^2}\{1, 1_A \circ T, 1_A \circ T^2, \dots\}$.
2. $\int 1_A f \circ T^n d\mu \xrightarrow[n\to\infty]{} \mu(A) \int f d\mu$ for all $f \in L^2$.
3. $\int g f \circ T^n d\mu \xrightarrow[n\to\infty]{} \int g d\mu \int f d\mu$ for all $f, g \in L^2$.

1.13. Show that a Kakutani skyscraper over an invertible transformation is invertible, and find a formula for its inverse.

1.14. Conservativity
Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving transformation on an *infinite* σ–finite, measure space.[6] A set $W \in \mathscr{B}$ is called *wandering*, if $\{T^{-n}W : n \geq 0\}$ are pairwise disjoint. A mpt is called *conservative*, if every wandering set has measure zero.

1. Show that any ppt is conservative. Give an example of a non-conservative mpt on a σ–finite infinite measure space.
2. Show that Poincaré's recurrence theorem extends to conservative mpt.
3. Suppose $(X, \mathscr{B}, \mu, T)$ is a conservative ergodic mpt, and let A be a set of finite positive measure. Show that the induced transformation $T_A : A \to A$ is well–defined a.e. on A, and is an ergodic ppt.
4. Prove Kac formula for conservative ergodic transformations under the previous set of assumptions.

[6] A measure space is called σ–finite, if its sample space is the countable union of finite measure sets.

Notes for chapter 1

The standard references for measure theory are [7] and [4], and the standard references for ergodic theory of probability preserving transformations are [6] and [8]. For ergodic theory on infinite measure spaces, see [1]. Our proof of the Perron-Frobenius theorem is taken from [3]. Kac's formula has very simple proof when T is invertible. The proof we use (taken from [5]) works for non-invertible transformations, and extends to the conservative infinite measure setting. The ergodicity of the geodesic flow was first proved by E. Hopf by other means. The short proof we gave is due to Gelfand & Fomin and is reproduced in [2].

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Chapter 2
Ergodic Theorems

2.1 The Mean Ergodic Theorem

Theorem 2.1 (von Neumann's Mean Ergodic Theorem). *Suppose* $(X,\mathscr{B},\mu,T)$ *is a ppt. If* $f\in L^2$*, then* $\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k \xrightarrow[n\to\infty]{L^2} \overline{f}$ *where* $\overline{f}\in L^2$ *is invariant. If T is ergodic, then* $\overline{f}=\int f d\mu$.

Proof. Observe that since T is measure preserving, then $\int f\circ T d\mu = \int f d\mu$ for every $f\in L^1$, and $\|f\circ T\|_2=\|f\|_2$ for all $f\in L^2$ (prove this, first for indicator functions, then for all L^2–functions).

Suppose $f=g-g\circ T$ where $g\in L^2$ (in this case we say that f is a *coboundary* with *transfer function* $g\in L^2$), then it is obvious that

$$\left\|\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k\right\|_2 = \frac{1}{N}\|g\circ T^n - g\|_2 \le 2\|g\|_2/N \xrightarrow[N\to\infty]{} 0.$$

Thus the theorem holds for all elements of $\mathscr{C}:=\{g-g\circ T : g\in L^2\}$.

We claim that the theorem holds for all elements of $\overline{\mathscr{C}}$ (L^2–closure). Suppose $f\in\overline{\mathscr{C}}$, then for every $\varepsilon>0$, there is an $F\in\mathscr{C}$ s.t. $\|f-F\|_2<\varepsilon$. Choose N_0 such that for every $N>N_0$, $\|\frac{1}{N}\sum_{k=0}^{N-1}F\circ T^k\|_2<\varepsilon$, then for all $N>N_0$

$$\begin{aligned}\left\|\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k\right\|_2 &\le \left\|\frac{1}{N}\sum_{k=0}^{N-1}(f-F)\circ T^k\right\|_2 + \left\|\frac{1}{N}\sum_{k=0}^{N-1}F\circ T^k\right\|_2\\ &\le \frac{1}{N}\sum_{k=0}^{N-1}\|(f-F)\circ T^k\|_2+\varepsilon<2\varepsilon.\end{aligned}$$

This shows that $\left\|\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k\right\|_2 \xrightarrow[N\to\infty]{} 0$.

Next we claim that $\overline{\mathscr{C}}^{\perp}=\{\text{invariant functions}\}$. Suppose $f\perp\overline{\mathscr{C}}$, then

$$\|f - f\circ T\|_2^2 = \langle f - f\circ T, f - f\circ T\rangle = \|f\|_2 - 2\langle f, f\circ T\rangle + \|f\circ T\|_2^2$$
$$= 2\|f\|_2^2 - 2\langle f, f - (f - f\circ T)\rangle = 2\|f\|_2^2 - 2\|f\|_2^2 = 0 \Longrightarrow f = f\circ T \text{ a.e.}$$

So $\overline{\mathscr{C}}^{\perp} \subseteq \{\text{invariant functions}\}$. Conversely, if f is invariant then for every $g \in L^2$

$$\langle f, g - g\circ T\rangle = \langle f, g\rangle - \langle f, g\circ T\rangle = \langle f, g\rangle - \langle f\circ T, g\circ T\rangle = \langle f, g\rangle - \langle f, g\rangle = 0,$$

so $f \perp \mathscr{C}$, whence $f \perp \overline{\mathscr{C}}$. In summary, $\overline{\mathscr{C}} = \{\text{invariant functions}\}$, and

$$L^2 = \overline{\mathscr{C}} \oplus \{\text{invariant functions}\}.$$

We saw above that the MET holds for all elements of $\overline{\mathscr{C}}$ with zero limit, and holds for all invariant functions f with limit f. Therefore the MET holds for all L^2–functions, and the limit $\overline{f}$ is the orthogonal projection of f on the space of invariant functions.

In particular $\overline{f}$ is invariant. If T is ergodic, then $\overline{f}$ is constant and $\overline{f} = \int \overline{f} d\mu$ almost everywhere. Also, since $\frac{1}{N}\sum_{n=1}^{N} f\circ T^n \to f$ in L^2, then

$$\int f d\mu = \frac{1}{N}\sum_{n=1}^{N}\langle 1, f\circ T^n\rangle = \langle 1, \frac{1}{N}\sum_{n=1}^{N} f\circ T^n\rangle \to \langle 1, \overline{f}\rangle = \int \overline{f} d\mu,$$

so $\int \overline{f} d\mu = \int f d\mu$ whence $\overline{f} = \int f d\mu$ almost everywhere. $\square$

Remark 1. The proof shows that the limit $\overline{f}$ is the projection of f on the space of invariant functions.

Remark 2. The proof only uses the fact that $Uf = f\circ T$ is an isometry of L^2. In fact it works for all linear operators $U : H \to H$ on separable Hilbert spaces s.t. $\|U\| \leq 1$, see problem 2.1.

Remark 3. If $f_n \xrightarrow[n\to\infty]{L^2} f$, then $\langle f_n, g\rangle \xrightarrow[n\to\infty]{} \langle f, g\rangle$ for all $g \in L^2$. Specializing to the case $f_n = \frac{1}{n}\sum_{k=0}^{n-1} 1_B \circ T^k, g = 1_A$ we obtain the following corollary of the MET:

Corollary 2.1. *A ppt $(X, \mathscr{B}, \mu, T)$ is ergodic iff for all $A, B \in \mathscr{B}$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k}B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

So ergodicity is mixing "on the average." We will return to this point when we discuss the definition of weak mixing in the next chapter.

2.2 The Pointwise Ergodic Theorem

Theorem 2.2 (Birkhoff's Pointwise Ergodic Theorem). *Let $(X,\mathscr{B},\mu,T)$ be a ppt. If $f\in L^1$, then the limit $\overline{f}(x):=\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{k=0}^{N-1}f(T^kx)$ exists for a.e. x, and $\frac{1}{N}\sum\limits_{k=0}^{N-1}f\circ T^k\to\overline{f}$ in L^1. The function $\overline{f}$ is T–invariant, absolutely integrable, and $\int\overline{f}d\mu=\int fd\mu$. If T is ergodic, then $\overline{f}=\int fd\mu$ almost everywhere.*

Proof. Since every $f\in L^1$ is the difference of two non-negative L^1-functions, there is no loss of generality in assuming that $f\geq 0$. Define

$$A_n(x):=\frac{1}{n}\sum_{k=0}^{n-1}f(T^kx)\ ,\ \overline{A}(x):=\limsup_{n\to\infty}A_n(x)\ ,\ \underline{A}(x):=\liminf_{n\to\infty}A_n(x).$$

$\overline{A}(x),\underline{A}(x)$ take values in $[0,\infty]$, are measurable, and are T-invariant, as can be easily checked by taking the limits on both sides of $A_n(x)=\frac{n-1}{n}A_{n-1}(Tx)+O(\frac{1}{n})$.

STEP 1. $\int fd\mu\geq\int\overline{A}d\mu$.

Proof. Fix $\varepsilon,L,M>0$, and set $\overline{A}_L(x):=\overline{A}(x)\wedge L=\min\{\overline{A}(x),L\}$ (an invariant function). The following function is well-defined and finite everywhere:

$$\tau_L(x):=\min\{n>0:A_n(x)>\overline{A}_L(x)-\varepsilon\}.$$

For a given N, we "color" the time interval $0,1,2,\dots,N-1$ as follows:

- If $\tau_L(T^0x)>M$, color 0 red; If $\tau_L(T^0x)\leq M$ color the next $\tau_L(x)$ times blue, and move to the first uncolored k
- If $\tau_L(T^kx)>M$, color k red; Otherwise color the next $\tau_L(T^kx)$ times blue, and move to the first uncolored k

Continue in this way until all times colored, or until $\tau_L(T^kx)>N-k$.

This partitions $\{0,1,\dots,N-1\}$ into red segments, and (possibly consecutive) blue segments of length $\leq M$, plus (perhaps) one last segment of length $\leq M$. Note:

- If k is red, then $T^kx\in[\tau_L>M]$, so $\boxed{\sum\limits_{\text{red }k\text{'s}}1_{[\tau_L>M]}(T^kx)\geq\text{number of red }k\text{'s}}$
- The average of f on each blue segment of length $\tau_L(T^kx)$ is larger than $\overline{A}_L(T^kx)-\varepsilon=\overline{A}_L(x)-\varepsilon$. So for each blue segment

$$\sum_{k\in\text{blue segment}}f(T^kx)\geq\text{length of segment}\times(\overline{A}_L(x)-\varepsilon).$$

Summing over all segments: $\boxed{\sum\limits_{\text{blue }k\text{'s}}f(T^kx)\geq(\overline{A}_L(x)-\varepsilon)\times\text{number of blue }k\text{'s.}}$

We can combine these two estimates as follows.

$$\sum_{k=0}^{N-1}(f+\overline{A}_L 1_{[\tau_L>M]})(T^k x)\geq \#(\text{blues})\cdot(\overline{A}_L(x)-\varepsilon)+\#(\text{reds})\cdot\overline{A}_L(x)$$
$$\geq \#(\text{blues and reds})\cdot(\overline{A}_L(x)-\varepsilon)\geq (N-M)(\overline{A}_L(x)-\varepsilon).$$

Next we divide by N, integrate, and obtain from the T-invariance of μ that

$$\int f d\mu+\int_{[\tau_L>M]}\overline{A}_L d\mu\geq (1-\tfrac{M}{N})\left(\int\overline{A}_L d\mu-\varepsilon\right)\overset{!}{\geq}\int\overline{A}_L d\mu-\frac{ML}{N}-\varepsilon,$$

where $\overset{!}{\geq}$ is because $0\leq\overline{A}_L\leq L$. Subtracting from both sides of the inequality the (finite) quantity $\int_{[\tau_L>M]}\overline{A}_L d\mu$, we find that

$$\int f d\mu\geq\int_{[\tau_L\leq M]}\overline{A}_L d\mu-\frac{ML}{N}-\varepsilon.$$

We now take the following limits in the following order: $N\to\infty$; $M\to\infty$; $L\to\infty$; and $\varepsilon\to 0$, using the monotone convergence theorem where appropriate. The result is $\int f d\mu\geq\int\overline{A}d\mu$.

STEP 2. $\int f d\mu\leq\int\underline{A}d\mu$

Proof. First observe, using Fatou's Lemma, that $\int\underline{A}d\mu\leq\|f\|_1<\infty$. So $\underline{A}(x)<\infty$ almost everywhere. For every $\varepsilon>0$,

$$\theta(x):=\min\{n>0:A_n(x)<\underline{A}(x)+\varepsilon\}$$

is well-defined and finite for a.e. x. We now repeat the coloring argument of step 1, with θ replacing τ and $\underline{A}$ replacing $\overline{A}_M$. Let $f_M(x):=f(x)\wedge M$, then as before

$$\sum_{k=0}^{N-1}f_M(T^k x)1_{[\theta\leq M]}(T^k x)\leq\sum_{k\text{ blue}}f(T^k x)+\sum_{k\text{ red}}0+\sum_{\text{no color}}M$$
$$\leq\#(\text{blues})(\underline{A}(x)+\varepsilon)+M^2\leq N(\underline{A}(x)+\varepsilon)+M^2.$$

Integrating and dividing by N, we obtain $\int 1_{[\theta\leq M]}f\wedge M\leq\int\underline{A}+\varepsilon+O(1/N)$. Passing to the limits $N\to\infty$ and then $M\to\infty$ (by the monotone convergence theorem), we get $\int f\leq\int\underline{A}+\varepsilon$. Since ε was arbitrary, $\int f\leq\int\underline{A}$.

STEP 3. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(T^k x)$ *exists almost everywhere*

Together, steps 1 and 2 imply that $\int(\overline{A}-\underline{A})d\mu\leq 0$. But $\overline{A}\geq\underline{A}$, so necessarily $\overline{A}=\underline{A}$ μ–almost everywhere (if $\mu[\overline{A}>\underline{A}]>0$, the integral would be strictly positive). So

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(T^k x)=\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(T^k x)\text{ a.e.},$$

which proves that the limit exists almost everywhere.

STEP 4. *$\overline{f}(x) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ is T-invariant, absolutely integrable, and $\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \to \overline{f}$ in L^1. Consequently, $\int \overline{f}d\mu = \int f d\mu$ and if T is ergodic, then $\overline{f} = \int f d\mu$ almost everywhere.*

Proof. The T-invariance of $\overline{f}$ follows by taking the limit as $n\to\infty$ in the identity $A_n(x) = \frac{n-1}{n}A_{n-1}(Tx) + \frac{1}{n}f(x)$. Absolute integrability is because of step 1.

To see that $\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \xrightarrow[n\to\infty]{L^1} \overline{f}$, fix $\varepsilon > 0$ and construct $\varphi \in L^\infty$ such that $\|f - \varphi\|_1 < \varepsilon$. Notice that $\|\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ T^k\|_\infty \le \|\varphi\|_\infty$, therefore by step 3 and the bounded convergence theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ T^k \xrightarrow[n\to\infty]{L^1} \overline{\varphi} \tag{2.1}$$

for some bounded invariant function $\overline{\varphi}$. So

$$\begin{aligned}
&\left\|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k - \overline{f}\right\|_1 \le \left\|\frac{1}{n}\sum_{k=0}^{n-1}(f-\varphi)\circ T^k\right\|_1 + \left\|\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ T^k - \overline{\varphi}\right\|_1 + \|\overline{\varphi} - \overline{f}\|_1 \\
&\le \|f-\varphi\|_1 + o(1) + \int \lim_{n\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}|(\varphi - f)\circ T^k| d\mu, \text{ by (2.1)} \\
&\le \|f-\varphi\|_1 + o(1) + \lim_{N\to\infty}\int \frac{1}{N}\sum_{k=0}^{n-1}|(\varphi - f)\circ T^k| d\mu, \text{ by Fatou's Lemma} \\
&\le 2\varepsilon + o(1).
\end{aligned}$$

Since ε was arbitrary, $\left\|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k - \overline{f}\right\|_1 \to 0$.

It is easy to see that if $g_n \xrightarrow[n\to\infty]{L^1} g$, then $\int g_n \to \int g$. So $\int \overline{f}d\mu = \lim A_n(f)d\mu = \int f d\mu$. Necessarily, in the ergodic case, $\overline{f} = const = \int \overline{f}d\mu = \int f d\mu$. □

2.3 The non-ergodic case

The almost sure limit in the pointwise ergodic theorem is clear when the map is ergodic: $\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k \xrightarrow[N\to\infty]{} \int f d\mu$. In this section we ask what is the limit in the non-ergodic case.

If f belongs to L^2, the limit is the projection of f on the space of invariant functions, because of the Mean Ergodic Theorem and the fact that every sequence of functions which converges in L^2 has a subsequence which converges almost everywhere to the same limit.[1] But if $f \in L^1$ we cannot speak of projections. The right notion in this case is that of the *conditional expectation*.

[1] Proof: Suppose $f_n \xrightarrow[n\to\infty]{L^2} f$. Pick a subsequence n_k s.t. $\|f_{n_k} - f\|_2 < 2^{-k}$. Then $\sum_{k\ge1}\|f_{n_k} - f\|_2 < \infty$. This means that $\|\sum|f_{n_k} - f|\|_2 < \infty$, whence $\sum(f_{n_k} - f)$ converges absolutely almost surely. It follows that $f_{n_k} - f \to 0$ a.e.

2.3.1 Conditional expectations and the limit in the ergodic theorem

Let $(X,\mathscr{B},\mu)$ be a probability space. Let $\mathscr{F}\subset\mathscr{B}$ be a σ–algebra. We think of $F\in\mathscr{F}$ as of the collection of all sets F for which we have sufficient information to answer the question " is $x\in F$?". The functions we have sufficient information to calculate are exactly the $\mathscr{F}$–measurable functions, as can be seen from the formula

$$f(x):=\inf\{t : x\in[f<t]\}.$$

Suppose g is *not* $\mathscr{F}$–measurable. What is the 'best guess' for $g(x)$ given the information $\mathscr{F}$?

Had g been in L^2, then the "closest" $\mathscr{F}$–measurable function (in the L^2–sense) is the projection of g on $L^2(X,\mathscr{F},\mu)$. The defining property of the projection Pg of g is $\langle Pg,h\rangle=\langle g,h\rangle$ for all $h\in L^2(X,\mathscr{F},\mu)$. The following definition mimics this case when g is not necessarily in L^2:

Definition 2.1. The *conditional expectation* of $f\in L^1(X,\mathscr{B},\mu)$ given $\mathscr{F}$ is the unique $L^1(X,\mathscr{F},\mu)$–element $\mathbb{E}(f|\mathscr{F})$ which is

1. $\mathbb{E}(f|\mathscr{F})$ is $\mathscr{F}$–measurable;
2. $\forall\varphi\in L^\infty$ $\mathscr{F}$–measurable, $\int\varphi\mathbb{E}(f|\mathscr{F})d\mu=\int\varphi f d\mu$.

Note: $\mathbb{E}(f|\mathscr{F})$ is only determined almost everywhere.

Proposition 2.1. *The conditional expectation exists for every L^1 element, and is unique up sets of measure zero.*

Proof. Consider the measures $\nu_f := f d\mu|_{\mathscr{F}}$ and $\mu|_{\mathscr{F}}$ on $(X,\mathscr{F})$. Then $\nu_f\ll\mu$. The function $\mathbb{E}(f|\mathscr{F}):=\frac{d\nu_f}{d\mu}$ (Radon-Nikodym derivative) is $\mathscr{F}$–measurable, and it is easy to check that it satisfies the conditions of the definition of the conditional expectation. The uniqueness of the conditional expectation is left as an exercise. $\square$

Proposition 2.2. *Suppose $f\in L^1$.*

1. *$f\mapsto\mathbb{E}(f|\mathscr{F})$ is linear, and a contraction in the L^1–metric;*
2. *$f\geq 0\Rightarrow\mathbb{E}(f|\mathscr{F})\geq 0$ a.e.;*
3. *if φ is convex and $\varphi\circ f\in L^1$, then $\mathbb{E}(\varphi\circ f|\mathscr{F})\geq\varphi(\mathbb{E}(f|\mathscr{F}))$;*
4. *if h is $\mathscr{F}$–measurable and bounded, then $\mathbb{E}(hf|\mathscr{F})=h\mathbb{E}(f|\mathscr{F})$;*
5. *If $\mathscr{F}_1\supset\mathscr{F}_2$, then $\mathbb{E}[\mathbb{E}(f|\mathscr{F}_)|\mathscr{F}_2]=\mathbb{E}(f|\mathscr{F}_2)$.*

We leave the proof as an exercise.

Theorem 2.3. *Let $(X,\mathscr{B},\mu,T)$ be a p.p.t, and $f\in L^1(X,\mathscr{B},\mu)$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}f\circ T^k=\mathbb{E}(f|\mathfrak{Inv}(T))\text{ a.e. and in }L^1,$$

where $\mathfrak{Inv}(T):=\{E\in\mathscr{B}:E=T^{-1}E\}$. Alternatively, $\mathfrak{Inv}(T)$ is the σ–algebra generated by all T–invariant functions.

Proof. Set $\overline{f} := \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k$ on the set where the limit exists, and zero otherwise. Then $\overline{f}$ is $\mathfrak{Inv}$–measurable and T-invariant. For every T-invariant $\varphi \in L^\infty$,

$$\int \varphi \overline{f} d\mu = \int \varphi \frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k d\mu + O\left(\|\varphi\|_\infty \left\| \frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k - \overline{f}\right\|_1\right) =$$
$$= \frac{1}{N}\sum_{k=0}^{N-1}\int \varphi\circ T^k f\circ T^k d\mu + o(1) \xrightarrow[N\to\infty]{} \int \varphi f d\mu,$$

because the convergence in the ergodic theorem is also in L^1. □

2.3.2 Conditional probabilities

Recall that a *standard probability space* is a probability space $(X,\mathscr{B},\mu)$ where X is a complete, metric, separable space, and $\mathscr{B}$ is its Borel σ–algebra.

Theorem 2.4 (Existence of Conditional Probabilities). *Let μ by a Borel probability measure on a standard probability space $(X,\mathscr{B},\mu)$, and let $\mathscr{F}\subset\mathscr{B}$ be a σ–algebra. There exist Borel probability measures $\{\mu_x\}_{x\in X}$ s.t.:*

1. $x\mapsto \mu_x(E)$ is $\mathscr{F}$–measurable for every $E\in\mathscr{B}$;
2. if f is μ–integrable, then $x\mapsto \int f d\mu_x$ is integrable, and $\int f d\mu = \int_X\left(\int_X f d\mu_x\right)d\mu$;
3. if f is μ–intergable, then $\int f d\mu_x = \mathbb{E}(f|\mathscr{F})(x)$ for μ–a.e. x.

Definition 2.2. The measures μ_x are called the *conditional probabilities* of $\mathscr{F}$. Note that they are only determined almost everywhere.

Proof. By the isomorphism theorem for standard spaces, there is no loss of generality in assuming that X is compact. Indeed, we may take X to be a compact interval. Recall that for compact metric spaces X, the space of continuous functions $C(X)$ with the maximum norm is separable.[2]

Fix a countable dense set $\{f_n\}_{n=0}^\infty$ in $C(X)$ s.t. $f_0\equiv 1$. Let $\mathscr{A}_\mathbb{Q}$ be the algebra generated by these functions over $\mathbb{Q}$. It is still countable.

Choose for every $g\in\mathscr{A}_\mathbb{Q}$ an $\mathscr{F}$–measurable version $\overline{\mathbb{E}}(g|\mathscr{F})$ of $\mathbb{E}(g|\mathscr{F})$ (recall that $\mathbb{E}(g|\mathscr{F})$ is an L^1–function, namely not a function at all but an equivalence class of functions). Consider the following collection of conditions:

1. $\forall\alpha,\beta\in\mathbb{Q}, g_{1,2}\in\mathscr{A}_\mathbb{Q}$, $\overline{\mathbb{E}}(\alpha g_1+\beta g_2|\mathscr{F})(x) = \alpha\overline{\mathbb{E}}(g_1|\mathscr{F})(x)+\beta\overline{\mathbb{E}}(g_2|\mathscr{F})(x)$
2. $\forall g\in\mathscr{A}_\mathbb{Q}$, $\min g\le \overline{\mathbb{E}}(g|\mathscr{F})(x)\le \max g$

[2] Proof: Compact metric space are separable because they have finite covers by balls of radius $1/n$. Let $\{x_n\}$ be a countable dense set of points, then $\varphi_n(\cdot) := \mathrm{dist}(x_n,\cdot)$ is a countable family of continuous functions, which separates points in X. The algebra which is generated over $\mathbb{Q}$ by $\{\varphi_n\}\cup\{1_X\}$ is countable. By the Stone-Weierstrass Theorem, it is dense in $C(X)$.

This is countable collection of $\mathscr{F}$–measurable conditions, each of which holds with full μ–probability. Let X_0 be the set of x's which satisfies all of them. This is an $\mathscr{F}$–measurable set of full measure.

We see that for each $x \in X_0$, $\varphi_x[g] := \overline{\mathbb{E}}(g|\mathscr{F})(x)$ is linear functional on $\mathscr{A}_{\mathbb{Q}}$, and $\|\varphi_x\| \leq 1$. It follows that φ_x extends uniquely to a positive bounded linear functional on $C(X)$. This is a measure μ_x.

Step 1. $\int(\int f d\mu_x)d\mu(x) = \int f d\mu$ for all $f \in C(X)$.

Proof. This is true for all $f \in \mathscr{A}_{\mathbb{Q}}$ by definition, and extends to all $C(X)$ because $\mathscr{A}_{\mathbb{Q}}$ is dense in $C(X)$. (But for $f \in L^1$ it is not even clear that the statement makes sense, because μ_x could live on a set with zero μ–measure!)

Step 2. $x \mapsto \mu_x(E)$ is $\mathscr{F}$–measurable for all $E \in \mathscr{B}$.

Exercise: Prove this using the following steps

1. The indicator function of any open set is the pointwise limit of a sequence of continuous functions $0 \leq h_n \leq 1$, thus the step holds for open sets.
2. The collection of sets whose indicators are pointwise limits of a bounded sequence of continuous functions forms an algebra. The step holds for every set in this algebra.
3. The collection of sets for which step 1 holds is a monotone class which contains a generating algebra.

Step 3. If $f = g$ μ–a.e., then $f = g$ μ_x–a.e. for μ–a.e. x.

Proof. Suppose $\mu(E) = 0$. Choose open sets $U_n \supseteq E$ such that $\mu(U_n) \to 0$. Choose continuous functions $0 \leq h_n^\varepsilon \leq 1$ s.t. h_n^ε vanish outside U_n, h_n^ε are non-zero inside U_n, and $h_n^\varepsilon \xrightarrow[\varepsilon \to 0^+]{} 1_{U_n}$ (e.g. $h_n^\varepsilon(\cdot) := [\mathrm{dist}(x, U_n^c)/\mathrm{diam}(X)]^\varepsilon$).

By construction $1_E \leq 1_{U_n} \equiv \lim\limits_{\varepsilon \to 0^+} h_n^\varepsilon$, whence

$$\begin{aligned}
\int \mu_x(E) d\mu(x) &\leq \int \int \lim_{\varepsilon \to 0^+} h_n^\varepsilon d\mu_x d\mu \leq \\
&\leq \lim_{\varepsilon \to 0^+} \int \int h_n^\varepsilon d\mu_x d\mu = \lim_{\varepsilon \to 0^+} \int h_n^\varepsilon d\mu \leq \lim_{\varepsilon \to 0^+} \int h_n^\varepsilon d\mu \leq \mu(U_n) \xrightarrow[n \to \infty]{} 0.
\end{aligned}$$

It follows that $\mu_x(E) = 0$ a.e.

Step 4. For all f μ–absolutely integrable, $\mathbb{E}(f|\mathscr{F})(x) = \int f d\mu_x$ μ–a.e.

Proof. Find $g_n \in C(X)$ such that

$$f = \sum_{n=1}^{\infty} g_n \ \mu\text{-a.e., and } \sum \|g_n\|_{L^1(\mu)} < \infty.$$

Then

$$\begin{aligned}
\mathbb{E}(f|\mathscr{F}) &= \sum_{n=1}^{\infty} \mathbb{E}(g_n|\mathscr{F}), \text{ because } \mathbb{E}(\cdot|\mathscr{F}) \text{ is a bounded operator on } L^1 \\
&= \sum_{n=1}^{\infty} \int_X g_n d\mu_x \text{ a.e., because } g_n \in C(X) \\
&= \int_X \sum_{n=1}^{\infty} g_n d\mu_x \text{ a.e., (justification below)} \\
&= \int_X f d\mu_x \text{ a.e.}
\end{aligned}$$

Here is the justification: $\int \sum |g_n| d\mu_x < \infty$, because the integral of this expression, by the monotone convergence theorem is less than $\sum \|g_n\|_1 < \infty$. ☐

2.3.3 The ergodic decomposition

Theorem 2.5 (The Ergodic Decomposition). *Let μ be an invariant Borel probability measure of a Borel map T on a standard probability space X. Let $\{\mu_x\}_{x\in X}$ be the conditional probabilities w.r.t. $\mathfrak{Inv}(T)$. Then*

1. *$\mu = \int_X \mu_x d\mu(x)$ (i.e. this holds when applies to L^1–functions or Borel sets);*
2. *μ_x is invariant for μ–a.e. $x \in X$;*
3. *μ_x is ergodic for μ–a.e. $x \in X$.*

Proof. By the isomorphism theorem for standard probability spaces, there is no loss of generality in assuming that X is a compact metric space, and that $\mathscr{B}$ is its σ–algebra of Borel sets.

For every $f \in L^1$, $\int f d\mu = \int \mathbb{E}(f|\mathfrak{Inv}(T))d\mu(x) = \int_X \int_X f d\mu_x d\mu(x)$. This shows (1). We have to show that μ_x is invariant and ergodic for μ–a.e. x.

Fix a countable set $\{f_n\}$ which is dense in $C(X)$, and choose Borel versions $\overline{\mathbb{E}}_\mu(f_n|\mathfrak{Inv}(T))(x)$. By the ergodic theorem, there is a set of full measure Ω such that for all $x \in \Omega$,

$$\int f_n d\mu_x = \overline{\mathbb{E}}_\mu(f_n|\mathfrak{Inv}(T))(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f_n(T^k x) \text{ for all } n.$$

Step 1. μ_x is T–invariant for a.e. $x \in \Omega$.

Proof. For every n,

$$\begin{aligned}
\int f_n \circ T d\mu_x &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f_n(T^{k+1}x) \text{ a.e. (by the PET)} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f_n(T^k x) = \overline{\mathbb{E}_\mu}(f_n|\mathfrak{Inv}(T))(x) = \int f_n d\mu_x.
\end{aligned}$$

Let Ω' be the set of full measure for which the above holds for all n, and fix $x \in \Omega'$. Since $\{f_n\}$ is $\|\cdot\|_\infty$–dense in $C(X)$, we have $\int f \circ T d\mu_x = \int f d\mu_x$ for all $f \in C(X)$. Using the density of $C(X)$ in $L^1(\frac{\mu_x + \mu_x \circ T^{-1}}{2})$, it is routine to see that $\int f \circ T d\mu_x = \int f d\mu_x$ for all μ_x–integrable functions. This means that $\mu_x \circ T^{-1} = \mu_x$ for all $x \in \Omega'$.

Step 2. μ_x is ergodic for all $x \in \Omega$.

Proof. With $\{f_n\}_{n=1}^\infty$ as above, let $\Omega'' := \{x \in \Omega' : \forall k,\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f_k(T^k x) = \int f_k d\mu_x\}$. This is a set of full measure because of the ergodic theorem. Now

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \left\| \frac{1}{N}\sum_{k=0}^{N-1} f_n \circ T^k - \int_X f_n d\mu_x \right\|_{L^1(\mu)} \quad (\because L^1\text{-convergence in the PET}) \\
&= \lim_{N\to\infty} \int_X \left\| \frac{1}{N}\sum_{k=0}^{N-1} f_n \circ T^k - \int_X f_n d\mu_x \right\|_{L^1(\mu_x)} d\mu(x) \ \ (\because \mu = \int_X \mu_x d\mu) \\
&\geq \int_X \liminf_{N\to\infty} \left\| \frac{1}{N}\sum_{k=0}^{N-1} f_n \circ T^k - \int_X f_n d\mu_x \right\|_{L^1(\mu_x)} d\mu(x) \ (\because \text{Fatou's Lemma})
\end{aligned}
$$

It follows that $\liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f_n \circ T^k - \int_X f_n d\mu_x \right\|_{L^1(\mu_x)} = 0$ μ–a.e. Let

$$
\Omega := \left\{ x \in \Omega'' : \liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f_n \circ T^k - \int_X f_n d\mu_x \right\|_{L^1(\mu_x)} = 0 \text{ for all } n, \right\},
$$

then Ω has full measure.

Fix $x \in \Omega$. Since $\{f_n\}_{n\geq 1}$ is dense in $C(X)$, and $C(X)$ is dense in $L^1(\mu_x)$, $\{f_n\}$ is dense in $L^1(\mu_x)$. A standard approximation argument shows that

$$
\liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k - \int_X f d\mu_x \right\|_{L^1(\mu_x)} = 0 \text{ for all } f \in L^1(\mu_x).
$$

In particular, every $f \in L^1(\mu_x)$ such that $f \circ T = f$ μ_x-a.e. must be constant μ_x-almost everywhere. So μ_x is ergodic for $x \in \Omega'''$. □

2.4 An Ergodic Theorem for $\mathbb{Z}^d$-actions

Let $T_1, \ldots, T_d$ denote d measure preserving transformations on a probability space $(\Omega, \mathscr{F}, \mu)$. We say that $T_1, \ldots, T_d$ *commute* if $T_i \circ T_j = T_j \circ T_i$ for all i, j. Let $\mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$ and define for $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$

$$
T^{\underline{n}} := T_1^{n_1} \circ \cdots \circ T_d^{n_d}.
$$

If $T_1, \dots, T_d$ commute, then $T^{\underline{n}} \circ T^{\underline{m}} = T^{\underline{n}+\underline{m}}$. An algebraist would say that the semigroup $(\mathbb{Z}_+^d, +)$ *acts* on $(\Omega, \mathscr{F}, \mu)$ by $\underline{n} \cdot x = T^{\underline{n}}(x)$. This is called the $\mathbb{Z}_+^d$*–semi-action* generated by $T_1, \dots, T_d$. If T_i are invertible, this extends naturally to a $\mathbb{Z}^d$–action.

A pointwise ergodic theorem for d–commuting maps is an almost sure convergence statement for averages of the type

$$\frac{1}{|I_r|} S_{I_r} f := \frac{1}{|I_r|} \sum_{\underline{n} \in I_r} f \circ T^{\underline{n}}$$

where I_r is a sequence of subsets of $\mathbb{Z}_+^d$ which "tends to $\mathbb{Z}_+^d$", and $|I_r|$ =cardinality of I_r. Such statements are not true for any choice of $\{I_r\}$ (even when $d = 1$). Here we prove the following pointwise ergodic theorem for *increasing boxes* : Boxes are sets of the form

$$[\underline{n}, \underline{m}) := \{\underline{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d : n_i \le k_i \lneqq m_i \ \ (1 \le i \le d)\} \quad (\underline{n}, \underline{m} \in \mathbb{Z}_+^d).$$

A sequence of boxes $\{I_r\}_{r \ge 1}$ is said to be *increasing* if $I_r \subset I_{r+1}$ for all r. An increasing sequence of boxes is said to *tend to* $\mathbb{Z}_+^d$ if $\mathbb{Z}_+^d = \bigcup_{r \ge 1} I_r$.

Theorem 2.6 (Tempelman). *Let $T_1, \dots, T_d$ be d–commuting probability preserving maps on a probability space $(\Omega, \mathscr{F}, \mu)$, and suppose $\{I_r\}_{r \ge 1}$ is an increasing sequence of boxes which tends to $\mathbb{Z}_+^d$. If $f \in L^1$, then*

$$\frac{1}{|I_r|} \sum_{\underline{n} \in I_r} f \circ T^{\underline{n}} \xrightarrow[r \to \infty]{} \mathbb{E}(f | \mathfrak{Inv}(T_1) \cap \cdots \cap \mathfrak{Inv}(T_d)) \text{ almost surely.}$$

For convergence along more general sequences of boxes, see problem 2.8.

Proof. Fix $f \in L^1$. Almost sure convergence is obvious in the following two cases:

1. *Invariant functions:* If $f \circ T_i = f$ for $i = 1, \dots, d$, then $\frac{1}{|I_r|} S_{I_r} f = f$ for all r, so the limit exists and is equal to f.
2. *Coboundries:* Suppose $f = g - g \circ T_i$ for some $g \in L^\infty$ and some i,

$$\begin{aligned}\left| \frac{1}{|I_r|} S_{I_r} f \right| &= \frac{1}{|I_r|} \left| S_{I_r} g_i - S_{I_r + \underline{e}_i} g_i \right|, \text{ where } \underline{e}_1, \dots \underline{e}_d \text{ is the standard basis of } \mathbb{R}^d \\ &= \frac{1}{|I_r|} \left| S_{I_r \setminus (I_r + \underline{e}_i)} g_i - S_{(I_r + \underline{e}_i) \setminus I_r} g_i \right| \le \frac{|I_r \triangle (I_r + \underline{e}_i)|}{|I_r|} \|g_i\|_\infty.\end{aligned}$$

Now $|I_r \triangle (I_r + \underline{e}_i)| / |I_r| \xrightarrow[r \to \infty]{} 0$, because the lengths $\ell_1(r), \dots, \ell_d(r)$ of the sides of the box I_r tend to infinity, and so

$$\frac{|I_r \triangle (I_r + \underline{e}_i)|}{|I_r|} = \frac{2}{\ell_i(r)} \xrightarrow[r \to \infty]{} 0.$$

So the limit exists and is equal to zero.

Step 1. Any $f \in L^1$ can put in the form $f = \sum_{i=1}^{d}(g_i - g_i \circ T_i) + h + \varphi$, where $g_i \in L^\infty$, h is T_i–invariant for all i, and $\|\varphi\|_1 < \varepsilon$ with ε arbitrarily small.

Proof. One checks, as in the proof of the Mean Value Theorem, that

$$\overline{\mathrm{span}}\{g - g \circ T_i : g \in L^2,\ (1 \le i \le d)\}^\perp = \{f \in L^2 : f \circ T_i = f\ (1 \le i \le d)\},$$

whence $L^2 = \{f \in L^2 : f \circ T_i = f\ (1 \le i \le d)\} \oplus \overline{\mathrm{span}}\{g - g \circ T_i : g \in L^2,\ (1 \le i \le d)\}$ (orthogonal sum).

This means that any $f' \in L^2$ can be put in the form $f' = \sum_{i=1}^{d}(g_i' - g_i' \circ T_i) + h + \varphi'$, where $g_i' \in L^2$, $h \in L^2$ is T_i–invariant for all i, and $\|\varphi'\|_2 < \varepsilon/3$.

Since L^∞ is dense in L^2, it is no problem to replace g_i' by L^∞–functions g_i so that $f' = \sum_{i=1}^{d}(g_i - g_i \circ T_i) + h + \varphi$, where $\|\varphi\|_2 < \varepsilon/2$. By Cauchy–Schwarz, $\|\varphi\|_1 < \varepsilon/2$. This proves the step when f is in L^2. If f is in L^1, find $f' \in L^2$ s.t. $\|f - f'\|_1 < \varepsilon/2$ and apply the above to f'.

Step 2 (**Maximal Inequality**). For every non-negative $\varphi \in L^1$ and $t > 0$,

$$\mu\left[\sup_r \frac{1}{|I_r|} S_{I_r}\varphi > t\right] \le \frac{2^d\|\varphi\|_1}{t}. \tag{2.2}$$

We give the proof later.

Step 3. How to use the maximal inequality to complete the proof.

For every $f \in L^1$, let $\Delta(f)(\omega) := \limsup\limits_{r\to\infty} \frac{1}{|I_r|}(S_{I_r}f)(\omega) - \liminf\limits_{r\to\infty} \frac{1}{|I_r|}(S_{I_r}f)(\omega)$. To show that $\lim \frac{1}{|I_r|} S_{I_r} f$ exists almost surely, one needs to show that $\Delta(f) = 0$ a.e.

Notice that Δ is subadditive: $\Delta(f_1 + f_2) \le \Delta(f_1) + \Delta(f_2)$. If we express f as in the first step, then we get $\Delta(f) \le \Delta(\varphi)$. It follows that for every $\delta > 0$,

$$\begin{aligned}\mu[\Delta(f) > \delta] \le \mu[\Delta(\varphi) > \delta] &\le \mu\left[2\sup_r \frac{1}{|I_r|} S_{I_r}|\varphi| > \delta\right] \\ &\le \frac{2^d\|\varphi\|_1}{(\delta/2)} \text{ by the maximal inequality.}\end{aligned}$$

Taking $\delta := \sqrt{\varepsilon}$ and recalling that φ was constructed so that $\|\varphi\|_1 < \varepsilon$, we see that

$$\mu\left[\Delta(f) > \sqrt{\varepsilon}\right] < 2^{d+1}\sqrt{\varepsilon}.$$

But ε was arbitrary, so we must have $\Delta(f) = 0$ almost everywhere. In other words, $\lim \frac{1}{|I_r|} S_{I_r} f$ exists almost everywhere. The proof also shows that the value of the limit $\overline{f}$ equals h, so it is an invariant function.

To identify h, we argue as in the proof of Birkhoff's theorem. First we claim that $\frac{1}{|I_r|} S_{I_r} f \xrightarrow[r\to\infty]{L^1} h$. If f is bounded, this is a consequence of pointwise convergence and the bounded convergence theorem. For general L^1–functions, write $f = f' + \varphi$ with $f' \in L^\infty$ and $\|\varphi\|_1 < \varepsilon$. The averages of f' converge in L^1, and the averages of

φ remain small in L^1 norm. It follows that $\frac{1}{|I_r|}S_{I_r}f$ converge in L^1. The limit must agree with the pointwise limit $\overline{f}$ (see the footnote on page 39).

Integrate $\frac{1}{|I_r|}S_{I_1}f$ against a bounded invariant function g and pass to the limit. By L^1–covergence, $\int fg = \int hg$. It follows that $h = \mathbb{E}(f|\bigcap_{i=1}^d \mathfrak{Inv}(T_i))$. In particular, if the $\mathbb{Z}_+^d$–action generated by $T_1,\dots,T_d$ is ergodic, then $h = \int f$. This finishes the proof, assuming the maximal inequality.

Proof of the maximal inequality. Let $\varphi \in L^1$ be a non-negative function, fix N and $\alpha > 0$. We will estimate the measure of $E_N(\alpha) := \{\omega \in \Omega : \max_{1\le k\le N} \frac{1}{|I_k|}S_{I_k}\varphi > \alpha\}$.

Pick a large box I, and let $A(\omega) := \{\underline{n} \in I : T^{\underline{n}}(\omega) \in E_N(\alpha)\}$. For every $\underline{n} \in A(\omega)$ there is a $1 \le k(\underline{n}) \le N$ such that $(S_{I_{k(\underline{n})}+\underline{n}}\varphi)(\omega) > \alpha|I_{k(\underline{n})}|$.

Imagine that we were able to find a disjoint subcollection $\{\underline{n}+I_{k(\underline{n})} : \underline{n} \in A'(\omega)\}$ which is "large" in the sense that there is some global constant K s.t.

$$\sum_{\underline{n}\in A'(\omega)} |\underline{n}+I_{k(\underline{n})}| \ge \frac{1}{K}|A(\omega)|. \tag{2.3}$$

The sets $\underline{n}+I_{k(\underline{n})}$ $(\underline{n} \in A'(\omega))$ are included in the box $J \supset I$ obtained by increasing the sides of I by $2\max\{\mathrm{diam}(I_1),\dots,\mathrm{diam}(I_N)\}$. This, the non-negativity of φ, and the invariance of μ implies that

$$\begin{aligned}\|\varphi\|_1 &\ge \int \frac{1}{|J|}(S_J\varphi)(\omega)\,d\mu \ge \frac{1}{|J|}\int \sum_{\underline{n}\in A'(\omega)} (S_{I_{k(\underline{n})}+\underline{n}}\varphi)(\omega)\,d\mu \\ &\ge \frac{1}{|J|}\int \sum_{\underline{n}\in A'(\omega)} \alpha|I_{k(\underline{n})}+\underline{n}|\,d\mu \\ &\ge \frac{\alpha}{K|J|}\int |A(\omega)|d\mu = \frac{\alpha}{K|J|}\int \sum_{\underline{n}\in I} 1_{E_N(\alpha)}(T^{\underline{n}}\omega)d\mu = \frac{\alpha|I|}{K|J|}\mu[E_N(\alpha)].\end{aligned}$$

It follows that $\mu[E_N(\alpha)] \le K\frac{|J|}{|I|}\|\varphi\|_1/\alpha$. Now I was arbitrary, and by construction $|J| \sim |I|$ as $|I| \to \infty$, so $\mu[E_N(\alpha)] \le K\|\varphi\|_1/\alpha$. In the limit $N \to \infty$, we get the maximal inequality (except for the identification $K = 2^d$).

We now explain how to find the disjoint subcollection $\{\underline{n}+I_{k(\underline{n})} : \underline{n} \in A'(\omega)\}$. We use the "greedy" approach by first adding as many translates of I_N (the largest of $I_1,\dots,I_N$) as possible, then as many translates of I_{N-1} as possible, and so on:

(N) Let $\mathscr{M}_N$ be a maximal disjoint collection of sets of the form $I_N+\underline{n}$ with $k(\underline{n}) = N$.

$(N-1)$ Let $\mathscr{M}_{N-1}$ be a maximal disjoint collection of sets of the form $I_{N-1}+\underline{n}$ with $k(\underline{n}) = N-1$ and such that all elements of $\mathscr{M}_{N-1}$ are disjoint from $\bigcup \mathscr{M}_N$.

....

(1) Let $\mathscr{M}_1$ be a maximal disjoint collection of sets of the form $I_1+\underline{n}$ where $k(\underline{n}) = 1$ and such that all elements of $\mathscr{M}_1$ are disjoint from $\bigcup(\mathscr{M}_N \cup\cdots\cup M_2)$.

Now let $A'(\omega) := \{\underline{n} : I_{k(\underline{n})} + \underline{n} \in \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_N\}$. This is a disjoint collection.

We show that $A(\omega) \subseteq \bigcup_{\underline{n}\in A'(\omega)} (\underline{n}+I_{k(\underline{n})} - I_{k(\underline{n})})$ (where $I - I = \{\underline{n}-\underline{m} : \underline{n},\underline{m} \in I\}$). Suppose $\underline{n} \in A(\omega)$. By the maximality of $\mathscr{M}_{k(\underline{n})}$, either $\underline{n}+I_{k(\underline{n})} \in \mathscr{M}_{k(\underline{n})}$, or $\underline{n}+I_{k(\underline{n})}$ intersects some $\underline{m}+I_{k(\underline{m})} \in \mathscr{M}_{k(\underline{m})}$ s.t. $k(\underline{m}) \geq k(\underline{n})$.

- In the first case $\underline{n} \in \underline{n}+I_{k(\underline{n})} - I_{k(\underline{n})} \in \bigcup_{\underline{n}\in A'(\omega)} (\underline{n}+I_{k(\underline{n})} - I_{k(\underline{n})})$
- In the second case there are $\underline{u} \in I_{k(\underline{n})}, \underline{v} \in I_{k(\underline{m})}$ s.t. $\underline{n}+\underline{u} = \underline{m}+\underline{v}$, and again we get $\underline{n} \in \underline{m}+I_{k(\underline{m})} - I_{k(\underline{n})} \subseteq \underline{m}+I_{k(\underline{m})} - I_{k(\underline{m})}$ (since $k(\underline{m}) \geq k(\underline{n})$ and I_r is increasing).

For a d–dimensional box I, $|I-I| = 2^d|I|$. Since $A(\omega) \subseteq \bigcup\limits_{\underline{n}\in A'(\omega)} (\underline{n}+I_{k(\underline{n})} - I_{k(\underline{n})})$, $|A(\omega)| \leq \sum_{\underline{n}\in A'(\omega)} 2^d |I_{k(\underline{n})}|$, and we get (2.3) with $K = 2^d$. $\square$

2.5 The Subadditive Ergodic Theorem

We begin with two examples.

Example 1 (Random walks on groups) Let $(X, \mathscr{B}, \mu, T)$ be the Bernoulli scheme with probability vector $\underline{p} = (p_1, \ldots, p_d)$. Suppose G is a group, and $f : X \to G$ is the function $f(x_0, x_2, \ldots) = g_{x_0}$, where $g_1, \ldots, g_n \in G$. The expression

$$f_n(x) := f(x)f(Tx)\cdots f(T^{n-1}x)$$

describes the position of a random walk on G, which starts at the identity, and whose steps have the distribution $\Pr[\text{step} = g_i] = p_i$. What can be said on the behavior of this random walk?

In the special case $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$, $f_n(x) = f(x) + f(Tx) + \cdots + f(T^{n-1}x)$, and the ergodic theorem[3] says that $\frac{1}{n} f_n(x)$ has an almost sure limit, equal to $\int f d\mu = \sum p_i g_i$. So: the random walk has speed $\|\sum p_i g_i\|$, and direction $\sum p_i g_i / \|\sum p_i g_i\|$. (Note that if $G = \mathbb{Z}^d$, the direction need not lie in G.)

Example 2 (The derivative cocycle) Suppose $T : V \to V$ is a diffeomorphism acting on an open set $V \subset \mathbb{R}^d$. The derivative of T at $x \in V$ is a linear transformation $(dT)(x)$ on $\mathbb{R}^d$, $\underline{v} \mapsto [(dT)(x)]\underline{v}$. By the chain rule,

$$(dT^n)(x) = (dT)(T^{n-1}x) \circ (dT)(T^{n-2}x) \circ \cdots \circ (dT)(x).$$

If we write $f(x) := (dT)(x) \in \mathrm{GL}(d, \mathbb{R})$, then we see that

$$(dT^n)(x) = f(T^{n-1}x) f(T^{n-2}x) \cdots f(Tx) f(x)$$

is a “random walk” on $\mathrm{GL}(d, \mathbb{R}) := \{\text{invertible } d \times d \text{ matrices with real entries}\}$. Notice the order of multiplication!

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[3] applied to the each coordinate of the vector valued function $f = (f^1, \ldots, f^d)$.

What is the "speed" of this random walk? Does it have an asymptotic "direction"? The problem of describing the "direction" of random walk on a group is deep, and remain somewhat mysterious to this day, even in the case of groups of matrices. We postpone it for the moment, and focus on the conceptually easier task of defining the "speed." Suppose G is a group of $d \times d$ matrices with real-entries. Then G can be viewed to be group of linear operators on $\mathbb{R}^d$, and we can endow $A \in G$ with the *operator norm* $\|A\| := \max\{\|Av\|_2/\|v\|_2 : 0 \neq v \in \mathbb{R}^d\}$. Notice that $\|AB\| \leq \|A\|\|B\|$. We will measure the *speed* of $f_n(x) := f(x)f(Tx)\cdots f(T^{n-1}x)$ by analyzing

$$g^{(n)}(x) := \log \|f_n(x)\| \text{ as } n \to \infty.$$

Key observation: $g^{(n+m)} \leq g^{(n)} + g^{(m)} \circ T^n$, because

$$\begin{aligned} g^{(n+m)}(x) &= \log\|f_{n+m}(x)\| = \log\|f_n(x)f_m(T^n x)\| \leq \log(\|f_n(x)\| \cdot \|f_m(T^n x)\|) \\ &\leq \log\|f_n(x)\| + \log\|f_m(T^n x)\| = g^{(n)}(x) + g^{(m)}(T^n x). \end{aligned}$$

We say that $\{g^{(n)}\}_n$ is a *subadditive cocycle*.

Theorem 2.7 (Kingman's Subadditive Ergodic Theorem). *Let $(X, \mathscr{B}, m, T)$ be a probability preserving transformation, and suppose $g^{(n)} : X \to \mathbb{R}$ is a sequence of measurable functions such that $g^{(n+m)} \leq g^{(n)} + g^{(m)} \circ T^n$ for all n, m, and $g^{(1)} \in L^1$. Then the limit $g := \lim_{n\to\infty} g^{(n)}/n$ exists almost surely, and is an invariant function.*

Proof. We begin by observing that it is enough to treat the case when $g^{(n)}$ are all non-positive. This is because $h^{(n)} := g^{(n)} - (g^{(1)} + g^{(1)} \circ T + \cdots + g^{(1)} \circ T^{n-1})$ are non–positive, sub-additive, and differ from $g^{(n)}$ by the ergodic sums of $g^{(1)}$ whose asymptotic behavior we know by Birkhoff's ergodic theorem.

Assume then that $g^{(n)}$ are all non-negative. Define $G(x) := \liminf_{n\to\infty} g^{(n)}(x)/n$ (the limit may be equal to $-\infty$). We claim that $G \circ T = G$ almost surely.

Starting from the subadditivity inequality $g^{(n+1)} \leq g^{(n)} \circ T + g^{(1)}$, we see that $G \leq G \circ T$. Suppose there were a set of positive measure E where $G \circ T > G + \varepsilon$. Then for every $x \in E$, $G(T^n x) \geq G(T^{n-1}x) \geq \cdots \geq G(Tx) > G(x) + \varepsilon$. But this is impossible, because by Poincaré's Recurrence Theorem, for a.e. x there is some $n > 0$ such that $G(T^n x) = \infty$ or $|G(T^n x) - G(x)| < \varepsilon$ (prove!). This contradiction shows that $G = G \circ T$ almost surely. Henceforth we work the set of full measure

$$X_0 := \bigcap_{n \geq 1} [G \circ T^n = G].$$

Fix $M > 0$, and define $G_M := G \vee (-M)$. This is an invariant function on X_0. We aim at showing $\limsup_{n\to\infty} \frac{g^{(n)}}{n} \leq G_M$ a.s.. Since M is arbitrary, this implies that $\limsup_{n\to\infty} g^{(n)}/n \leq G = \liminf_{n\to\infty} g^{(n)}/n$, whence almost sure convergence.

Fix $x \in X_0$, $N \in \mathbb{N}$, and $\varepsilon > 0$. Call $k \in \mathbb{N} \cup \{0\}$

- "good", if $\exists \ell \in \{1, \ldots, N\}$ s.t. $g^{(\ell)}(T^k x)/\ell \leq G_M(T^k x) + \varepsilon = G_M(x) + \varepsilon$;

- "bad", if it's not good: $g^{(\ell)}(T^k x)/\ell > G_M(x) + \varepsilon$ for all $\ell = 1, \dots N$.

Color the integers $0, \dots, n-1$ inductively as follows, starting from $k = 1$. Let k be the smallest non-colored integer,

(a) If $k \leq n - N$ and k is "bad", color it red;
(b) If $k \leq n - N$ and k is "good", find the smallest $1 \leq \ell \leq N$ s.t. $g^{(\ell)}(T^k x)/\ell \leq G_M(T^k x) + \varepsilon$ and color the segment $[k, k+\ell)$ blue;
(c) If $k > n - N$, color k white.

Repeat this procedure until all integers $0, \dots, n-1$ are colored.

The "blue" part can be decomposed into segments $[\tau_i, \tau_i + \ell_i)$, with ℓ_i s.t. $g^{(\ell_i)}(T^{\tau_i} x)/\ell_i \leq G_M(x) + \varepsilon$. Let b denote the number of these segments.

The "red" part has size $\leq \sum_{k=1}^{n} 1_{B(N,M,\varepsilon)}(T^k x)$, where

$$B(N, M, \varepsilon) := \{x \in X_0 : g^{(\ell)}(x)/\ell > G_M(x) + \varepsilon \text{ for all } 1 \leq \ell \leq N\}.$$

Let r denote the size of the red part. The "white" part has size $w \leq N$.

By the sub-additivity condition

$$\begin{aligned}
\frac{g^{(n)}(x)}{n} &\leq \frac{1}{n}\sum_{i=1}^{b} g^{(\ell_i)}(T^{\tau_i} x) + \underbrace{\frac{1}{n}\sum_{k \text{ red}} g^{(1)}(T^k x) + \frac{1}{n}\sum_{k \text{ white}} g^{(1)}(T^k x)}_{\text{non-positive}} \\
&\leq \frac{1}{n}\sum_{i=1}^{b} g^{(\ell_i)}(T^{\tau_i} x) \leq \frac{1}{n}\sum_{i=1}^{b} (G_M(x) + \varepsilon)\ell_i = \frac{\#\{\text{blues}\}}{n}(G_M(x) + \varepsilon).
\end{aligned}$$

Now $\#\{\text{blues}\} \leq n - (r + w) = n - \sum_{k=1}^{n} 1_{B(N,M,\varepsilon)}(T^k x) + O(1)$, so by the Birkhoff ergodic theorem, for almost every x, $\#\{\text{blues}\}/n \xrightarrow[n\to\infty]{} 1 - \mathbb{E}(1_{B(N,M,\varepsilon)}|\mathfrak{Inv})$. Thus

$$\limsup_{n\to\infty} \frac{g^{(n)}(x)}{n} \leq (G_M(x) + \varepsilon)(1 - \mathbb{E}(1_{B(N,M,\varepsilon)}|\mathfrak{Inv})) \text{ almost surely.}$$

Now N was arbitrary, and for fixed M and ε, $B(N, M, \varepsilon) \downarrow \varnothing$ as $N \uparrow \infty$, because $G_M \geq G = \liminf_{\ell\to\infty} g^{(\ell)}/\ell$. It is not difficult to deduce from this that $\mathbb{E}(1_{B(N,M,\varepsilon)}|\mathfrak{Inv}) \xrightarrow[N\to\infty]{} 0$ almost surly.[4] Thus

$$\limsup_{n\to\infty} \frac{g^{(n)}(x)}{n} \leq G_M(x) + \varepsilon \text{ almost surely.}$$

Since ε was arbitrary, $\limsup_{n\to\infty} g^{(n)}/n \leq G_M$ almost surely, which proves almost sure convergence by the discussion above. $\square$

[4] Suppose $0 \leq f_n \leq 1$ and $f_n \downarrow 0$. The conditional expectation is monotone, so $\mathbb{E}(f_n|\mathscr{F})$ is decreasing at almost every point. Let φ be its almost sure limit, then $0 \leq \varphi \leq 1$ a.s., and by the BCT, $\mathbb{E}(\varphi) = \mathbb{E}(\lim \mathbb{E}(f_n|\mathscr{F})) = \lim \mathbb{E}(\mathbb{E}(f_n|\mathscr{F})) = \lim \mathbb{E}(f_n) = \mathbb{E}(\lim f_n) = 0$, whence $\varphi = 0$ almost everywhere.

Proposition 2.3. *Suppose m is ergodic, and $g^{(n)} \in L^1$ for all n, then the limit in Kingman's ergodic theorem is the constant* $\inf[(1/n)\int g^{(n)}dm] \in [-\infty,\infty)$.

Proof. Let $G := \lim g^{(n)}/n$. Subadditivity implies that $G\circ T \leq G$. Recurrence implies that $G\circ T = T$. Ergodicity implies that $G = c$ a.e., for some constant $c = c(g) \geq -\infty$. We claim that $c \leq \inf[(1/n)\int g^{(n)}dm]$. This is because

$$\begin{aligned} c &= \lim_{k\to\infty}\frac{1}{kn}g^{(kn)} \leq \lim_{k\to\infty}\frac{1}{k}\left(\frac{g^{(n)}}{n}+\frac{g^{(n)}}{n}\circ T^n+\cdots\frac{g^{(n)}}{n}\circ T^{n(k-1)}\right)\\ &= \frac{1}{n}\int g^{(n)}dm \ \ \text{(Birkhoff's ergodic theorem)}, \end{aligned}$$

proving that $c \leq (1/n)\int g^{(n)}dm$ for all n.

To prove the other inequality we first note (as in the proof of Kingman's subadditive theorem) that it is enough to treat the case when $g^{(n)}$ are all non-positive. Otherwise work with $h^{(n)} := g^{(n)} - (g^{(1)}+\cdots+g^{(1)}\circ T^{n-1})$. Since $g^{(1)} \in L^1$,

$$\frac{1}{n}(g^{(1)}+\cdots+g^{(1)}\circ T^{n-1}) \xrightarrow[n\to\infty]{} \int g^{(1)}dm \text{ pointwise and in } L^1.$$

Thus $c(g) = \lim \frac{g^{(n)}}{n} = c(h)+\int g^{(1)} = \inf(1/n)[\int h^{(n)}+\int S_n g^{(1)}] = \inf[(1/n)\int g^{(n)}]dm$.

Suppose then that $g^{(n)}$ are all non-positive. Fix N, and set $g_N^{(n)} := \max\{g^{(n)}, -nN\}$. This is, again, subadditive because

$$\begin{aligned} g_N^{(n+m)} &= \max\{g^{(n+m)}, -(n+m)N\} \leq \max\{g^{(n)}+g^{(m)}\circ T^n, -(n+m)N\}\\ &\leq \max\{g_N^{(n)}+g_N^{(m)}\circ T^n, -(n+m)N\} \equiv g_N^{(n)}+g_N^{(m)}\circ T^n. \end{aligned}$$

By Kingman's theorem, $g_N^{(n)}/n$ converges pointwise to a constant $c(g_N)$. By definition, $-N \leq g_N^{(n)}/n \leq 0$, so by the bounded convergence theorem,

$$c(g_N) = \lim_{n\to\infty}\frac{1}{n}\int g_N^{(n)}dm \geq \inf\frac{1}{n}\int g_N^{(n)}dm \geq \inf\frac{1}{n}\int g^{(n)}dm. \tag{2.4}$$

Case 1: $c(g) = -\infty$. In this case $g^{(n)}/n \to -\infty$, and for every N there exists $N(x)$ s.t. $n > N(x) \Rightarrow g_N^{(n)}(x) = -N$. Thus $c(g_N) = -N$, and (2.4) implies $\inf[(1/n)\int g^{(n)}dm] = -\infty = c(g)$.

Case 2: $c(g)$ is finite. Take $N > |c(g)|+1$, then for a.e. x, if n is large enough, then $g^{(n)}/n > c(g)-\varepsilon > -N$, whence $g_N^{(n)} = g^{(n)}$. Thus $c(g) = c(g_N) \geq \inf\frac{1}{n}\int g^{(n)}dm$ and we get the other inequality. $\square$

Here is a direct consequence of the subadditive ergodic theorem (historically, it predates the subadditive ergodic theorem):

Theorem 2.8 (Furstenberg–Kesten). *Let $(X,\mathscr{B},\mu,T)$ be a ppt, and $A : X \to \mathrm{GL}(d,R)$ be a measurable function s.t. $\log\|A^{\pm 1}\| \in L^1$. If $A_n(x) := A(T^{n-1}x)\cdots A(x)$, then the following limit exists a.e. and is invariant: $\lim\limits_{n\to\infty}\frac{1}{n}\log\|A_n(x)\|$.*

The following immediate consequence will be used in the proof of the Oseledets theorem for invertible cocycles:

Remark: *Suppose $(X,\mathscr{B},m,T)$ is invertible, and let $g^{(n)}$ be a subadditive cocycle s.t. $g^{(1)} \in L^1$. Then for a.e. x, $\lim\limits_{n\to\infty} g^{(n)}\circ T^{-n}/n$ exists and equals $\lim\limits_{n\to\infty} g^{(n)}/n$.*

Proof. Since $g^{(n)}$ is subadditive, $g^{(n)}\circ T^{-n}$ is subadditive:

$$g^{(n+m)}\circ T^{-(n+m)} \le [g^{(n)}\circ T^m + g^{(m)}]\circ T^{-(n+m)} = g^{(n)}\circ T^{-n} + [g^{(m)}\circ T^{-m}]\circ T^{-n}.$$

Let $m = \int m_y d\pi(y)$ be the ergodic decomposition of m. Kingman's ergodic theorem and the previous remark say that for π–a.e. y,

$$\begin{aligned}\lim_{n\to\infty}\frac{g^{(n)}\circ T^{-n}}{n} &= \inf\frac{1}{n}\int g^{(n)}\circ T^{-n}dm_y = \inf\frac{1}{n}\int g^{(n)}dm_y \quad m_y \text{ a.e.}\\ &= \lim_{n\to\infty}\frac{g^{(n)}}{n} \quad m_y \text{ a.e.}\end{aligned}$$

Thus the set where the statement of the remark fails has zero measure with respect to all the ergodic components of m, and this means that the statement is satisfied on a set of full m–measure. ☐

2.6 The Multiplicative Ergodic Theorem

2.6.1 Preparations from Multilinear Algebra

Multilinear forms. Let $V = \mathbb{R}^n$ equipped with the Euclidean inner product $\langle v,w\rangle = \sum v_k w_k$. A *linear functional* on V is a linear map $\omega : V \to \mathbb{R}$. The set of linear functionals is denoted by V^*. Any $v \in V$ determines $v^* \in V^*$ via $v^* = \langle v,\cdot\rangle$. Any linear function is of this form.

A *k–multilinear function* is a function $T : V^k \to \mathbb{R}$ such that for all i and $v_1,\dots,v_{i-1},v_{i+1},\dots,v_k \in V$, $T(v_1,\dots,v_{i-1},\cdot,v_{i+1},\dots,v_k)$ is a linear functional.

The set of all k–multilinear functions on V is denoted by $T^k(V)$. The *tensor product* of $\omega \in T^k(V)$ and $\eta \in T^\ell(V)$ is $\omega\otimes\eta \in T^{k+\ell}(V)$ given by

$$(\omega\otimes\eta)(v_1,\dots,v_{k+l}) := \omega(v_1,\dots,v_k)\eta(v_{k+1},\dots,v_{k+l}).$$

The tensor product is bilinear and associative, but it is not commutative.

The dimension of $T^k(V)$ is n^k. Here is a basis: $\{e^*_{i_1}\otimes\cdots\otimes e^*_{i_k} : 1 \le i_1,\dots,i_k \le n\}$. To see this note that every element in $T^k(\Omega)$ is completely determined by its action on $\{(e_{i_1},\cdots,e_{i_k}) : 1 \le i_1,\dots,i_k \le n\}$.

Define an *inner product* on $T^k(V)$ by declaring the above basis to be an orthogonal collection of vectors of length $\frac{1}{\sqrt{k!}}$ (the reason for the normalization will become clear later).

Alternating multilinear forms. A multilinear form ω is called *alternating*, if it satisfies $\exists i \neq j (v_i = v_j) \Rightarrow \omega(v_1,\dots,v_n) = 0$. Equivalently,

$$\omega(v_1,\dots,v_i,\dots,v_j,\dots v_n) = -\omega(v_1,\dots,v_j,\dots,v_i,\dots v_n).$$

(to see the equivalence, expand $\omega(v_1,\dots,v_i+v_j,\dots,v_j+v_i,\dots v_n)$). The set of all k–alternating forms is denoted by $\Omega^k(V)$.

Any multilinear form ω gives rise to an alternating form $\mathrm{Alt}(\omega)$ via

$$\mathrm{Alt}(\omega) := \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\sigma\cdot\omega,$$

where S_k is the group of k–permutations, and the action of a permutation σ on $\omega \in T^k(V)$ is given by $(\sigma\cdot\omega)(v_1,\dots,v_k) = \omega(v_{\sigma(1)},\dots,v_{\sigma(k)})$. The normalization $k!$ is to guarantee $\mathrm{Alt}|_{\Omega^k(V)} = id$, and $\mathrm{Alt}^2 = \mathrm{Alt}$. Note that Alt is linear.

Lemma 2.1. $\mathrm{Alt}[\mathrm{Alt}(\omega_1\otimes\omega_2)\otimes\omega_3] = \mathrm{Alt}(\omega_1\otimes\omega_2\otimes\omega_3)$.

Proof. We show that if $\mathrm{Alt}(\omega) = 0$, then $\mathrm{Alt}(\omega\otimes\eta) = 0$ for all η. Specializing to the case $\omega = \mathrm{Alt}(\omega_1\otimes\omega_2) - \omega_1\otimes\omega_2$ and $\eta = \omega_3$, we get (since $\mathrm{Alt}^2 = \mathrm{Alt}$)

$$\mathrm{Alt}[(\mathrm{Alt}(\omega_1\otimes\omega_2) - \omega_1\otimes\omega_2)\otimes\omega_3] = 0,$$

which is equivalent to the statement of the lemma.

Suppose $\omega \in T^k(V)$, $\eta \in T^\ell(V)$, and $\mathrm{Alt}(\omega) = 0$. Let $G := \{\sigma \in S_{k+\ell} : \sigma(i) = i \text{ for all } i = k+1,\dots,k+\ell\}$. This is a subgroup of S_{k+l}, and there is natural isomorphism $\sigma \mapsto \sigma' := \sigma|_{\{1,\dots,k\}}$ from G to S_k. Let $S_{k+l} = \biguplus_j G\sigma_j$ be the corresponding coset decomposition, then

$$\begin{aligned}
&(k+\ell)!\mathrm{Alt}(\omega\otimes\eta)(v_1,\dots,v_{k+\ell}) = \\
&= \sum_j\sum_{\sigma\in G}\mathrm{sgn}(\sigma\sigma_j)(\sigma\sigma_j)\cdot(\omega\otimes\eta)(v_1,\dots,v_{k+\ell}) \\
&= \sum_j \mathrm{sgn}(\sigma_j)\eta(v_{\sigma_j(k+1)},\dots,v_{\sigma_j(k+\ell)})\sum_{\sigma\in G}\mathrm{sgn}(\sigma)(\sigma\cdot\omega)(v_{\sigma_j(1)},\dots,v_{\sigma_j(k)}) \\
&= \sum_j \mathrm{sgn}(\sigma_j)\eta(v_{\sigma_j(k+1)},\dots,v_{\sigma_j(k+\ell)})\sum_{\sigma'\in S_k}\mathrm{sgn}(\sigma')(\sigma'\cdot\omega)(v_{\sigma_j(1)},\dots,v_{\sigma_j(k)}) \\
&= \sum_j \mathrm{sgn}(\sigma_j)\eta(v_{\sigma_j(k+1)},\dots,v_{\sigma_j(k+\ell)})k!\mathrm{Alt}(\omega)(v_{\sigma_j(1)},\dots,v_{\sigma_j(k)}) = 0.
\end{aligned}$$

□

Using this "antisymmetrization operator", we define the following product, called the *exterior product* or the *wedge product*: If $\omega \in \Omega^k(V), \eta \in \Omega^l(V)$, then

$$\omega \wedge \eta := \frac{(k+l)!}{k!\ell!} \mathrm{Alt}(\omega \otimes \eta).$$

The wedge product is bilinear, and the previous lemma shows that it is associative. It is almost anti commutative: If $\omega \in \Omega^k(V), \eta \in \Omega^\ell(V)$, then

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

We'll see the reason for the peculiar normalization later.

Proposition 2.4. $\{e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ *is an orthonormal basis for* $\Omega^k(V)$*, whence* $\dim \Omega^k(V) = \binom{n}{k}$.

Proof. Suppose $\omega \in \Omega^k(V)$, then $\omega \in T^k(V)$ and so $\omega = \sum a_{i_1,\dots,i_k} e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}$, where the sum ranges over all k–tuples of numbers between 1 and n. If $\omega \in \Omega^k(V)$, then $\mathrm{Alt}(\omega) = \omega$ and so

$$\omega = \sum a_{i_1,\dots,i_k} \mathrm{Alt}(e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}).$$

Fix $\xi := e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}$. If $i_\alpha = i_\beta$ for some $\alpha \neq \beta$, then the permutation σ_0 which switches $\alpha \leftrightarrow \beta$ preserves ξ. Thus for all $\sigma \in S_k$,

$$\mathrm{sgn}(\sigma\sigma_0)(\sigma\sigma_0) \cdot \xi = -\mathrm{sgn}(\sigma)\sigma \cdot \xi$$

and we conclude that $\mathrm{Alt}(\xi) = 0$. If, on the other hand, $i_1, \dots, i_k$ are all different, then it is easy to see using lemma 2.1 that $\mathrm{Alt}(e^*_{i_1} \otimes e^*_{i_2} \otimes \cdots \otimes e^*_{i_k}) = \frac{1}{k!} e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$.

Thus $\omega = \frac{1}{k!} \sum a_{i_1,\dots,i_k} e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$, and we have proved that the set of forms in the statement spans $\Omega^k(V)$.

To see that this set is independent, we show that it is orthonormal. Suppose $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$, then the sets $\{\sigma \cdot e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}\}$, $\{\sigma \cdot e^*_{j_1} \otimes \cdots \otimes e^*_{j_k}\}$ are disjoint, so $\mathrm{Alt}(e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}) \perp \mathrm{Alt}(e^*_{j_1} \otimes \cdots \otimes e^*_{j_k})$. This proves orthogonality. Orthonormality is because $\|e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}\|_2 = k!\|\mathrm{Alt}(e^*_{i_1} \wedge \cdots \wedge e^*_{i_k})\|_2 = k! \left\| \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\sigma \cdot (e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}) \right\|_2 = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)^2 (\frac{1}{\sqrt{k!}})^2 = 1$. (This explains why we chose to define $\|e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}\|_2 := \frac{1}{\sqrt{k!}}$) ☐

Corollary 2.2. $e^*_1 \wedge \cdots \wedge e^*_n$ *is the determinant. This is the reason for the peculiar normalization in the definition of* $\wedge$.

Proof. The determinant is an alternating n–form, and $\dim \Omega^n(V) = 1$, so the determinant is proportional to $e^*_1 \wedge \cdots \wedge e^*_n$. Since the values of both forms on the standard basis is one (because $e^*_1 \wedge \cdots \wedge e^*_n = n!\mathrm{Alt}(e^*_1 \otimes \cdots \otimes e^*_n)$), they are equal. ☐

We define an inner product on $\Omega^k(V)$ by declaring the basis in the proposition to be orthonormal. Let $\|\cdot\|$ be the resulting norm.

Lemma 2.2. *For $v \in V$, let $v^* := \langle v, \cdot \rangle$, then*

(a) $\|\omega \wedge \eta\| \le \|\omega\| \|\eta\|$.
(b) $\langle v_1^* \wedge \cdots \wedge v_k^*, w_1^* \wedge \cdots \wedge w_k^* \rangle = \det(\langle v_i, w_j \rangle)$.
(c) If $\{u_1, \ldots, u_n\}$ is an orthonormal basis for V, then $\{u_{i_1}^* \wedge \cdots \wedge u_{i_k}^* : 1 \le i_1 < \cdots < i_k \le n\}$ is an orthonormal basis for $\Omega^k(V)$.
(d) If $\mathrm{span}\{v_1, \ldots, v_k\} = \mathrm{span}\{u_1, \ldots, u_k\}$, *then $v_1^* \wedge \cdots \wedge v_k^*$ and $u_1^* \wedge \cdots \wedge u_k^*$ are proportional.*

Proof. Write for $I = (i_1, \ldots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le n$, $e_I^* := e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$.
Represent $\omega := \sum \alpha_I e_I^*$, $\eta := \sum \beta_J e_J^*$, then

$$\|\omega \wedge \eta\|^2 = \left\| \sum_{I,J} \alpha_I \beta_J e_I^* \wedge e_J^* \right\|^2 = \left\| \sum_{I \cap J = \varnothing} \pm \alpha_I \beta_J e_{I \cup J}^* \right\|^2 = \sum_{I \cap J = \varnothing} \alpha_I^2 \beta_J^2 \le \|\omega\|^2 \|\eta\|^2.$$

Take two multi indices I, J. If $I = J$, then the inner product matrix is the identity matrix. If $I \ne J$, then $\exists \alpha \in I \setminus J$ and then the α–row and column of the inner product matrix will be zero. Thus the formula holds for any pair e_I^*, e_J^*. Since part (b) of the lemma holds for all basis vectors, it holds for all vectors. Part (c) immediately follows.

Next we prove part (d). Represent $v_i = \sum \alpha_{ij} u_j$, then

$$\begin{aligned} v_1^* \wedge \cdots \wedge v_k^* &= \mathrm{const.}\, \mathrm{Alt}(v_1^* \otimes \cdots \otimes v_k^*) = \mathrm{const.}\, \mathrm{Alt}\left(\sum_j \alpha_{1j} u_j^* \otimes \cdots \otimes \sum_j \alpha_{kj} u_j^* \right) \\ &= \mathrm{const.} \sum \alpha_{1 j_1} \cdots \alpha_{k j_k} \mathrm{Alt}(u_{j_1}^* \otimes \cdots \otimes u_{j_k}^*). \end{aligned}$$

The terms where $j_1, \ldots, j_k$ are not all different are annihilated by Alt. The terms where $j_1, \ldots, j_k$ are all different are mapped by Alt to a form which proportional to $u_1^* \wedge \cdots \wedge u_k^*$. Thus the result of the sum is proportional to $u_1^* \wedge \cdots \wedge u_k^*$. ☐

Exterior product of linear operators Let $A : V \to V$ be a linear operator. The $k-th$ *exterior product* of A is $A^{\wedge k} : \Omega^k(V) \to \Omega^k(V)$ given by

$$(A^{\wedge k} \omega)(v_1, \ldots, v_k) := \omega(A^t v_1, \ldots, A^t v_k).$$

The transpose is used to get $A^{\wedge k}(v_1^* \wedge \cdots \wedge v_k^*) = (Av_1)^* \wedge \cdots \wedge (Av_k)^*$.

Theorem 2.9. $\|A^{\wedge k}\| = \lambda_1 \cdots \lambda_k$, *where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of $(A^t A)^{1/2}$, listed in decreasing order with multiplicities.*

Proof. The matrix AA^t is symmetric, so it can be orthogonally diagonalized. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of eigenvectors, listed so that $(AA^t) v_i = \lambda_i^2 v_i$. Then $\{v_I^* : I \subseteq \{1, \ldots, d\}, |I| = k\}$ is an orthonormal basis for $\Omega^k(\mathbb{R}^d)$, where we are using the multi index notation

$$v_I^* = v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*,$$

where $i_1 < \cdots < i_k$ is an ordering of I.

Given $\omega \in \Omega^k(\mathbb{R}^d)$, write $\omega = \sum \omega_I v_I^*$, then

$$\|A^{\wedge k}\omega\|^2 = \langle A^{\wedge k}\omega, A^{\wedge k}\omega\rangle = \left\langle \sum_I \omega_I A^{\wedge k} v_I^*, \sum_J \omega_J A^{\wedge k} v_J^* \right\rangle$$
$$= \sum_{I,J} \omega_I \omega_J \left\langle A^{\wedge k} v_I^*, A^{\wedge k} v_J^* \right\rangle.$$

Now,
$$\left\langle A^{\wedge k} v_I^*, A^{\wedge k} v_J^* \right\rangle = \langle (Av_{i_1})^* \wedge \cdots \wedge (Av_{i_k})^*, (Av_{j_1})^* \wedge \cdots \wedge (Av_{j_k})^* \rangle$$
$$= \det\left(\langle Av_{i_\alpha}, Av_{j_\beta}\rangle\right) \qquad \text{(Lemma 2.2(b))}$$
$$= \det\left(\langle v_{i_\alpha}, A^t A v_{j_\beta}\rangle\right) = \det\left(\langle v_{i_\alpha}, \lambda_{i_\beta}^2 v_{j_\beta}\rangle\right)$$
$$= \prod_{j\in J} \lambda_j^2 \det\left(\langle v_{i_\alpha}, v_{i_\beta}\rangle\right)$$
$$= \prod_{j\in J} \lambda_j^2 \langle v_I^*, v_J^*\rangle = \begin{cases} \prod_{i\in I} \lambda_i^2 & I = J \\ 0 & I \neq J \end{cases},$$

Thus $\|A^{\wedge k}\omega\|^2 = \sum_I \omega_I^2 \prod_{i\in I} \lambda_i^2 \leq \|\omega\|^2 \prod_{i=1}^k \lambda_i^2$. It follows that $\|A^{\wedge k}\| \leq \lambda_1 \cdots \lambda_k$.

To see that the inequality is in fact an equality, consider the case $\omega = v_I^*$ where $I = \{1, \ldots, k\}$: $\|A^{\wedge k}\omega\| = \langle v_I^*, v_I^*\rangle = (\lambda_1 \cdots \lambda_k)^2 = (\lambda_1 \cdots \lambda_k)^2 \|\omega\|^2$. □

Exterior products and angles between vector spaces The *angle between vector spaces* $V, W \subset \mathbb{R}^d$ is

$$\measuredangle(V,W) := \min\{\arccos\langle v, w\rangle : v \in V, w \in W, \|v\| = \|w\| = 1\}.$$

So if $V \cap W \neq \{0\}$ iff $\measuredangle(V,W) = 0$, and $V \perp W$ iff $\measuredangle(V,W) = \pi/2$.

Proposition 2.5. *If $(w_1, \ldots, w_k)$ is a basis of W, and $(v_1, \ldots, v_\ell)$ is a basis of W, then* $\|(v_1^* \wedge \cdots \wedge v_\ell^*) \wedge (w_1^* \wedge \cdots \wedge w_k^*)\| \leq \|v_1^* \wedge \cdots \wedge v_\ell^*\| \cdot \|w_1^* \wedge \cdots \wedge w_k^*\| \cdot |\sin\measuredangle(V,W)|$.

Proof. If $V \cap W \neq \{0\}$ then both sides are zero, so suppose $V \cap W = \{0\}$, and pick an orthonormal basis $e_1, \ldots, e_{n+k}$ for $V \oplus W$. Let $w \in W, v \in V$ be unit vectors s.t. $\measuredangle(V,W) = \measuredangle(v,w)$, and write $v = \sum v_i e_i$, $w = \sum w_j e_j$, then

$$\begin{aligned}
\|v^* \wedge w^*\|^2 &= \left\| \sum_{i,j} v_i w_j e_i^* \wedge e_j^* \right\|^2 = \left\| \sum_{i<j} (v_i w_j - v_j w_i) e_i^* \wedge e_j^* \right\|^2 = \sum_{i<j} (v_i w_j - v_j w_i)^2 \\
&= \frac{1}{2} \sum_{i,j} (v_i w_j - v_j w_i)^2 \qquad \text{(the terms where } i = j \text{ vanish)} \\
&= \frac{1}{2} \sum_{i,j} \left(v_i^2 w_j^2 + v_j^2 w_i^2 - 2 v_i w_i \cdot v_j w_j \right) = \frac{1}{2} \left[2 \sum_i v_i^2 \sum_j w_j^2 - 2 \left(\sum_i v_i w_i \right)^2 \right] \\
&= \|v\|^2 \|w\|^2 - \langle v, w \rangle^2 = 1 - \cos^2 \measuredangle(v,w) = \sin^2 \measuredangle(V,W).
\end{aligned}$$

Complete v to an orthonormal basis $(v, v_2', \dots, v_\ell')$ of V, and complete w to an orthonormal basis $(w, w_2', \dots, w_k')$ of W. Then

$$\begin{aligned}
&\|(v^* \wedge {v_2'}^* \wedge \cdots \wedge v_\ell') \wedge (w^* \wedge {w_2'}^* \wedge \cdots \wedge {v_k'}^*)\| \\
&\qquad \le \|v^* \wedge w^*\| \cdot \|{v_2'}^* \wedge \cdots \wedge {v_\ell'}^*\| \cdot \|{w_2'}^* \wedge \cdots \wedge {v_k'}^*\| = |\sin \measuredangle(V,W)| \cdot 1 \cdot 1,
\end{aligned}$$

because of orthonormality. By lemma 2.2

$$\begin{aligned}
v_1^* \wedge \cdots \wedge v_\ell^* &= \pm \|v_1^* \wedge \cdots \wedge v_\ell^*\| \cdot v^* \wedge {v_2'}^* \wedge \cdots \wedge {v_\ell'}^* \\
w_1^* \wedge \cdots \wedge w_k^* &= \pm \|w_1^* \wedge \cdots \wedge w_k^*\| \cdot w^* \wedge {w_2'}^* \wedge \cdots \wedge {w_k'}^*
\end{aligned}$$

and the proposition follows. □

2.6.2 Proof of the Multiplicative Ergodic Theorem

Let $(X, \mathscr{B}, m, f)$ be a ppt, and $A : X \to \mathrm{GL}(d, \mathbb{R})$ some Borel map. We define $A_n := A \circ f^{n-1} \cdots A$, then the *cocycle identity* holds: $A_{n+m}(x) = A_n(f^m x) A_m(x)$.

Theorem 2.10 (Multiplicative Ergodic Theorem). *Let $(X, \mathscr{B}, T, m)$ be a ppt, and $A : X \to \mathrm{GL}(d, \mathbb{R})$ a Borel function s.t. $\ln \|A(x)^{\pm 1}\| \in L^1(m)$, then*

$$\Lambda(x) := \lim_{n \to \infty} [A_n(x)^t A_n(x)]^{1/2n}$$

exists a.e., and $\lim_{n\to\infty} \frac{1}{n} \ln \|A_n(x) \Lambda(x)^{-n}\| = \lim_{n\to\infty} \frac{1}{n} \ln \|(A_n(x)\Lambda(x)^{-n})^{-1}\| = 0$ *a.s.*

Proof. The matrix $B_n(x) := \sqrt{A_n(x)^t A_n(x)}$ is symmetric, therefore it can be orthogonally diagonalized. Let $\exists \lambda_n^1(x) < \cdots < \lambda_n^{s_n(x)}(x)$ be its different eigenvalues, and $\mathbb{R}^d = W_n^{\lambda_n^1(x)}(x) \oplus \cdots \oplus W_n^{\lambda_n^{s_n(x)}(x)}(x)$ the orthogonal decomposition ot $\mathbb{R}^d$ into the corresponding eigenspaces. The proof has the following structure:

Part 1: Let $t_n^1(x) \le \cdots \le t_n^d(x)$ be a list of the eigenvalues of $B_n(x) := \sqrt{A_n(x)^t A_n(x)}$ *with multiplicities*, then for a.e. x, there is a limit $t_i(x) = \lim_{n\to\infty} [t_n^i(x)]^{1/n}$, $i = 1, \dots, d$.

Part 2: Let $\lambda_1(x) < \cdots < \lambda_{s(x)}(x)$ be a list of the different values of $\{t_i(x)\}_{i=1}^d$. Divide $\{t_n^i(x)\}_{i=1}^d$ into $s(x)$ subsets of values $\{t_n^i(x) : i \in I_n^j\}$, $(1 \leq j \leq s(x))$ in such a way that $t_n^i(x)^{1/n} \to \lambda_j(x)$ for all $i \in I_n^j$. Let

$$
\begin{aligned}
U_n^j(x) &:= \text{sum of the eigenspaces of } t_n^i(x),\ i \in I_n^j. \\
&= \text{the part of the space where } B_n(x) \text{ dilates by approximately } \lambda_j(x)^n.
\end{aligned}
$$

We show that the spaces $U_n^j(x)$ converge as $n \to \infty$ to some limiting spaces $U^j(x)$ (in the sense that the orthogonal projections on $U_n^j(x)$ converge to the orthogonal projection on $U^j(x)$).

Part 3: The theorem holds with $\Lambda(x) : \mathbb{R}^d \to \mathbb{R}^d$ given by $v \mapsto \lambda_i(x)v$ on $U^i(x)$.

Part 1 is proved by applying the sub additive ergodic theorem for a cleverly chosen sub-additive cocyle ("Raghunathan's trick"). Parts 2 and 3 are (non-trivial) linear algebra.

Part 1: Set $g_i^{(n)}(x) := \sum_{j=d-i+1}^d \ln t_n^j(x)$. This quantity is finite, because $A_n^t A_n$ is invertible, so none of its eigenvalues vanish.

The sequence $g_i^{(n)}$ is subadditive! This is because the theory of exterior products says that $\exp g_i^{(n)} =$ product of the i largest e.v.'s of $\sqrt{A_n(x)^t A_n(x)} = \|A_n(x)^{\wedge i}\|$, so

$$
\begin{aligned}
\exp g_i^{(n+m)}(x) &= \|A_{n+m}^{\wedge i}(x)\| = \|A_m(T^n x)^{\wedge i} A_n(x)^{\wedge i}\| \leq \|A_m(T^n x)^{\wedge i}\| \|A_n(x)^{\wedge i}\| \\
&= \exp[g_i^{(m)}(T^n x) + g_i^{(n)}(x)],
\end{aligned}
$$

whence $g_i^{(n+m)} \leq g_i^{(n)} + g_i^{(m)} \circ T^n$.

We want to apply Kingman's subadditive ergodic theorem. First we need to check that $g_i^{(1)} \in L^1$. We use the following fact from linear algebra: if λ is an eigenvalue of a matrix B, then $\|B^{-1}\|^{-1} \leq |\lambda| \leq \|B\|$.[5] Therefore

$$
\begin{aligned}
|\ln t_n^i(x)| &\leq \frac{1}{2} \max\{|\ln \|A_n^t A_n\||, |\ln \|(A_n^t A_n)^{-1}\||\} \\
&\leq \max\{|\ln \|A_n(x)\||, |\ln \|A_n(x)^{-1}\||\} \\
&\leq \sum_{k=0}^{n-1} \Big(|\ln \|A(T^k x)\| + |\ln \|A(T^k x)^{-1}\|| \Big) \\
\therefore |g_i^{(n)}(x)| &\leq i \sum_{k=0}^{n-1} \Big(|\ln \|A(T^k x)\| + |\ln \|A(T^k x)^{-1}\|| \Big). \qquad (2.5)
\end{aligned}
$$

So $\|g^{(n)}\|_1 \leq n(\big\|\ln \|A\|\big\|_1 + \big\|\ln \|A^{-1}\|\big\|_1) < \infty$.

Thus Kingman's ergodic theorem says that $\lim \frac{1}{n} g_i^{(n)}(x)$ exists almost surely, and belongs to $[-\infty, \infty)$. In fact the limit is finite almost everywhere, because (2.5) and

[5] Proof: Let v be an eigenvector of λ with norm one, then $|\lambda| = \|Bv\| \leq \|B\|$ and $1 = \|B^{-1}Bv\| \leq \|B^{-1}\| \|Bv\| = \|B^{-1}\| |\lambda|$.

the Pointwise Ergodic Theorem imply that

$$\lim_{n\to\infty}\frac{1}{n}|g_i^{(n)}| \leq i\mathbb{E}\left(|\ln\|A\|| + |\ln\|A^{-1}\||\,\big|\,\mathfrak{Inv}\right) < \infty \text{ a.e.}$$

Taking differences, we see that the following limit exists a.e.:

$$\ln t_i(x) := \lim_{n\to\infty}\frac{1}{n}[g_{d-i+1}^{(n)}(x) - g_{d-i}^{(n)}(x)] = \lim_{n\to\infty}\frac{1}{n}\ln t_n^i(x).$$

Thus $[t_n^i(x)]^{1/n} \xrightarrow[n\to\infty]{} t_i(x)$ almost surely, for some $t_i(x) \in \mathbb{R}$.

Part 2: Fix x s.t. $[t_n^i(x)]^{1/n} \xrightarrow[n\to\infty]{} t_i(x)$ for all $1 \leq i \leq d$. Henceforth we work with this x only, and write for simplicity $A_n = A_n(x)$, $t_i = t_i(x)$ etc.

Let $s = s(x)$ be the number of the different t_i. List the *different* values of these quantities an increasing order: $\lambda_1 < \lambda_2 < \cdots < \lambda_s$. Set $\chi_j := \log\lambda_j$. Fix $0 < \delta < \frac{1}{2}\min\{\chi_{j+1} - \chi_j\}$. Since for all i there is a j s.t. $(t_n^i)^{1/n} \to \lambda_j$, the following sets eventually stabilize and are independent of n:

$$I_j := \{i : |(t_n^i)^{1/n} - \lambda_j| < \delta\} \;\; (j = 1,\ldots,s).$$

Define, relative to $\sqrt{A_n^t A_n}$,

- $U_n^j := \sum_{i\in I_j}$[eigenspace of $t_n^i(x)$] (this sum is not necessarily direct);
- $V_n^r := \oplus_{j\leq r} U_n^j$
- $\widetilde{V}_n^r := \oplus_{j\geq r} U_n^j$

The linear spaces $U_n^1,\ldots,U_n^s$ are orthogonal, since they are eigenspaces of different eigenvalues for a symmetric matrix $(\sqrt{A_n^t A_n})$. We show that they converge as $n\to\infty$, in the sense that their orthogonal projections converge.

The proof is based on the following technical lemma. Denote the projection of a vector v on a subspace W by $v|W$, and write $\chi_i := \log\lambda_i$.

Technical lemma: For every $\delta > 0$ there exists constants $K_1,\ldots,K_s > 1$ and N s.t. for all $n > N$, $t = 1,\ldots,s$, $k \in \mathbb{N}$, and $u \in V_n^r$,

$$\left\|u|\widetilde{V}_{n+k}^{r+t}\right\| \leq K_t\|u\|\exp(-n(\chi_{r+t} - \chi_r - \delta t))$$

We give the proof later. First we show how it can be used to finish parts 2 and 3.

We show that V_n^r converge as $n\to\infty$. Since the projection on U_n^i is the projection on V_n^i minus the projection on V_n^{i-1}, it will then follow that the projections of U_n^i converge.

Fix N large. We need it to be so large that

1. I_j are independent of n for all $n > N$;
2. The technical lemma works for $n > N$ with δ as above.

There will be other requirements below.

Fix an orthonormal basis $(v_n^1,\dots,v_n^{d_r})$ for V_n^r ($d_r=\dim(V_n^r)=\sum_{j\le r}|I_j|$). Write

$$v_n^i=\alpha_n^i w_{n+1}^i+u_{n+1}^i,\text{ where } w_{n+1}^i\in V_{n+1}^r,\ \|w_{n+1}^i\|=1,\ u_{n+1}^i\in\widetilde{V}_{n+1}^{r+1}.$$

Note that $\|u_{n+1}^i\|=\|v_n^i|\widetilde{V}_{n+1}^{r+1}\|\le K_1\exp(-n(\chi_{r+1}-\chi_r-\delta))$. Using the identity $\alpha_n^i=\sqrt{1-\|u_{n+1}^i\|^2}$, it is easy to see that for some constants C_1 and $0<\theta<1$ independent of n and (v_n^i),

$$\|v_n^i-w_{n+1}^i\|\le C_1\theta^n.$$

($\theta:=\max_r\exp[-(\chi_{r+1}-\chi_r-\delta)]$ and $C_1:=2K_1$ should work.)

The system $\{w_{n+1}^i\}$ is very close to being orthonormal:

$$\langle w_{n+1}^i,w_{n+1}^j\rangle=\langle w_{n+1}^i-v_n^i,w_{n+1}^j\rangle+\langle v_n^i,v_n^j\rangle+\langle v_n^i,w_{n+1}^j-v_n^j\rangle=\delta_{ij}+O(\theta^n),$$

because $\{v_n^i\}$ is an orthonormal system. It follows that for all n large enough, w_{n+1}^i are linearly independent. A quick way to see this is to note that

$$\begin{aligned}\|(w_{n+1}^1)^*\wedge\cdots\wedge(w_{n+1}^{d_r})^*\|^2&=\det(\langle w_{n+1}^i,w_{n+1}^j\rangle)\quad\text{(lemma 2.2)}\\&=\det(I+O(\theta^n))\neq 0,\text{ provided } n \text{ is large enough,}\end{aligned}$$

and to observe that wedge produce of a linearly dependent system vanishes.

It follows that $\{w_{n+1}^1,\dots,w_{n+1}^{d_r}\}$ is a linearly independent subset of V_{n+1}^r. Since $\dim(V_{n+1}^r)=\sum_{j\le r}|I_j|=d_r$, this is a basis for V_{n+1}^r.

Let (v_{n+1}^i) be the orthonormal basis obtained by applying the Gram–Schmidt procedure to (w_{n+1}^i). We claim that there is a global constant C_2 such that

$$\|v_n^i-v_{n+1}^i\|\le C_2\theta^n. \tag{2.6}$$

Write $v_i=v_{n+1}^i, w_i=w_{n+1}^i$, then the Gram–Schmidt process is to set $\overline{v}_i=u_i/\|u_i\|$, where u_i are defined by induction by $u_1:=w_1$, $u_i:=w_i-\sum_{j<i}\langle w_i,\overline{v}_j\rangle\overline{v}_j$. We construct by induction global constants C_2^i s.t. $\|\overline{v}_i-w_i\|\le C_2^i\theta^n$, and then take $C_2:=\max\{C_2^i\}$. When $i=1$, we can take $C_2^1:=C_1$, because $\overline{v}_1=w_1$, and $\|w_1-v_n^1\|\le C_1\theta^n$. Suppose we have constructed $C_2^1,\dots,C_2^{i-1}$. Then

$$\|u_i-w_i\|\le\sum_{j<i}|\langle w_i,\overline{v}_j\rangle|\le\sum_{j<i}|\langle w_i,w_j\rangle|+\|w_j-\overline{v}_j\|\le\left(2C_1(i-1)+\sum_{j<i}C_2^j\right)\theta^n,$$

because $|\langle w_i,w_j\rangle|=|\langle w_i-v_n^i,w_j\rangle+\langle v_n^i,w_j-v_n^j\rangle+\langle v_n^i,v_n^j\rangle|\le 2C_1\theta^n$. Call the term in the brackets K, and assume n is so large that $K\theta^n<1/2$, then $|\|u_i\|-1|\le\|u_i-w_i\|\le K\theta^n$, whence

$$\|\overline{v}_i-w_i\|=\left\|\frac{u_i-\|u_i\|w_i}{\|u_i\|}\right\|\le\frac{\|u_i-w_i\|+|1-\|u_i\||}{\|u_i\|}\le 4K\theta^n$$

and we can take $C_2^i := 4K$. This proves (2.6).

Starting from the orthonormal basis (v_n^i) for V_n^r, we have constructed an orthonormal basis (v_{n+1}^i) for V_{n+1}^r such that $\|v_n^i - v_{n+1}^i\| \le C_2\theta^n$. Continue this procedure by induction, and construct the orthonormal bases (v_{n+k}^i) for V_{n+k}^r. By (2.6), these bases form Cauchy sequences: $v_{n+k}^i \xrightarrow[k\to\infty]{} v^i$.

The limit vectors must also be orthonormal. Denote their span by V^r. The projection on V^r takes the form

$$\sum_{i=1}^{d_r} \langle v^i, \cdot\rangle v^i = \lim_{k\to\infty} \sum_{i=1}^{d_r} \langle v_{n+k}^i, \cdot\rangle v_{n+k}^i = \lim_{k\to\infty} \mathrm{proj}_{V_{n+k}^r}.$$

Thus $V_{n+k}^r \to V^r$.

Part 3: We saw that $\mathrm{proj}_{U_n^i(x)} \xrightarrow[n\to\infty]{} \mathrm{proj}_{U^i(x)}$ for some linear spaces $U_i(x)$. Set $\Lambda(x) \in \mathrm{GL}(\mathbb{R}^d)$ to be the matrix representing

$$\Lambda(x) = \sum_{j=1}^{s(x)} e^{\chi_j(x)} \mathrm{proj}_{U_i(x)}.$$

Since $U_i(x)$ are limits of U_n^i, they are orthogonal, and they sum up to $\mathbb{R}^d$. It follows that Λ is invertible, symmetric, and positive.

Choose an orthogonal basis $\{v_n^1(x), \dots, v_n^d(x)\}$ of $B_n(x) := \sqrt{A_n(x)^t A_n(x)}$ so that $B_n v_n^i = t_n^i v_n^i$ for all i, and let $W_n^i := \mathrm{span}\{v_n^i\}$. Then for all $v \in \mathbb{R}^d$,

$$\begin{aligned}
(A_n^t A_n)^{1/2n} v &= (\sqrt{A_n^t A_n})^{1/n} v = \sum_{i=1}^{d} t_n^i(x)^{1/n} \mathrm{proj}_{W_n^i}(v) \\
&= \sum_{j=1}^{s} \sum_{i\in I_j} t_n^i(x)^{1/n} \mathrm{proj}_{W_n^i}(v) \\
&= \sum_{j=1}^{s} e^{\chi_j(x)} \sum_{i\in I_j} \mathrm{proj}_{W_n^i}(v) + o(\|v\|), \\
&\qquad\qquad \text{where } o(\|v\|) \text{ denotes a vector with norm } o(\|v\|) \\
&= \sum_{j=1}^{s} e^{\chi_j(x)} \mathrm{proj}_{U_n^j}(v) + o(\|v\|) \xrightarrow[n\to\infty]{} \Lambda(x) v.
\end{aligned}$$

Thus $(A_n^t A_n)^{1/2n} \to \Lambda$.

We show that $\frac{1}{n}\log\|(A_n \Lambda^{-n})^{\pm 1}\| \xrightarrow[n\to\infty]{} 0$. It's enough to show that

$$\lim_{n\to\infty} \frac{1}{n} \log\|A_n v\| = \chi_r := \log\lambda_r \text{ uniformly on the unit ball in } U_r. \tag{2.7}$$

To see that this is enough, note that $\Lambda v = \sum_{r=1}^{s} e^{\chi_r}(v|U_r)$; for all $\delta > 0$, if n is large enough, then for every v,

$$\|A_n\Lambda^{-n}v\| \le \sum_{r=1}^{s} e^{-n\chi_r}\|A_n(v|U_r)\| = \sum_{r=1}^{s} e^{-n\chi_r}e^{n(\chi_r+\delta)}\|v\| \le se^{n\delta}\|v\| \quad (v\in\mathbb{R}^d)$$
$$\|A_n\Lambda^{-n}v\| = e^{-n\chi_r}\|A_nv\| = e^{\pm n\delta}\|v\| \qquad (v\in U_r)$$

Thus $\|A_n\Lambda^{-n}\| \asymp e^{-n\delta}$ for all δ, whence $\frac{1}{n}\log\|A_n\Lambda^{-n}\| \to 0$ a.e.

To see that $\frac{1}{n}\log\|(A_n\Lambda^{-n})^{-1}\| \to 0$, we use a duality trick.

Define for a matrix C, $C^\# := (C^{-1})^t$, then $(C_1C_2)^\# = C_1^\# C_2^\#$. Thus $(A^\#)_n = (A_n)^\#$, and $B_n^\# := \sqrt{(A^\#)_n^t(A^\#)_n} = (\sqrt{A_n^tA_n})^\# = (\sqrt{A_n^tA_n})^{-1}$. Thus we have the following relation between the objects associated to $A^\#$ and A:

1. the eigenvalues of $B_n^\#$ are $1/t_n^d \le \cdots \le 1/t_n^1$ (the order is flipped)
2. the eigenspace of $1/t_n^i$ for $B_n^\#$ is the eigenspace of t_n^i for B_n
3. $\chi_j^\# = -\chi_{s-j+1}$
4. $(U_n^j)^\# = U_n^{s-j+1}$, $(V_n^r)^\# = \widetilde{V}_n^{s-r+1}$, $(\widetilde{V}_n^r)^\# = V_n^{s-r+1}$
5. $\Lambda^\# = \Lambda^{-1}$.

Thus $\|(\Lambda^nA_n^{-1})\| = \|(\Lambda^nA_n^{-1})^t\| = \|A_n^\#(\Lambda^\#)^{-n}\|$, so the claim $\frac{1}{n}\log\|\Lambda^nA_n^{-1}\| \xrightarrow[n\to\infty]{} 0$ a.e. follows from what we did above, applied to $A^\#$.

Here is another consequence of this duality: There exist $K_1^\#,\dots,K_t^\#$ s.t. for all δ, there is an N s.t. for all $n>N$, if $u\in U_n^r$, then for all k

$$\|u|V_{n+k}^{r-t}\| \le K_t^\# \exp[-n(\chi_r-\chi_{r-t}-\delta)]. \tag{2.8}$$

To see this note that $V_{n+k}^{r-t} = (\widetilde{V}_{n+k}^{s-r+t+1})^\#$ and $U_n^r \subset (V_n^{s-r+1})^\#$, and apply the technical lemma to the cocycle generated by $A^\#$.

We prove (2.7). Fix $\delta>0$ and N large (we see how large later), and assume $n>N$. Suppose $v\in U_r$ and $\|v\|=1$. Write $v=\lim v_{n+k}$ with $v_{n+k} := v|U_{n+k}^k \in U_{n+k}^r$. Note that $\|v_{n+k}\|\le 1$. We decompose v_{n+k} as follows

$$v_{n+k} = \left(v_{n+k}|V_n^{r-1}\right) + \left(v_{n+k}|U_n^r\right) + \sum_{t=1}^{s-r}\left(v_{n+k}|U_n^{r+t}\right),$$

and estimate the size of the image of each of the summands under A_n.

First summand:

$$\begin{aligned}\|A_n(v_{n+k}|V_n^{r-1})\|^2 &\equiv \langle B_n^2(v_{n+k}|V_n^{r-1}),(v_{n+k}|V_n^{r-1})\rangle\\ &= e^{2n(\chi_{r-1}+o(1))}\|v_{n+k}|V_n^{r-1}\| \le e^{2n(\chi_{r-1}+o(1))}.\end{aligned}$$

Thus the first summand is less than $\exp[n(\chi_{r-1}+o(1))]$.

Second Summand:

$$\begin{aligned}
\|A_n(v_{n+k}|U_n^r)\|^2 &= \langle B_n^2(v_{n+k}|U_n^r),(v_{n+k}|U_n^r)\rangle\\
&= e^{2n(\chi_r+o(1))}\|v_{n+k}|U_n^r\|^2 = e^{2n(\chi_r+o(1))}\left(\|v|U_n^r\| \pm (\|(v_{n+k}-v)|U_n^r\|\right)^2\\
&= e^{2n(\chi_r\pm\delta)}\left(\|v|U_n^r\| \pm \|v_{n+k}-v\|\right)^2 = e^{2n(\chi_r\pm\delta)}[1+o(1)] \text{ uniformly in } v.
\end{aligned}$$

Thus the second summand is $[1+o(1)]\exp[n(\chi_r\pm\delta)]$ uniformly in $v\in U_r$, $\|v\|=1$.

Third Summand: For every t,

$$\begin{aligned}
\|A_n(v_{n+k}|U_n^t)\|^2 &= \langle B_n^2(v_{n+k}|U_n^t),(v_{n+k}|U_n^t)\rangle\\
&\le e^{2n(\chi_{r+t}+o(1))}\|v_{n+k}|U_n^{r+t}\|^2\\
&\equiv e^{2n(\chi_{r+t}+o(1))}\left(\sup_{u\in U_n^{r+t},\|u\|=1}\langle v_{n+k},u\rangle\right)^2, \text{ because } \|x|W\| = \sup_{w\in W,\|w\|=1}\langle x,w\rangle\\
&\le e^{2n(\chi_{r+t}+o(1))}\left(\sup_{u\in U_n^{r+t},\|u\|=1}\ \sup_{v\in V_{n+k}^r,\|v\|\le 1}\langle v,u\rangle\right)^2\\
&= e^{2n(\chi_{r+t}+o(1))}\sup_{u\in U_n^{r+t},\|u\|=1}\|u|V_{n+k}^r\|^2\\
&\le (K_t^{\#})^2 e^{2n(\chi_{r+t}+o(1))}\exp[-2n(\chi_{r+t}-\chi_r-o(1))], \text{ by (2.8)}\\
&= (K_t^{\#})^2 e^{2n(\chi_r+o(1))}.
\end{aligned}$$

Note that the cancellation of χ_{r+t} — this is the essence of the technical lemma. We get: $\|A_n(v_{n+k}|U_n^t)\| = O(\exp[n(\chi_r+o(1))])$. Summing over $t=1,\dots,s-r$, we get that third summand is $O(\exp[n(\chi_r+o(1))])$.

Putting these estimates together, we get that

$$\|A_n v_{n+k}\| \le \text{const.}\exp[n(\chi_r+o(1))] \text{ uniformly in } k, \text{ and on the unit ball in } U_r.$$

"Uniformity" means that the $o(1)$ can be made independent of v and k. It allows us to pass to the limit as $k\to\infty$ and obtain

$$\|A_n v\| \le \text{const.}\exp[n(\chi_r+o(1))] \text{ uniformly on the unit ball in } U_r.$$

On the other hand, an orthogonality argument shows that

$$\begin{aligned}
\|A_n v_{n+k}\|^2 &= \langle B_n^2 v_{n+k}, v_{n+k}\rangle\\
&= \|\text{1st summand}\|^2 + \|\text{2nd summand}\|^2 + \|\text{3rd summand}\|^2\\
&\ge \|\text{2nd summand}\|^2 = [1+o(1)]\exp[2n(\chi_r+o(1))].
\end{aligned}$$

Thus $\|A_n v_{n+k}\| \ge [1+o(1)]\exp[n(\chi_r+o(1))]$ uniformly in v,k. Passing to the limit as $k\to\infty$, we get $\|A_n v\| \ge \text{const.}\exp[n(\chi_r+o(1))]$ uniformly on the unit ball in U_r. These estimates imply (2.7).

Proof of the technical lemma: We are asked to estimate the norm of the projection of a vector in V_n^r on V_{n+k}^{r+t}. We so this in three steps:

1. $V_n^r \to V_{n+1}^{r+t}$, all $t > 0$;
2. $V_n^r \to V_{n+k}^{r+1}$, all $k > 0$;
3. $V_n^r \to V_{n+k}^{r+t}$, all $t, k > 0$.

Step 1. The technical lemma for $k = 1$: Fix $\delta > 0$, then for all n large enough and for all $r' > r$, if $u \in V_n^r$, then $\|u|V_{n+1}^{r'}\| \le \|u\| \exp(-n(\chi_{r'} - \chi_r - \delta))$.

Proof. Fix ε, and choose $N = N(\varepsilon)$ so large that $t_n^i = e^{\pm n\varepsilon} t_i$ for all $n > N$, $i = 1, \ldots, d$. For every $t = 1, \ldots, s$, if $u \in V_n^r$, then

$$\begin{aligned}
\|A_{n+1}u\| &= \sqrt{\langle A_{n+1}^t A_{n+1}u, u\rangle} \\
&= \sqrt{\langle A_{n+1}^t A_{n+1}(u|\widetilde{V}_{n+1}^{r+t}), (u|\widetilde{V}_{n+1}^{r+t})\rangle + \langle A_{n+1}^t A_{n+1}(u|V_{n+1}^{r+t-1}), (u|V_{n+1}^{r+t-1})\rangle} \\
&\quad \text{(because } V_{n+1}^{r+t-1}, \widetilde{V}_{n+1}^{r+t} \text{ are orthogonal, } A_{n+1}^t A_{n+1}\text{–invariant,} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \mathbb{R}^d = V_{n+1}^{r+t-1} \oplus \widetilde{V}_{n+1}^{r+t}) \\
&= \sqrt{\|A_{n+1}(u|\widetilde{V}_{n+1}^{r+t})\|^2 + \|A_{n+1}(u|V_{n+1}^{r+t-1})\|^2} \\
&\ge \|A_{n+1}(u|\widetilde{V}_{n+1}^{r+t})\| = e^{(\chi_{r+t} \pm \varepsilon)(n+1)} \|u|\widetilde{V}_{n+1}^{r+t}\|.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|A_{n+1}u\| = \|A(T^n x)A_n(x)u\| &\le \|A(T^n x)\| \sqrt{\langle A_n^t A_n u, u\rangle} \\
&\le \|A(T^n x)\| e^{n(\chi_r \pm \varepsilon)} \|u\| \\
&= e^{n(\chi_r \pm \varepsilon) + o(n)} \|u\|,
\end{aligned}$$

because by the ergodic theorem

$$\frac{1}{n} \log \|A(T^n x)\| = \frac{1}{n} \sum_{k=0}^{n} \log \|A(T^k x)\| - \frac{1}{n} \sum_{k=0}^{n-1} \log \|A(T^k x)\| \xrightarrow[n\to\infty]{} 0 \text{ a.e.}$$

By further increasing N, we can arrange $|o(n)| < n\varepsilon$, which gives

$$e^{(\chi_{r+t} - \varepsilon)(n+1)} \|u|\widetilde{V}_{n+1}^{r+t}\| \le e^{n(\chi_r + 2\varepsilon)},$$

whence $\|u|\widetilde{V}_{n+1}^{r+t}\| \le e^{-n(\chi_{r+t} - \chi_r - 3\varepsilon)}$. Now take $\varepsilon := \delta/3$.

Step 2. Fix $\delta > 0$. Then for all n large enough and for all k, if $u \in V_n^r$, then $\|u|\widetilde{V}_{n+k}^{r+1}\| \le \|u\| \sum_{j=0}^{k-1} \exp(-(n+j)(\chi_{r+1} - \chi_r - \delta))$. Thus $\exists K_1$ s.t.

$$\|u|\widetilde{V}_{n+k}^{r+1}\| \le K_1 \|u\| \exp[-n(\chi_{r+1} - \chi_r - \delta)].$$

Proof. We use induction on k. The case $k=1$ is dealt with in step 1. We assume by induction that the statement holds for $k-1$, and prove it for k. Decompose

$$u|\widetilde{V}^{r+1}_{n+k} = [(u|V^r_{n+k-1})|\widetilde{V}^{r+1}_{n+k}] + [(u|\widetilde{V}^{r+1}_{n+k-1})|\widetilde{V}^{r+1}_{n+k}].$$

- *First summand*: $u|V^r_{n+k-1} \in V^r_{n+k-1}$, so by step 1 the norm of the first summand is less than $\|u|V^r_{n+k-1}\|\exp[-(n+k-1)(\chi_{r+1}-\chi_r-\delta)]$, whence less than $\|u\|\exp[-(n+k-1)(\chi_{r+1}-\chi_r-\delta)]$.
- *Second summand:* The norm is at most $\|u|\widetilde{V}^{r+1}_{n+k-1}\|$. By the induction hypothesis, this is less than $\|u\|\sum_{j=0}^{k-2}\exp(-(n+j)(\chi_{r+1}-\chi_r-\delta))$.

We get the statement for k, and step 2 follows by induction.

As a result, we obtain the existence of a constant $K_1>1$ for which $u\in V^r_n$ implies $\|u|\widetilde{V}^{r+1}_{n+k}\| \le K_1\|u\|\exp(-n(\chi_{r+1}-\chi_r-\delta))$.

Step 3. $\exists K_1,\dots,K_{s-1}>1$ s.t. for all n large enough and for all k, $u\in V^r_n$ implies $\|u|\widetilde{V}^{r+\ell}_{n+k}\| \le K_\ell\|u\|\exp(-n(\chi_{r+\ell}-\chi_r-\ell\delta))$ ($\ell=1,\dots,s-r$).

Proof. We saw that K_1 exists. We assume by induction that $K_1,\dots,K_{t-1}$ exist, and construct K_t. Fix $0<\delta_0<\min_j\{\chi_{j+1}-\chi_j\}-\delta$; the idea is to first prove that if $u\in V^r_n$, then

$$\|u|\widetilde{V}^{r+t}_{n+k}\| \le \|u\|\left(\sum_{\tau=1}^{t-1}K_\tau\right)\left(\sum_{j=0}^{k-1}e^{-\delta_0 j}\right)\left(\sum_{j=0}^{k-1}\exp[-(n+j)(\chi_{r+t}-\chi_r-t\delta)]\right) \tag{2.9}$$

Once this is done, step 3 follows with $K_t := \left(\sum_{\tau=1}^{t-1}K_\tau\right)\left(\sum_{j\ge 0}e^{-\delta_0 j}\right)^2$.

We prove (2.9) using induction on k. When $k=1$ this is because of step 1. Suppose, by induction, that (2.9) holds for $k-1$. Decompose:

$$u|\widetilde{V}^{r+t}_{n+k} = \underbrace{u|V^r_{n+k-1}|\widetilde{V}^{r+t}_{n+k}}_{A} + \underbrace{\sum_{r<r'<r+t} u|U^{r'}_{n+k-1}|\widetilde{V}^{r+t}_{n+k}}_{B} + \underbrace{u|\widetilde{V}^{r+t}_{n+k-1}|\widetilde{V}^{r+t}_{n+k}}_{C}$$

- *Estimate of* $\|A\|$: By step 1, $\|A\| \le \|u\|\exp(-(n+k-1)(\chi_{r+t}-\chi_r-\delta))$.
- *Estimate of* $\|B\|$: By step 1, and the induction hypothesis (on t):

$$\begin{aligned}
\|B\| &\le \sum_{r<r'<r+t}\|u|U^{r'}_{n+k-1}\|\exp(-(n+k-1)(\chi_{r+t}-\chi_{r'}-\delta))\\
&\le \sum_{r<r'<r+t}\|u|\widetilde{V}^{r'}_{n+k-1}\|\exp(-(n+k-1)(\chi_{r+t}-\chi_{r'}-\delta))\\
&\le \sum_{r<r'<r+t}K_{r'-r}\|u\|\exp(-n(\chi_{r'}-\chi_r-(r'-r)\delta))\times\\
&\qquad\qquad\times\exp(-(n+k-1)(\chi_{r+t}-\chi_{r'}-\delta))
\end{aligned}$$

$$\leq \|u\| \left(\sum_{t'=1}^{t-1} K_{t'}\right) e^{-(k-1)(\chi_{r+t}-\chi_{r'}-\delta)} \exp(-n(\chi_{r+t}-\chi_r-t\delta))$$
$$\leq \|u\| \left(\sum_{t'=1}^{t-1} K_{t'}\right) e^{-\delta_0(k-1)} \exp(-n(\chi_{r+t}-\chi_r-t\delta)).$$

- *Estimate of* $\|C\|$: $\|C\| \leq \|u|\widetilde{V}_{n+k-1}^{r+t}\|$. By the induction hypothesis on k,

$$\|C\| \leq \|u\| \left(\sum_{t'=1}^{t-1} K_{t'}\right) \left(\sum_{j=0}^{k-2} e^{-\delta_0 j}\right) \left(\sum_{j=0}^{k-2} \exp[-(n+j)(\chi_{r+t}-\chi_r-t\delta)]\right).$$

It is not difficult to see that when we add these bounds for $\|C\|, \|B\|$ and $\|A\|$, the result is smaller than the RHS of (2.9) for k. This completes the proof by induction of (2.9). As explained above, step 3 follows by induction. □

Corollary 2.3. *Let $\chi_1(x) < \cdots < \chi_{s(x)}(x)$ denote the logarithms of the (different) eigenvalues of $\Lambda(x)$. Let U_{χ_i} be the eigenspace of $\Lambda(x)$ corresponding to* $\exp \chi_i$. *Set $V_\chi := \bigoplus_{\chi' \leq \chi} U_{\chi'}$.*

1. *$\chi(x,v) := \lim\limits_{n\to\infty} \frac{1}{n} \log \|A_n(x)v\|$ exists a.s, and is invariant.*
2. *$\chi(x,v) = \chi_i$ on $V_{\chi_i} \setminus V_{\chi_{i-1}}$*
3. *If $\|A^{-1}\|, \|A\| \in L^\infty$, then $\frac{1}{n} \log|\det A_n(x)| = \sum k_i \chi_i$, where $k_i = \dim U_{\chi_i}$.*

$\{\chi_i(x)\}$ are called the Lyapunov exponents *of x. $\{V_{\chi_i}\}$ is called the* Lyapunov filtration *of x. Property (2) implies that $\{V_\chi\}$ is A–invariant: $A(x)V_\chi(x) = V_\chi(Tx)$. Property (3) is sometimes called* regularity.

Remark: $V_{\chi_i} \setminus V_{\chi_{i-1}}$ is A–invariant, but if $A(x)$ is not orthogonal, then U_{χ_i} doesn't need to be A–invariant. When T is invertible, there is a way of writing $V_{\chi_i} = \bigoplus_{j \leq i} H_j$ so that $A(x)H_j(x) = H_j(Tx)$ and $\chi(x,\cdot) = \chi_j$ on $H_j(x)$, see the next section.

2.6.3 The Multiplicative Ergodic Theorem for Invertible Cocycles

Suppose $A : X \to \mathrm{GL}(n,\mathbb{R})$. There is a unique extension of the definition of $A_n(x)$ to non-positive n's, which preserves the cocycle identity: $A_0 := id$, $A_{-n} := (A_n \circ T^{-n})^{-1}$. (Start from $A_{n-n} = A_0 = id$ and use the cocycle identity.)

The following theorem establishes a compatibility between the Lyapunov spectra and filtrations of A_n and A_{-n}.

Theorem 2.11. *Let $(X, \mathscr{B}, m, T)$ be an invertible probability preserving transformation, and $A : X \to \mathrm{GL}(d,\mathbb{R})$ a Borel function s.t.* $\ln\|A(x)^{\pm 1}\|$. *There are invariant Borel functions $p(x)$, $\chi_1(x) < \cdots < \chi_{p(x)}(x)$, and a splitting $\mathbb{R}^d = \bigoplus_{i=1}^{p(x)} H^i(x)$ s.t.*

1. *$A_n(x)H^i(x) = H^i(T^n x)$ for all $n \in \mathbb{Z}$*

2. $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|A_n(x)v\| = \pm\chi_i(x)$ *on the unit sphere in* $H^i(x)$.
3. $\frac{1}{n} \log \sin \measuredangle(H^i(T^n x), H^j(T^n x)) \xrightarrow[n\to\infty]{} 0$.

Proof. Fix x, and let Let $t_n^1 \leq \cdots \leq t_n^d$ and $\bar{t}_n^1 \leq \cdots \leq \bar{t}_n^d$ be the eigenvalues of $(A_n^t A_n)^{1/2}$ and $(A_{-n}^t A_{-n})^{1/2}$. Let $t_i := \lim (t_n^i)^{1/n}$, $\bar{t}_i = \lim (\bar{t}_n^i)^{1/n}$. These limits exists almost surely, and $\{\log t_i\}, \{\log \bar{t}_i\}$ are lists of the Lyapunov exponents of A_n and A_{-n}, repeated with multiplicity. The proof of the Oseledets theorem shows that

$$\begin{aligned}
\sum_{k=d-i+1}^{d} \log \bar{t}_i &\equiv \lim_{n\to\infty} \frac{1}{n} \log \|A_{-n}^{\wedge i}\| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(|\det A_{-n}| \cdot \|((A_{-n})^{-1})^{\wedge(d-i)}\| \right) \quad \text{(write using e.v.'s)} \\
&\equiv \lim_{n\to\infty} \frac{1}{n} \log \left(|(\det A_n \circ T^{-n})|^{-1} \cdot \|A_n^{\wedge(d-i)} \circ T^{-n}\| \right) \\
&\equiv \lim_{n\to\infty} \frac{1}{n} \log \left(|\det A_n|^{-1} \cdot \|A_n^{\wedge(d-i)}\| \right) \quad \text{(remark after Kingman's Theorem)} \\
&= \sum_{k=d-i+1}^{d} \log t_k - \sum_{k=1}^{d} \log t_k = -\sum_{k=1}^{d-i} \log t_k.
\end{aligned}$$

Since this is true for all i, $\log t_i = -\log \bar{t}_{d-i+1}$.

It follows that if the Lyapunov exponents of A_n are $\chi_1 < \ldots < \chi_s$, then the Lyapunov exponents of A_{-n} are $-\chi_s < \cdots < -\chi_1$.

Let $V^1(x) \subset V^2(x) \subset \cdots \subset V^s(x)$ be the Lyapunov filtration of A_n:

$$V^i(x) := \{v : \lim_{n\to\infty} \frac{1}{n} \log \|A_n(x)v\| \leq \chi_i(x)\}.$$

Let $\overline{V}^1(x) \supset \overline{V}^2(x) \supset \cdots \supset \overline{V}^s(x)$ be the following *decreasing* filtration, given by

$$\overline{V}^i(x) := \{v : \lim_{n\to\infty} \frac{1}{n} \log \|A_{-n}(x)v\| \leq -\chi_i(x)\}.$$

These filtrations are invariant: $A(x)V^i(x) = V^i(Tx)$, $A(x)\overline{V}^i(x) = \overline{V}^i(Tx)$.

Set $H^i(x) := V^i(x) \cap \overline{V}^i(x)$. We must have $A(x)H^i(x) = H^i(Tx)$.

We claim that $\mathbb{R}^d = \bigoplus H^i(x)$ almost surely. It is enough to show that for a.e. x, $\mathbb{R}^d = V^i(x) \oplus \overline{V}^{i+1}(x)$, because

$$\begin{aligned}
\mathbb{R}^d \equiv \overline{V}^1 &= \overline{V}^1 \cap [V^1 \oplus \overline{V}^2] && (V^1 \oplus \overline{V}^2 = \mathbb{R}^d) \\
&= H^1 \oplus [\overline{V}^1 \cap \overline{V}^2] = H^1 \oplus \overline{V}^2 && (\overline{V}^1 \supseteq \overline{V}^2) \\
&= H^1 \oplus [\overline{V}^2 \cap (V^2 \oplus \overline{V}^3)] && (V^2 \oplus \overline{V}^3 = \mathbb{R}^d) \\
&= H^1 \oplus H^2 \oplus \overline{V}^3 = \cdots = H^1 \oplus \cdots \oplus H^s.
\end{aligned}$$

Since the spectra of $\Lambda, \overline{\Lambda}$ agree with matching multiplicities, $\dim V^i + \dim \overline{V}^{i+1} = d$. Thus it is enough to show that $E := \{x : V^i(x) \cap \overline{V}^{i+1}(x) \neq \{0\}\}$ has zero measure for all i.

Assume otherwise, then by the Poincaré recurrence theorem, for almost every $x \in E$ there is a sequence $n_k \to \infty$ for which $T^{n_k}(x) \in E$. By the Oseledets theorem, for every $\delta > 0$, there is $N_\delta(x)$ such that for all $n > N_\delta(x)$,

$$\|A_n(x)u\| \leq \|u\| \exp[n(\chi_i + \delta)] \quad \text{for all } u \in V^i \cap \overline{V}^{i+1}, \tag{2.10}$$

$$\|A_{-n}(x)u\| \leq \|u\| \exp[-n(\chi_{i+1} - \delta)] \quad \text{for all } u \in V^i \cap \overline{V}^{i+1}. \tag{2.11}$$

If $n_k > N_\delta(x)$, then $A_{n_k}(x)u \in V^i(T^{n_k}x) \cap \overline{V}^{i+1}(T^{n_k}x)$ and $T^{n_k}(x) \in E$, so

$$\|u\| = \|A_{-n_k}(T^{n_k}x)A_{n_k}(x)u\| \leq \|A_{n_k}(x)u\| \exp[-n_k(\chi_{i+1} - \delta)],$$

whence $\|A_{n_k}(x)u\| \geq \|u\| \exp[n_k(\chi_{i+1} - \delta)]$. By (2.10),

$$\exp[n_k(\chi_{i+1} + \delta)] \leq \exp[n_k(\chi_i + \delta)],$$

whence $|\chi_{i+1} - \chi_i| < 2\delta$. But δ was arbitrary, and could be chosen to be much smaller than the gaps between the Lyapunov exponents. With this choice, we get a contradiction which shows that $m(E) = 0$.

Thus $\mathbb{R}^d = \bigoplus H^i(x)$. Evidently, $V^i = V^i \supseteq \bigoplus_{j \leq i} H^j$ and $V^i \cap \bigoplus_{j > i} H^j \subseteq V^i \cap \overline{V}^{i+1} = \{0\}$, so $V^i = \bigoplus_{j \leq i} H^j$. In the same way $\overline{V}^i = \bigoplus_{j \geq i} H^j$. It follows that $H^i \subset (V^i \setminus V^{i-1}) \cap (\overline{V}^i \setminus \overline{V}^{i+1})$. Thus $\lim\limits_{n \to \pm\infty} \frac{1}{|n|} \log \|A_n v\| = \pm \chi_i$ on the unit sphere in H^i.

Next we study the angle between $H^i(x)$ and $\widetilde{H}^i(x) := \bigoplus_{j \neq i} H^j(x)$. Pick a basis $(v_1^i, \ldots, v_{m_i}^i)$ for $H^i(x)$. Pick a basis $(w_1^i, \ldots, w_{m_i}^i)$ for $\widetilde{H}^i(x)$. Since $A_n(x)$ is invertible, $A_k(x)$ maps $(v_1^i, \ldots, v_{m_i}^i)$ onto a basis of $H^i(T^k x)$, and $(w_1^i, \ldots, w_{m_i}^i)$ onto a basis of $\widetilde{H}^i(T^k x)$. Thus if $v := \bigwedge v_j^i$, $w := \bigwedge w_j^i$, then

$$|\sin \measuredangle(H^i(T^k x), \widetilde{H}(T^k x))| \geq \frac{\|A_n(x)^{\wedge d}(v \wedge w)\|}{\|A_n(x)^{\wedge m_i} v\| \cdot \|A_n(x)^{\wedge (d-m_i)} w\|}.$$

We view $A_n^{\wedge p}$ as an invertible matrix acting on $\mathrm{span}\{e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* : i_1 < \cdots < i_p\}$ via $(A_n(x)e_{i_1})^* \wedge \cdots \wedge (A_n(x)e_{i_p})^*$. It is clear

$$\Lambda_p(x) := \lim_{n \to \infty} ((A_n^{\wedge p})^*(A_n^{\wedge p}))^{1/2n} = \left(\lim_{n \to \infty} (A_n^* A_n)^{1/2n}\right)^{\wedge p} = \Lambda(x)^{\wedge p},$$

thus the eigenspaces of $\Lambda_p(x)$ are the tensor products of the eigenspaces of $\Lambda(x)$. This determines the Lyapunov filtration of $A_n(x)^{\wedge p}$, and implies – by Oseledets theorem – that if $v_j \in V_{\chi_{k(j)}} \setminus V_{\chi_{k(j)-1}}$, and $v_1, \ldots, v_k$ are linearly independent, then

$$\lim_{n\to\infty}\frac{1}{n}\log\|A_n(x)^{\wedge p}\omega\| = \sum_{j=1}^{p}\chi_{k(j)},\ \text{for } \omega := v_1\wedge\cdots\wedge v_p.$$

It follows that $\lim\limits_{n\to\infty}\frac{1}{n}\log|\sin\measuredangle(H^i(T^n x),\widetilde{H}^i(T^n x))| \geq 0$. $\square$

2.7 The Karlsson-Margulis ergodic theorem

Suppose (X,d) is a "nice" metric space, and $\mathrm{Isom}(X)$ is the group of isometries of X. Let $(\Omega,\mathscr{F},\mu,T)$ be a ppt and $f:\Omega\to\mathrm{Isom}(X)$ be a measurable map such that for some (any) $x_0\in X$, $\int_X d(f(\omega)\cdot x,x_0)d\mu(\omega)<\infty$.

Consider the "random walk" $x_n(\omega) := f(T^{n-1}\omega)\circ\cdots\circ f(\omega)\cdot x_0$. It is not difficult to see that $g^{(n)}(\omega) := d(x_0,x_n(\omega))$ is a sub-additive cocycle. The sub-additive ergodic theorem, implies the existence of almost sure *asymptotic speed* $s(\omega) = \lim\limits_{n\to\infty}\frac{1}{n}d(x_0,x_n(\omega))$. The Karlsson-Margulis theorem provides (under additional assumptions on (X,d)), the existence of an *asymptotic velocity*: A geodesic ray $\gamma_\omega(t)\subset X$ which starts at x_0 s.t. $d(x_n(\omega),\gamma_\omega(s(\omega)n)) = o(n)$ as $n\to\infty$.

Example 1 (Birkhoff Ergodic Theorem): Take an ergodic ppt $(\Omega,\mathscr{F},\mu,T)$, $X=\mathbb{R}^d$ and $f(\omega)\cdot\underline{v} := \underline{v}+\underline{f}(\omega)$ where $\underline{f}(\omega) := (f_1(\omega),\ldots,f_d(\omega))$ and $f_i:\Omega\to\mathbb{R}$ are absolutely integrable with non-zero integral. Then

$$\underline{x}_n(\omega) = \underline{x}_0 + n\Big(\frac{1}{n}\sum_{i=0}^{n-1}f_1(T^i\omega),\ldots,\frac{1}{n}\sum_{i=0}^{n-1}f_d(T^i(\omega))\Big)$$

so $s(\omega) = \|\underline{v}\|_2$ a.e. and $\gamma_\omega(t) = t\underline{v}/\|\underline{v}\|$ a.e., where $\underline{v} := (\int f_1 d\mu,\ldots,\int f_d d\mu)$.

Example 2 (Multiplicative Ergodic Theorem): See section 2.7.3 below.

2.7.1 The boundary of a non-compact proper metric space

Let (X,d) be a metric space. We need some terminology:

1. X is called *proper* is every closed bounded subset of X is compact.
2. A *curve* is a continuous function $\gamma:[a,b]\to X$. A curve is called *rectifiable* if

$$\ell(g) := \sup\left\{\sum_{i=0}^{n-1}d(\gamma(t_i),\gamma(t_{i+1})) : a=t_0<t_1<\cdots<t_n=b, n\geq 1\right\} < \infty.$$

   The number $\ell(\gamma)$ is called the *length* of γ.
3. A *geodesic segment* from A to B $(A,B\in X)$ is a curve $\gamma:[0,L]\to X$ s.t. $\gamma(0)=A$, $\gamma(L)=B$, and $d(\gamma(t),\gamma(t')) = |t-t'|$ for all $0\leq t,t'\leq L$. In particular, $L=d(x,y)$. We denote such segments by AB.

4. A *geodesic ray* is a curve $\gamma\colon[0,\infty)\to X$ s.t. $d(\gamma(t),\gamma(t'))=|t-t'|$ for all t,t'.
5. A metric space is called a *geodesic space*, if every $A,B\in X$ are connected by a geodesic segment.
6. A geodesic metric space is called *complete* if every geodesic segment can be extended to a geodesic ray in the positive direction.

Suppose X is a non–compact proper geodesic metric space. We are interested in describing the different ways of "escaping to infinity" in X.

The idea is to compactify X by adding to it a "boundary" so that tending to infinity in X in a certain "direction" corresponds to tending to a point in the boundary of X in $\widehat{X}$.

Fix once and for all a reference point $x_0\in X$ (the "origin"), and define for each $x\in X$ the function

$$D_x(z):=d(z,x)-d(x_0,x).$$

$D_x(\cdot)$ is has Lipschitz constant 1, and $D_x(x_0)=0$. It follows that $\{D_x(\cdot):x\in X\}$ is equicontinuous and uniformly bounded on compact subsets of X.

By the Arzela–Ascoli theorem, every sequence $\{D_{x_n}\}_{n\geq 1}$ has a subsequence $\{D_{x_{n_k}}\}_{k\geq 1}$ which converges pointwise (in fact uniformly on compacts). Let

$$\widehat{X}:=\{\lim_{n\to\infty}D_{x_n}(\cdot):\{x_n\}_{n\geq 1}\subset X,\{D_{x_n}\}_{n\geq 1}\text{ converges uniformly on compacts}\}.$$

We put a metric on $\widehat{X}$ as follows:

$$d(f,g):=\sum_{n=1}^{\infty}\frac{1}{2^n}\varphi(\max_{d(x_0,x)\leq n}|f(x)-g(x)|),$$

where $\varphi(t):=\frac{1}{\pi}\arctan(t)+\frac{1}{2}$ (a homeomorphism $\varphi:\mathbb{R}\to(0,1)$). This metric is well-defined, since X is proper. The resulting topology is the topology of uniform convergence on compacts.

Theorem 2.12. *Suppose (X,d) is a proper geodesic space, then $\widehat{X}$ is a compact metric space, and if $\imath:X\hookrightarrow\widehat{X}$ is the map $\imath(x)=D_x(\cdot)$, then*

1. $\imath:X\to\imath(X)$ is a homeomorphism;
2. $\imath(X)$ is dense in $\widehat{X}$.

Proof. $\widehat{X}$ is compact, because it is the closure of $\{D_x(\cdot):x\in X\}$, which is precompact by Arzela–Ascoli.

The map $\imath:x\mapsto D_x(\cdot)$ is one-to-one because x can be read of $D_x(\cdot)$ as the unique point where that function attains its minimum. The map $\imath$ is continuous, because if $d(x_n,x)\to 0$, then

$$\begin{aligned}|D_{x_n}(z)-D_x(z)|&\leq|d(z,x_n)-d(z,x)|+|d(x_n,x_0)-d(x,x_0)|\\&\leq 2d(x_n,x)\xrightarrow[n\to\infty]{}0.\end{aligned}$$

To see that ι^{-1} is continuous, we first note that since $\widehat{X}$ is metrizable, it is enough to show that if $D_{x_n} \to D_x$ uniformly on compacts, then $x_n \to x$ in X. Suppose $D_{x_n} \to D_x$ uniformly on compacts, and fix $\varepsilon > 0$. Suppose by way of contradiction that $\exists n_k \uparrow \infty$ s.t. $d(x_{n_k}, x) \geq \varepsilon$. Construct y_{n_k} on the geodesic segment connecting x to x_{n_k} s.t. $d(x, y_{n_k}) = \varepsilon/2$. We have

$$\begin{aligned} D_{x_{n_k}}(y_{n_k}) &= d(y_{n_k}, x_{n_k}) - d(x_{n_k}, x_0) = d(x, x_{n_k}) - d(x, y_{n_k}) - d(x_{n_k}, x_0) \\ &= D_{x_{n_k}}(x) - \frac{\varepsilon}{2}. \end{aligned}$$

Since $d(y_{n_k}, x) = \varepsilon/2$ and X is proper, y_{n_k} lie in a compact subset of X. W.l.o.g. $y_{n_k} \xrightarrow[k\to\infty]{} y \in X$. Passing to the limit we see that $D_x(y) = D_x(x) - \frac{\varepsilon}{2} < D_x(x)$. But this is absurd, since D_x attains its minimum at x. It follows that $x_n \to x$. $\square$

Terminology: $\widehat{X}$ is called the *horofunction compactification* of X, and $\partial X := \widehat{X} \setminus \iota(X)$ is called the *horofunction boundary* of X. Elements of ∂X are called *horofunctions*.

The horofunction compactification has a very nice geometric interpretation in case (X, d) has "non–positive curvature", a notion we now proceed to make precise.

Suppose (X, d) is a geodesic space, then any three points $A, B, C \in X$ determine a *geodesic triangle* $\triangle ABC$ obtained by connecting A, B, C by geodesic segments. A euclidean triangle $\triangle\overline{ABC} \subset \mathbb{R}^2$ is called a (euclidean) *comparison triangle* for $\triangle ABC$ if it has the same lengths:

$$d(A,B) = d_{\mathbb{R}^2}(A,B), d(B,C) = d_{\mathbb{R}^2}(B,C), d(C,A) = d_{\mathbb{R}^2}(C,A).$$

A point $\bar{x} \in \overline{AB}$ is called a *comparison point* for $x \in AB$, if $d(x, A) = d_{\mathbb{R}^2}(\bar{x}, \overline{A})$ and $d(x, B) = d_{\mathbb{R}^2}(\bar{x}, \overline{B})$.

Definition 2.3. A geodesic metric space (X, d) is called a *CAT(0)* space if for any geodesic triangle $\triangle ABC$ in X and points $x \in AC$, $y \in BC$, if $\triangle\overline{A}\,\overline{B}\,\overline{C}$ is a euclidean comparison triangle for $\triangle ABC$, and $\bar{x} \in \overline{A}\,\overline{C}$ and $\bar{y} \in \overline{B}\,\overline{C}$ are comparison points to $x \in AB$ and $y \in BC$, then $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$. (See figure 2.1.)

Theorem 2.13. *Suppose (X, d) is a CAT(0) complete proper geodesic space.*

1. *If γ is a geodesic ray s.t. $\gamma(0) = x_0$, then the following limit exists, and is a horofunction:*
$$B_\gamma(z; x_0) = \lim_{t\to\infty}[d(\gamma(t), z) - d(\gamma(t), x_0)].$$
2. *Every horofunction arises this way.*
3. *If two geodesic rays γ, γ' s.t. $\gamma(0) = \gamma'(0) = x_0$ determine the same horofunction, then they are equal.*

Thus horofunctions are represented by geodesic rays emanating from x_0.

Proof.

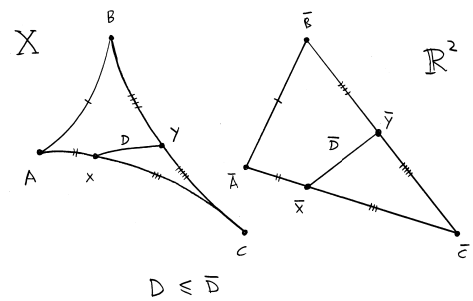

Fig. 2.1 The CAT(0) inequality

Part 1. Existence of $B_\gamma(z;x_0)$.

Suppose γ is a geodesic ray s.t. $\gamma(0) = x_0$.

1. $t \mapsto d(\gamma(t),z) - d(\gamma(t),x_0)$ is decreasing: If $t < s$, then

$$\begin{aligned} d(\gamma(t),z) - d(\gamma(t),x_0) &\geq [d(\gamma(s),z) - d(\gamma(s),\gamma(t))] - d(\gamma(t),x_0) \quad \text{(triangle ineq.)} \\ &= d(\gamma(s),z) - (s-t) - t = d(\gamma(s),z) - s \\ &= d(\gamma(s),z) - d(\gamma(s),x_0). \end{aligned}$$

2. $t \mapsto d(\gamma(t),z) - d(\gamma(t),x_0)$ is bounded below, by $d(x_0,z)$.

It follows that the limit which defines $B_\gamma(z;x_0)$ exists pointwise. By the Arzela–Ascoli theorem, this limit holds uniformly on compacts, so $B_\gamma \in \widehat{X}$.

To see that $B_\gamma \in \partial X$, we note that $B_\gamma(\gamma(s);x_0) = -s \xrightarrow[s\to\infty]{} -\infty$ whereas every function of the form $D_x(\cdot)$ is bounded from below. Thus $B_\gamma \in \widehat{X} \setminus \imath(X) = \partial X$.

Part 2. Every horofunction is equal to B_γ for some geodesic ray γ.

Suppose D is a horofunction, and write $D = \lim\limits_{k\to\infty} D_{x_k}$. We must have $x_k \to \infty$ (i.e. x_k leaves every compact set), otherwise, since X is proper, there is a convergent subsequence $x_{k_i} \to x$. But in this case (cf. the proof of the previous theorem) $D = \lim D_{x_{k_i}} = D_x \in \imath(X)$, whereas we are assuming that D is a horofunction.

We show that the geodesic segments x_0x_n converge to a geodesic ray γ s.t. $D(\cdot) = B_\gamma(\,\cdot\,;x_0)$, and then prove that $D = B_\gamma(\,\cdot\,;x_0)$.

Step 1. Let $\gamma_n(t)$ denote the geodesic ray which starts at x_0 and passes through x_n (it exists since X is complete) , then $\gamma_n(t) \to \gamma(t)$ uniformly on compacts in $[0,\infty)$, where $\gamma(t)$ is geodesic ray.

Proof. Fix $\varepsilon > 0$ and N so large that if $k > N$, then $d(x_k, x_0) > t$ and $|D_{x_k}(z) - D(z)| < \varepsilon$ for all z s.t. $d(z, x_0) \leq t$. Let $y_k := \gamma_k(t)$, the point on the geodesic segment $x_0 x_k$ at distance t from x_0. We show that $\{y_k\}_{k \geq 1}$ is a Cauchy sequence.

Fix $m, n > N$, and construct the geodesic triangle $\triangle x_m y_n x_0$. Let $\triangle \overline{x}_m \overline{y}_n \overline{x}_0$ be its euclidean comparison triangle. Let $\overline{y}_m \in [\overline{x}_m, \overline{x}_0]$ be the comparison point to y_m on the geodesic segment from x_m to x_0. By the CAT(0) property,

$$d(y_m, y_n) \leq |\overline{y}_m \overline{y}_n|.$$

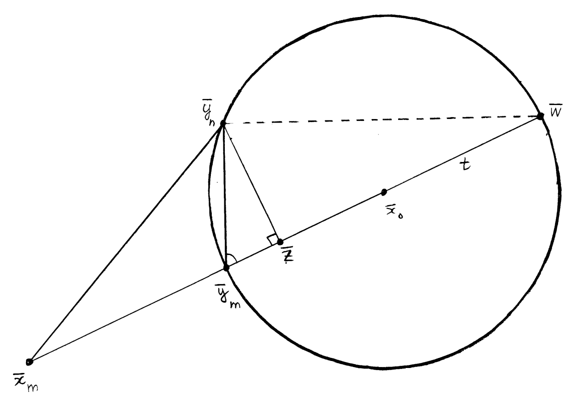

Fig. 2.2

Working in the euclidean plane, we drop a height $\overline{y}_n \overline{z}$ to the line connecting $\overline{x}_m$ to $\overline{x}_0$, and mark the point $\overline{w}$ at distance $2t$ from $\overline{y}_m$ on the line passing through $\overline{x}_m$ and $\overline{x}_0$ (figure 2.2). Let $\theta := \measuredangle \overline{y}_n \overline{y}_m \overline{x}_0$, then

$$\frac{|\overline{y}_n \overline{y}_m|}{\overline{y}_m \overline{z}} = \frac{1}{\cos\theta} = \frac{2t}{|\overline{y}_n \overline{y}_m|}.$$

It follows that $|\overline{y}_n \overline{y}_m| \leq \sqrt{2t|\overline{y}_m \overline{z}|}$.

$$\begin{aligned}
|\bar{y}_m\bar{z}| &= |\bar{x}_m\bar{z}| - |\bar{x}_m\bar{y}_m| \le |\bar{x}_m\bar{y}_n| - |\bar{x}_m\bar{y}_m| \quad (\because |\bar{x}_m\bar{z}| \text{ is the hypotenuse in } \triangle\bar{x}_m\bar{z}\bar{y}_n)\\
&= d(x_m,y_n) - d(x_m,y_m) = d(x_m,y_n) - d(x_m,x_0) + d(x_m,x_0) - d(x_m,y_m)\\
&= d(x_m,y_n) - d(x_m,x_0) + t\\
&= d(x_m,y_n) - d(x_m,x_0) - [d(x_n,y_n) - d(x_n,x_0)]\\
&= D_{x_m}(y_n) - D_{x_n}(y_n)\\
&\le \sup_{d(y,x_0)\le t} |D_{x_m}(y) - D_{x_n}(y)| \xrightarrow[m,n\to\infty]{} 0 \text{ by assumption.}
\end{aligned}$$

This shows that $\{\gamma_n(t)\}_{n\ge1} = \{y_n\}_{n\ge1}$ is a Cauchy sequence. Moreover, the Cauchy criterion holds uniformly on compact subsets of $t \in [0,\infty)$.

The limit $\gamma(t) = \lim \gamma_n(t)$ must be a geodesic ray emanating from x_0 (exercise).

Step 2. $D(z) = B_\gamma(z;x_0)$.

Proof. Let $t_n := d(x_0,x_n)$, and define $\xi_n := \begin{cases} \gamma(t_n) & n \text{ is odd} \\ x_n & n \text{ is even.} \end{cases}$ Then $D_{\xi_{2n}}(z) \to D(z)$ and $D_{\xi_{2n-1}}(z) \to B_\gamma(z;x_0)$.

We use the fact that the geodesic segments $x_0\xi_n$ converge to $\gamma(t)$ to show that $|D_{\xi_n} - D_{\xi_{n+1}}| \to 0$ uniformly on compacts. It will follow that $D(z) = B_\gamma(z;x_0)$.

Fix $\varepsilon > 0$ small and $r > \rho > 0$ large. Let η_k denote the point on the segment $\xi_k x_0$ at distance r from x_0, then

$$\begin{aligned}
|D_{\xi_k}(z) - D_{\xi_{k+1}}(z)| &= |d(\xi_k,z) - d(\xi_k,x_0) - d(\xi_{k+1},z) + d(\xi_{k+1},x_0)|\\
&= \big|d(\xi_k,z) - [d(\xi_k,\eta_k) + r] - d(\xi_{k+1},z) + [d(\xi_{k+1},\eta_{k+1}) + r]\big|\\
&= \big|d(\xi_k,z) - d(\xi_k,\eta_k) - d(\xi_{k+1},z) + d(\xi_{k+1},\eta_{k+1})\big|\\
&\le \big|d(\xi_k,z) - [d(\xi_k,\eta_k) + d(\eta_k,z)]\big|\\
&\quad + \big|[d(\eta_{k+1},z) + d(\xi_{k+1},\eta_{k+1})] - d(\xi_{k+1},z)\big|\\
&\quad + |d(\eta_k,z) - d(\eta_{k+1},z)|
\end{aligned}$$

The last summand tends to zero because $\eta_k \to \gamma(r)$. We show that the other two summands are small for all z s.t. $d(z,x_0) \le \rho$.

Let $\triangle\overline{\xi}_k\bar{x}_0\bar{z}$ be a euclidean comparison triangle for $\triangle\xi_k x_0 z$, and let $\overline{\eta}_k \in \overline{\xi}_k\bar{x}_0$ be a comparison point to $\eta_k \in \xi_k x_0$. Let $\bar{z}'$ be the projection of $\bar{z}$ in the line passing through $\overline{\xi}_k\bar{x}_0$ (figure 2.3).

By the CAT(0) property, $d(\eta_k,z) \le d(\overline{\eta}_k,\bar{z})$, and so

$$\begin{aligned}
[d(\xi_k,\eta_k) + d(\eta_k,z)] - d(\xi_k,z) &\le [d(\overline{\xi}_k,\overline{\eta}_k) + d(\overline{\eta}_k,\bar{z})] - d(\overline{\xi}_k,\bar{z})\\
&= [d(\overline{\xi}_k,\bar{z}') - d(\overline{\xi}_k,\bar{z})] + [d(\overline{\eta}_k,\bar{z}) - d(\overline{\eta}_k,\bar{z}')]
\end{aligned}$$

We now appeal to the following simple consequence of the Pythagorean Theorem: In a triangle $\triangle ABC$ s.t. $\measuredangle ABC = 90^\circ$, if $|AB| > r$ and $|BC| \le \rho$, then $0 \le |AC| - |AB| < \rho^2/r$. Applying this to $\triangle\overline{\xi}_k\bar{z}'\bar{z}$ and $\triangle\overline{\eta}_k\bar{z}'\bar{z}$, we see that

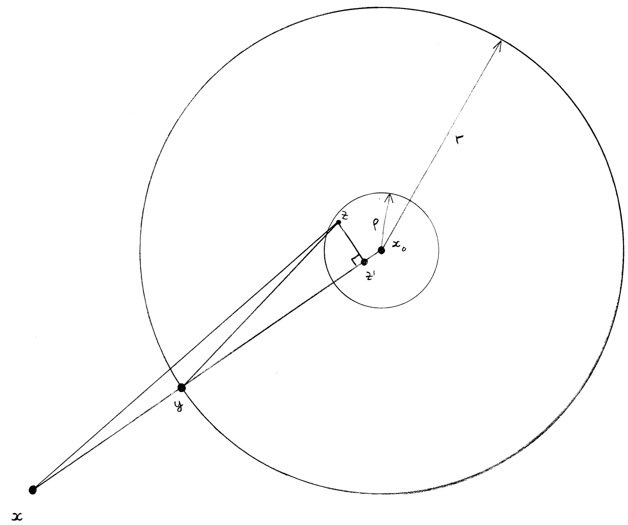

Fig. 2.3

$$d(\xi_k,\eta_k)+d(\eta_k,z)-d(\xi_k,z)\leq 2\rho^2/r.$$

Similarly, one shows that $d(\xi_{k+1},\eta_{k+1})+d(\eta_{k+1},z)-d(\xi_{k+1},z)\leq 2\rho^2/r$. Choosing $r>2\rho^2/\varepsilon$ sufficiently large, and k large enough so that $d(x_k,x_0)>r$ we see that the two remaining terms are less than ε, with the result that $|D_{\xi_{k+1}}(z)-D_{\xi_k}(z)|<3\varepsilon$ for all z s.t. $d(z,x_0)<\rho$.

It follows that $B_\gamma(z;x_0)=\lim D_{\xi_{2n-1}}(z)=\lim D_{\xi_{2n}}(z)=D(z)$.

Part 3. If $B_{\gamma_1}(\cdot\,;x_0)=B_{\gamma_2}(\cdot\,;x_0)$, then $\gamma_1=\gamma_2$.

Proof. Fix $t_n\uparrow\infty$, and set $x_n:=\begin{cases}\gamma_1(t_n) & n \text{ is odd}\\ \gamma_2(t_n) & n \text{ is even}\end{cases}$. The sequences $D_{x_{2n}}(\cdot),D_{x_{2n-1}}(\cdot)$ have the same limit, $B_{\gamma_1}(\cdot\,;x_0)=B_{\gamma_2}(\cdot\,;x_0)$, therefore $\lim D_{x_n}$ exists. Step 1 in Part 2 shows that the geodesic segments x_0x_n must converge uniformly to a geodesic ray $\gamma(t)$. But these geodesic segments lie on γ_1 for n odd and on γ_2 for n even; it follows that $\gamma_1=\gamma_2$. ☐

Proposition 2.6. *Let (X,d) be a proper geodesic space with the CAT(0) property, and suppose $x_n\in X$ tend to infinity at speed s, i.e. $\frac{1}{n}d(x_0,x_n)\xrightarrow[n\to\infty]{}s$. If*

$$\frac{1}{n}D(x_n)\xrightarrow[n\to\infty]{}-s\quad\textit{for}\quad D\in\partial X,$$

then $d(x_n,\gamma(sn))=o(n)$, where γ is the geodesic ray s.t. $D=B_\gamma$.

Proof. Fix an origin $x_0\in X$, and suppose γ is the geodesic ray so that

$$D(z) = B_\gamma(z;x_0) := \lim_{t\to\infty}[d(\gamma(t),z) - d(\gamma(t),x_0)].$$

Fix n and t large, and consider the geodesic triangle $\triangle x_n x_0 \gamma(st)$ and the point $\gamma(sn)$ on the segment from x_0 to $\gamma(st)$. Let $\triangle \overline{x}_n \overline{x}_0 \overline{\gamma(st)}$ be the euclidean comparison triangle, and let $\overline{\gamma}(sn)$ be the comparison point to $\gamma(st)$. By the CAT(0) property,

$$d(x_n,\gamma(sn)) \le d(\overline{x}_n,\overline{\gamma}(sn)).$$

Let $\overline{w}_t$ be the point on the segment connecting $\overline{\gamma(st)}$ to $\overline{x}_0$ at the same distance from $\overline{\gamma(st)}$ as $\overline{x}_n$. Drop a height $\overline{x}_n\overline{z}$ to that segment (figure 2.4).

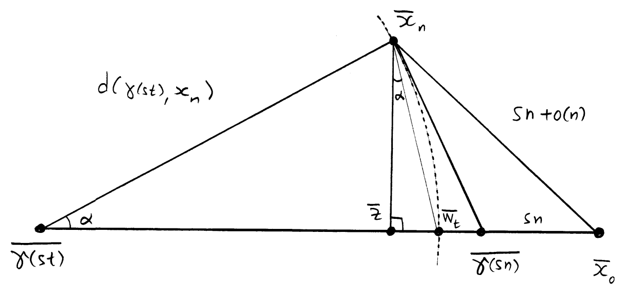

Fig. 2.4

1. • $|\overline{w}_t\,\overline{\gamma(sn)}| = \big||\overline{\gamma(st)}\,\overline{w}_t| + |\overline{\gamma(sn)}\,\overline{x}_0| - |\overline{\gamma(st)}\,\overline{x}_0|\big| = |d(\gamma(st),x_n) + sn - st| \xrightarrow[t\to\infty]{} D(x_n) + sn$. By assumption, $D(x_n)/n \to -s$, so $\exists N_0$ so that for all $n \ge N_0$, for every $t > n$, $|\overline{w}_t\,\overline{\gamma(sn)}| = o(n)$.
   • For fixed n, it is easy to see that $\alpha(t) := \measuredangle \overline{x}_n \overline{\gamma(st)}\overline{z} \xrightarrow[t\to\infty]{} 0$. It follows that $|\overline{z}\,\overline{w}_t| = |\overline{x}_n\,\overline{z}|\tan\frac{\alpha(t)}{2} \xrightarrow[t\to\infty]{} 0$. So $\exists T(n)$ s.t. for all $t > T(n)$, $|\overline{z}\,\overline{w}_t| = o(n)$.

   We see that for all $n > N_0$ and $t > T(n)$, $|\overline{z}\overline{x}_0| = sn + o(n)$.
2. By assumption $|\overline{x}_n\overline{x}_0| = d(x_n,x_0) = sn + o(n)$, so if $t > T(n)$ then

$$|\overline{x}_n\,\overline{z}|^2 = |\overline{x}_n\overline{x}_0|^2 - |\overline{x}_0\overline{z}|^2 = [sn+o(n)]^2 - [sn+o(n)]^2 = o(n^2),$$

   whence $|\overline{x}_n\,\overline{z}| = o(n)$.

3. $|\bar{z}\overline{\gamma(sn)}| = |\bar{z}\bar{x}_0| - |\overline{\gamma(sn)}x_0| = [sn + o(n)] - sn = o(n)$.

It follows from the above that if $t > T(n)$, then

$$d(x_n, \gamma(sn)) \le |\bar{x}_n\overline{\gamma(sn)}| = \sqrt{|\bar{x}_n\bar{z}|^2 + |\bar{z}\overline{\gamma(sn)}|^2} = \sqrt{o(n)^2 + o(n)^2} = o(n). \qquad \square$$

Corollary 2.4. $\partial X = \{D \in \widehat{X} : D$ *is not bounded below*$\}$.

Proof. $D_x(z) = d(z, x_0) - d(x_0, x) \ge -d(x, x_0)$, but $B_\gamma(\gamma(s); x_0) = -s \to -\infty$. $\square$

2.7.2 An ergodic theorem for isometric actions on CAT(0) spaces

Throughout this section, (X,d) is a metric space which is proper, geodesic, geodesically complete, and which has the CAT(0) property. We fix once and for all some point $x_0 \in X$ ("the origin").

A map $\varphi : X \to X$ is called an *isometry*, if it is invertible, and $d(\varphi(x), \varphi(y)) = d(x,y)$ for all $x, y \in X$. The collection of isometries is a group, which we will denote by $\mathrm{Isom}(X)$.

Suppose $(\Omega, \mathscr{B}, \mu, T)$ is a ppt and $f : \Omega \to \mathrm{Isom}(X)$ is measurable, in the sense that $(\omega, x) \mapsto f(\omega)(x)$ is a measurable map $\Omega \times X \to X$. We study the behavior of $f_n(\omega)x_0$ as $n \to \infty$, where

$$f_n(\omega) := f(\omega) \circ f(T\omega) \circ \cdots \circ f(T^{n-1}\omega).$$

The subadditive theorem implies that $\{f_n(\omega)x_0\}_{n \ge 1}$ has "asymptotic speed":

$$s(\omega) := \lim_{n\to\infty} \frac{1}{n} d(x_0, f_n(\omega)x_0).$$

We are interested in the existence of an "asymptotic direction."

Theorem 2.14 (Karlsson–Margulis). *Suppose $(\Omega, \mathscr{B}, \mu, T)$ is a ppt on a standard probability space, and $f : \Omega \to \mathrm{Isom}(X)$ is a measurable map. If $\int_\Omega d(x_0, f(\omega)x_0)d\mu$ is finite, then for a.e. $\omega \in \Omega$ there exists a geodesic ray $\gamma_\omega(t)$ emanating from x_0 s.t.*

$$\frac{1}{n} d(f(\omega)f(T\omega)\cdots f(T^{n-1}\omega)x_0, \gamma_\omega(ns(\omega))) \xrightarrow[n\to\infty]{} 0.$$

Proof (Karlsson and Ledrappier). Some reductions: w.l.o.g. Ω is a compact metric space and $\mathscr{B}$ is the Borel σ-algebra of Ω (cf. appendix A). W.l.o.g. μ is ergodic (otherwise work with its ergodic components). In the ergodic case, $s(\omega) = s$ a.e. where s is a constant. W.l.o.g. $s > 0$, otherwise the theorem holds trivially.

By Proposition 2.6, to prove the theorem it is enough to find a horofunction $D_\omega(\cdot)$ with the property that

$$\lim_{n\to\infty} \frac{1}{n} D_\omega(f_n(\omega)x_0) = -s(\omega). \tag{2.12}$$

The trick is to write the expression $D(f_n(\omega)x_0)$ as an ergodic sum for some other dynamical system, and then apply the pointwise ergodic theorem.

We first extend the action of an isometry φ on X to an action on $\widehat{X}$. Recall that every $D \in \widehat{X}$ equals $\lim_{n\to\infty} D_{x_n}(\cdot)$ where $D_{x_n}(\cdot) = d(x_n,\cdot) - d(x_n,x_0)$. Define

$$\begin{aligned}
\varphi(D)(z) &:= \lim_{n\to\infty} D_{\varphi(x_n)}(z)\\
&= \lim_{n\to\infty} d(\varphi(x_n),z) - d(\varphi(x_n),x_0) = \lim_{n\to\infty} d(x_n,\varphi^{-1}(z)) - d(x_n,\varphi^{-1}(x_0))\\
&= \lim_{n\to\infty} d(x_n,\varphi^{-1}(z)) - d(x_n,x_0) + d(x_n,x_0) - d(x_n,\varphi^{-1}(x_0))\\
&= D(\varphi^{-1}(z)) - D(\varphi^{-1}(x_0)).
\end{aligned}$$

In particular, the definition is independent of the choice of D_{x_n}, so the definition is proper. The identity $\varphi(D) := \lim D_{\varphi(x_n)}$ shows that $\varphi(D) \in \widehat{X}$. We write

$$(\varphi \cdot D)(z) := \varphi(D)(z) := D(\varphi^{-1}(z)) - D(\varphi^{-1}(x_0)).$$

This action preserves ∂X, because by Corollary 2.4, $D \in \partial X$ iff $\inf_X D = -\infty$ iff $\inf_X \varphi \cdot D = \inf D - D(\varphi^{-1}(x_0)) = -\infty$.

Define a map $S : \Omega \times \widehat{X} \to \Omega \times \widehat{X}$ by

$$S : (\omega, D) \mapsto (T(\omega), f(\omega)^{-1} \cdot D).$$

Notice that the second coordinate of the iterate $S^k(\omega,D) = (T^k(\omega), f_k(\omega)^{-1} \cdot D)$ is the horofunction $D(f_k(\omega)z) - D(f_k(\omega)x_0)$. Define $F : \Omega \times \widehat{X} \to \mathbb{R}$ by

$$F(\omega,D) := D(f(\omega)x_0),$$

then $(F \circ S^k)(\omega,D) = F(T^k(\omega), f_k(\omega)^{-1} \cdot D) = (f_k(\omega)^{-1} \cdot D)(f(T^k\omega)x_0)$
$= D(f_{k+1}(\omega)x_0) - D(f_k(\omega)x_0)$. Summing over k, we see that

$$\sum_{k=0}^{n-1} (F \circ S^k)(\omega,D) = D(f_n(\omega)x_0).$$

Technical step: *To construct a probability measure $\widehat{\mu}_0$ on $\Omega \times \widehat{X}$ such that*

1. *$\widehat{\mu}_0$ is S–invariant, S–ergodic, and $\widehat{\mu}_0(\Omega \times \partial X) = 1$;*
2. *$\widehat{\mu}_0$ projects to μ in the sense that $\widehat{\mu}_0(E \times \widehat{X}) = \mu(E)$ for all $E \subset \Omega$ measurable;*
3. *$F \in L^1(\widehat{\mu}_0)$ and $\int F d\widehat{\mu}_0 = -s$.*

We give the details later. First we explain how to use such $\widehat{\mu}_0$ to prove the theorem. By the pointwise ergodic theorem,

$$U := \left\{(\omega, D) \in \Omega \times \partial X : \frac{1}{n} D(f_n(\omega)x_0) \xrightarrow[n\to\infty]{} \int F d\widehat{\mu}_0 = -s\right\}$$

has full $\widehat{\mu}_0$–measure. Let $E := \{\omega \in \Omega : \exists D \in \partial X \text{ s.t. } (\omega, D) \in U\}$, then $E \times \partial X \supseteq U$, whence $\widehat{\mu}_0(E \times \partial X) = 1$. Since $\widehat{\mu}_0$ projects to μ, $\mu(E) = 1$. So for a.e. $\omega \in \Omega, \exists D_\omega \in \partial X$ with property (2.12). The theorem now follows from Proposition 2.14.

We now explain how to carry out the technical step. This requires some functional-theoretic machinery which we now develop.

Let $C(\widehat{X})$ denote the space of continuous functions on $\widehat{X}$ equipped with the supremum norm. Since $\widehat{X}$ is compact and metrizable, $C(\widehat{X})$ is a separable Banach space. By the Riesz representation theorem, the dual space $C(\widehat{X})^*$ can be identified with the space of signed measures on $\widehat{X}$ with finite total variation.

A function $\varphi : \Omega \to C(\widehat{X})$ will be called *measurable*, if $(\omega, D) \mapsto \varphi(\omega)(D)$ is measurable with respect to the Borel structures of $C(\widehat{X})$ and $\Omega \times \widehat{X}$. It is easy to check using the separability of $\widehat{X}$ that in this case $\omega \mapsto \|\varphi(\omega)\|$ is Borel.

Let $\mathscr{L}^1(\Omega, C(\widehat{X}))$ denote the space of measurable functions $\varphi : \Omega \to C(\widehat{X})$ such that $\int \|\varphi(\omega)\| d\mu < \infty$. Identifying φ, ψ s.t. $\int \|\varphi - \psi\| d\mu = 0$, we obtain a linear vector space

$$L^1(\Omega, C(\widehat{X}))$$

of equivalence classes of measurable functions $\varphi : \Omega \to C(\widehat{X})$ with norm $\|\varphi\|_1 := \int \|\varphi(\omega)\| d\mu < \infty$. Here are some basic facts on $L^1(\Omega, C(\widehat{X}))$ [1, chapter 1]:

1. $L^1(\Omega, C(\widehat{X}))$ is a Banach space. We leave this as an exercise.
2. $L^1(\Omega, C(\widehat{X}))$ is separable: Every $\varphi \in L^1(\Omega, C(\widehat{X}))$ can be approximated in norm by a function of the form $\sum_{i=1}^n \varphi_i(D) 1_{E_i}(\omega)$ where φ_i belong to a countable dense subset of $C(\widehat{X})$ and E_i belongs to a countable subset of $\mathscr{F}$.
3. The unit ball in $L^1(\Omega, C(\widehat{X}))^*$ is sequentially compact with respect to the weak-star topology on $L^1(\Omega, C(\widehat{X}))^*$. This follows from the Banach-Alaoglu Theorem.

Notice that $F(\omega, D) := D(f(\omega)x_0)$ belongs to $\mathscr{L}^1(\Omega, C(\widehat{X}))$: (a) For fixed ω, $F(\omega, \cdot)$ is continuous on $\widehat{X}$, because if $D_k \to D$ in $\widehat{X}$ then $D_k \to D$ pointwise (uniformly on compacts); (b) $F(\omega, D)$ is measurable, because it is not difficult to see that $F(\omega, D)$ is a limit of simple functions; (c) $F \in L^1$, because $|D(z)| = |D(z) - D(x_0)| \leq d(z, x_0)$, so $\|F\|_1 \leq \int_\Omega d(f(\omega)x_0, x_0) d\mu(\omega) < \infty$.

We are now ready to construct $\widehat{\mu}_0$.

We begin by noting that $\int (-F) d\widehat{\nu}_0 \leq s$ for all S-ergodic invariant measures $\widehat{\nu}$ which project to μ, so the $\widehat{\mu}_0$ we seek maximizes $\int (-F) d\widehat{\nu}_0$. To see this note that for every $x, z \in X$, $D_x(z) = d(z,x) - d(x_0,x) \geq [d(x_0,x) - d(x_0,z)] - d(x,x_0)$, so

$$D_x(z) \geq -d(x_0,z) \text{ with equality iff } z = x. \tag{2.13}$$

Passing to limits we find that $D(z) \geq -d(x_0,z)$ for all $D \in \widehat{X}$. In particular,

$$\int (-F) d\widehat{\nu} = \int -\frac{1}{n}\sum_{k=0}^{n-1}(F \circ S^k) d\widehat{\nu} = -\frac{1}{n}\int D(f_n(\omega)\cdot x_0) d\widehat{\nu}(\omega, D)$$
$$\leq \frac{1}{n}\int d(x_0, f_n(\omega)\cdot x_0) d\mu, \text{ by (2.13) and the assumption that } \widehat{\nu} \text{ projects to } \mu.$$

As this holds for all n, $\int(-F)d\widehat{\nu} \leq \inf \frac{1}{n}\int d(x_0, f_n(\omega)\cdot x_0) d\mu \overset{!}{=} s$. Equality $\overset{!}{=}$ is due to the subadditive ergodic theorem for the cocycle $d(x_0, f_n(\omega)\cdot x_0)$.

We see that the measure we seek maximizes the expectation value of $(-F)$, or equivalently, of $-\frac{1}{n}\sum_{k=0}^{n-1} F \circ S^k$.

Recall that $-\frac{1}{n}\sum_{k=0}^{n-1} F \circ S^k \equiv -\frac{1}{n} D(f_n(\omega)x_0)$. By (2.13), this expression is maximized at $D = D_{f_n(\omega)x_0}$. Let

$$\widehat{\eta}_n := \int_\Omega \delta_{(\omega, D_{f_n(\omega)x_0})} d\mu(\omega), \text{ where } \delta_{(\omega,D)} := \text{point mass at } (\omega, D).$$

We have $-\frac{1}{n}\int \sum_{k=0}^{n-1} F \circ S^k d\widehat{\eta}_n = -\frac{1}{n}\int D_{f_n(\omega)x_0}(f_n(\omega)x_0) d\mu = \frac{1}{n}\int d(x_0, f_n(\omega)x_0) d\mu$. So $-\frac{1}{n}\int \sum_{k=0}^{n-1} F \circ S^k d\widehat{\eta}_n \geq s$. It is easy to see that $\widehat{\eta}_n$ projects to μ.

But $\widehat{\eta}_n$ is not S–invariant. Let $\widehat{\mu}_n := \frac{1}{n}\sum_{k=0}^{n-1} \widehat{\eta}_n \circ S^{-k}$. This measure still projects to μ (check!); $-\int F d\widehat{\mu}_n = -\frac{1}{n}\int \sum_{k=0}^{n-1} F \circ S^k d\widehat{\eta}_n \geq s$; and $|\widehat{\mu}_n \circ S^{-1} - \widehat{\mu}| \leq 2/n$ (total variation norm).

We may think of $\widehat{\mu}_n$ as of positive bounded linear functional on $L^1(\Omega, \widehat{X})$ with norm 1. By the Banach-Alaoglu Theorem, there is a subsequence μ_{n_k} which converges weak-star in $L^1(\Omega, \widehat{X})^*$ to some positive $\widehat{\mu} \in L^1(\Omega, \widehat{X})^*$. This functional restricts to a functional in $C(\Omega \times \widehat{X})^*$, and therefore can be viewed as a positive finite measure on $\Omega \times \widehat{X}$. Abusing notation, we call this measure $\widehat{\mu}$.

- $\widehat{\mu}$ is a probability measure, because $\widehat{\mu}(1) = \lim \widehat{\mu}_{n_k}(1) = 1$
- $\widehat{\mu}$ projects to μ, because for every $E \in \mathscr{F}$, $g_E(\omega)(D) := 1_E(\omega)$ belongs to $\mathscr{L}^1(\Omega, C(\widehat{X}))$, whence $\widehat{\mu}(E) = \widehat{\mu}(g_E) = \lim \widehat{\mu}_{n_k}(\varphi_E) = \lim \widehat{\mu}_{n_k}(E) = \mu(E)$.
- $\widehat{\mu}$ is S-invariant, because if $g \in C(\Omega \times \widehat{X})$, then $g, g \circ S \in \mathscr{L}^1(\Omega, C(\widehat{X}))$, whence $|\widehat{\mu}(g) - \widehat{\mu}(g \circ S)| = \lim |\widehat{\mu}_{n_k}(g) - \widehat{\mu}_{n_k}(g \circ S)| = 0$.
- $\widehat{\mu}(-F) = s$: The inequality $\leq$ is because of the S-invariance of $\widehat{\mu}$. The inequality $\geq$ is because $F \in \mathscr{L}^1(\Omega, C(\widehat{X}))$, whence $\widehat{\mu}(-F) = \lim \widehat{\mu}_{n_k}(-F) \geq s$.

(These arguments require weak* convergence in $L^1(\Omega, C(\widehat{X}))^*$, not just weak* convergence of measures, because in general $g_E, F, g \circ S \notin C(\Omega \times \widehat{X})$.)

We found an S-invariant probability measure $\widehat{\mu}$ which projects to μ and so that $\widehat{\mu}(-F) = s$. But $\widehat{\mu}$ is not necessarily ergodic.

Let $\widehat{\mu} = \int_{\Omega \times \partial X} \widehat{\mu}_{(\omega,D)} d\widehat{\mu}$ be its ergodic decomposition. Then: (a) almost every ergodic component is ergodic and invariant; (b) Almost every ergodic component projects to μ (prove using the extremality μ); (c) Almost every ergodic component gives $(-F)$ integral s, because $\widehat{\mu}_{(\omega,D)}(-F) \leq s$ for a.e. (ω,D) by S-invariance, and the inequality cannot be strict on a set with positive measure as this would imply that $\widehat{\mu}(-F) < s$.

So a.e. ergodic component $\widehat{\mu}_0$ of $\widehat{\mu}$ is ergodic, invariant, projects to μ, and satisfied $\int F d\widehat{\mu}_0 = -s$. It remains to check that $\widehat{\mu}_0$ is carried by $\Omega \times \partial X$. This is because by the ergodic theorem, for $\widehat{\mu}$-a.e. (ω,D)

$$\frac{1}{n}D(f_n(\omega)x_0) \equiv -\frac{1}{n}\sum_{k=0}^{n-1}(F \circ S^k)(\omega,D) \xrightarrow[n\to\infty]{} -s < 0,$$

whence $\inf_X D = -\infty$. By corollary 2.4, $D \in \partial X$. The construction of $\widehat{\mu}_0$ is complete. As explained above, the theorem follows. $\square$

2.7.3 A geometric proof of the multiplicative ergodic theorem

We explain how to obtain the multiplicative ergodic theorem as a special case of the Karlsson-Margulis ergodic theorem. We begin with some notation and terminology.

- $\log := \ln$, $\log^+ t = \max\{\log t, 0\}$
- Vectors in $\mathbb{R}^d$ are denoted by v, w etc. $\langle v,w\rangle = \sum v_i w_i$ and $\|v\| = \sqrt{\langle v,v\rangle}$.
- The space of $d \times d$ matrices with real entries will be denote by $M_d(\mathbb{R})$.
- $\mathrm{diag}(\lambda_1,\ldots,\lambda_d) := (a_{ij}) \in M_d(\mathbb{R})$ where $a_{ii} := \lambda_i$ and $a_{ij} = 0$ for $i \neq j$
- $I :=$identity matrix$= \mathrm{diag}(1,\ldots,1)$
- For a matrix $A = (a_{ij}) \in M_d(\mathbb{R})$, $\mathrm{Tr}(A) := \sum a_{ii}$ (the trace), and $\mathrm{Tr}^{1/2}(A) := \sqrt{\mathrm{Tr}(A)}$. $A^t := (a_{ji})$ (the transpose). $\|A\| := \sup_{\|v\|\leq 1}\|Av\|$.

$$\exp(A) := I + \sum_{k=1}^{\infty} A^k/k!$$

- $\mathrm{GL}(d,\mathbb{R}) := \{A \in M_d(\mathbb{R}) : A \text{ is invertible}\}$
- $\mathrm{Sym}(d,\mathbb{R}) := \{A \in M_d(\mathbb{R}) : A \text{ is symmetric, i.e. } A^t = A\}$
- $O_d(\mathbb{R}) := \{A \in M_d(\mathbb{R}) : A \text{ is orthogonal, i.e. } A^t A = i\}$
- $\mathrm{Pos}_d(\mathbb{R}) := \{A \in \mathrm{Sym}_d(\mathbb{R}) : A \text{ positive definite, i.e. } \forall v \in \mathbb{R}^d \setminus \{0\},\ \langle Av,v\rangle \gneqq 0\}$.
- *Orthogonal diagonalization:* $P \in \mathrm{Pos}_d(\mathbb{R}) \Leftrightarrow \exists O \in O_d(\mathbb{R})$ such that $OPO^t = \mathrm{diag}(\lambda_1,\ldots,\lambda_d)$ and $\lambda_i > 0$ for all i.
- *Positive definite matrices have roots:* $\forall P \in \mathrm{Pos}_d(\mathbb{R}), \forall m \in \mathbb{N}\ \exists Q \in \mathrm{Pos}_d(\mathbb{R})$ s.t. $P = Q^m$. Take $Q := O^t \mathrm{diag}(\lambda_1^{1/m},\ldots,\lambda_d^{1/m})O$. We write $Q = P^{1/m}$
- *Positive definite matrices have logarithms:* $\forall P \in \mathrm{Pos}_d(\mathbb{R})$, $\exists!\ S \in \mathrm{Sym}_d(\mathbb{R})$ s.t. $P = \exp(S)$. Take $S := O^t \mathrm{diag}(\log\lambda_1,\ldots,\log\lambda_d)O$. We write $S := \log P$

Proposition 2.7. $\mathrm{GL}(d,\mathbb{R})$ *acts transitively on* $\mathrm{Pos}_d(\mathbb{R})$ *by* $A \cdot P = APA^t$*, i.e.:*

1. $\forall A \in \mathrm{GL}(d,\mathbb{R}), \forall P \in \mathrm{Pos}_d(\mathbb{R})$*,* $A \cdot P \in \mathrm{Pos}_d(\mathbb{R})$

2. $\forall A,B \in \mathrm{GL}(d,\mathbb{R}), \forall P \in \mathrm{Pos}_d(\mathbb{R})$, $(AB)\cdot P = A\cdot(B\cdot P)$
3. $\forall P,Q \in \mathrm{Pos}_d(\mathbb{R})$, $\exists A \in \mathrm{GL}(d,\mathbb{R})$ *s.t.* $A\cdot P = Q$

Proof. Suppose $A \in \mathrm{GL}(d,\mathbb{R}), P \in \mathrm{Pos}_d(\mathbb{R})$. Clearly APA^t is symmetric. It is positive definite, because for every $v \neq 0$, $\langle APA^t v, v\rangle = \langle PA^t v, A^t v\rangle > 0$ by the positive definiteness of P and the invertibility of A^t. This proves (1). (2) is trivial. (3) is because $A\cdot P = Q$ for $A := Q^{1/2}(P^{-1})^{1/2}$. Here we use the elementary fact that $P \in \mathrm{Pos}_d(\mathbb{R}) \Rightarrow P$ is invertible and $P^{-1} \in \mathrm{Pos}_d(\mathbb{R})$. ☐

Theorem 2.15. *There exists a unique metric d on $\mathrm{Pos}_d(\mathbb{R})$ as follows:*

1. *For all $A \in \mathrm{GL}(d,\mathbb{R})$, $P \mapsto A\cdot P$ is an isometry of $\mathrm{Pos}_d(\mathbb{R})$ i.e.*
$$d(APA^t, AQA^t) = d(P,Q) \quad (P,Q \in \mathrm{Pos}_d(\mathbb{R}))$$
2. *If S is symmetric and $\mathrm{Tr}(S^2) = 1$, then $\gamma(t) := \exp(tS)$ is a geodesic ray. Any geodesic which starts from the identity matrix has this form.*

This metric turns $\mathrm{Pos}_d(\mathbb{R})$ into a proper $CAT(0)$ geodesic space.

For details and proof, see Bridson & Haefliger: *Metric spaces of non-positive curvature*, Springer 1999, chapter II.10.

Lemma 2.3. *For every $A \in \mathrm{GL}(d,\mathbb{R})$, $d(I, A\cdot I) = 2\sqrt{\sum_{i=1}^d (\log\lambda_i)^2}$, where λ_i are singular values of A (the eigenvalues of $\sqrt{AA^t}$).*

Proof. The idea is to find the geodesic from I to $A\cdot I$. Such geodesics take the form $\gamma(\tau) = \exp(\tau S)$ for $S \in \mathrm{Sym}_d(\mathbb{R})$ s.t. $\mathrm{Tr}(S^2) = 1$. To find S we write $A\cdot I = AA^t = \exp(\log AA^t)$. This leads naturally to the solution
$$S := t^{-1}(\log AA^t), \text{ and } t := \mathrm{Tr}^{1/2}[\log AA^t]^2.$$
It follows that $A\cdot I = \exp(tS)$ whence $d(I, A\cdot I) = t = \mathrm{Tr}^{1/2}[\log AA^t]^2$.

To calculate the trace we use the well-known *singular value decomposition* of A: $A = O_1 \mathrm{diag}(\lambda_1,\ldots,\lambda_d) O_2^t$ for some $O_1, O_2 \in O_d(\mathbb{R})$. So
$$(\log AA^t)^2 = O_1 \mathrm{diag}(4(\log\lambda_1)^2,\ldots,4(\log\lambda_d)^2)O_1^t,$$
whence $d(I, A\cdot I) = t = \mathrm{Tr}^{1/2}[\log AA^t]^2 = 2\sqrt{\sum_{i=1}^d (\log\lambda_i)^2}$. ☐

Corollary 2.5. *For every $A \in \mathrm{GL}(d,\mathbb{R})$ we have the estimate*
$$\max\{|\log\|A\||, |\log\|A^{-1}\||\} \le d(I, A\cdot I) \le 2\sqrt{d}\max\{|\log\|A\||, |\log\|A^{-1}\||\}$$

Proof. Let $\lambda_1,\ldots,\lambda_d$ denote the singular values of A (the eigenvalues of $\sqrt{A^tA}$), and let $\lambda_{\max} := \max\{\lambda_1,\ldots,\lambda_d\}$, $\lambda_{\min} := \min\{\lambda_1,\ldots,\lambda_d\}$.

It is well-known that $\|A\| = \lambda_{\max}$, $\|A^{-1}\| = 1/\lambda_{\min}$.[6] The corollary follows from the lemma 2.3 and the identity $\max\{(\log\lambda_i)^2\} = 2\max\{|\log\lambda_{\max}|,|\log\lambda_{\min}|\}$. $\square$

Lemma 2.4. *Suppose $P_n, P \in \mathrm{Pos}_d(\mathbb{R})$. If $d(P_n,P) \xrightarrow[n\to\infty]{} 0$ where d is the metric in Theorem 2.15, then $\|P_n - P\| \xrightarrow[n\to\infty]{} 0$.*

Proof. For every $Q \in \mathrm{Pos}_d(\mathbb{R})$, $Q = Q^{1/2}\cdot I$. So $d(P_n,P) \to 0$ implies that

$$\delta_n := d(I, P^{-1/2}P_nP^{-1/2}) = d(I, P^{-1/2}P_n^{1/2}\cdot I) = d(P^{1/2}\cdot I, P_n^{1/2}\cdot I) = d(P,P_n) \xrightarrow[n\to\infty]{} 0.$$

Let $\gamma_n(t)$ denote the geodesic ray from I to $P^{-1/2}P_nP^{-1/2}$, then $\gamma_n(t) = \exp(tS_n)$ for some symmetric matrix S_n such that $\mathrm{Tr}[S_n^2] = 1$. This gives us the identity

$$P^{-1/2}P_nP^{-1/2} = \exp[\delta_n S_n], \text{ with } \delta_n \to 0, S_n \in \mathrm{Sym}_d(\mathbb{R}), \mathrm{Tr}[S^2] = 1.$$

We claim that $\exp[\delta_n S_n] \to I$ in norm. To see this we first diagonalize S_n orthogonally: $S_n = O_n \mathrm{diag}(\lambda_1(n),\dots,\lambda_d(n))O_n^t$ where $O_n \in O_d(\mathbb{R}), \sum_{i=1}^d \lambda_i(n)^2 = \mathrm{Tr}[S_n^2] = 1$. Using the identities $\exp\mathrm{diag}(\lambda_1,\dots,\lambda_d) = \mathrm{diag}(e^{\lambda_1},\dots,e^{\lambda_d})$ and $\exp(OAO^t) = O(\exp A)O^t$ for $O \in O_d(\mathbb{R})$, we find that

$$P^{-1/2}P_nP^{-1/2} = \exp[\delta_n S_n] = O_n \mathrm{diag}(e^{\delta_n\lambda_1(n)},\dots,e^{\delta_n\lambda_d(n)})O_n^t.$$

Since $\sum \lambda_i(n)^2 = 1$, the middle term tends to I in norm, whence by the orthogonality of O_n, $\|P^{-1/2}P_nP^{-1/2} - I\| \to 0$. It follows that $\|P_n - P\| \to 0$. $\square$

Multiplicative Ergodic Theorem: *Let $(\Omega,\mathscr{B},\mu,T)$ be a ppt on a standard probability space, and $A : \Omega \to \mathrm{GL}(d,\mathbb{R})$ a measurable function s.t. $\log\|A\|, \log\|A^{-1}\|$ are absolutely integrable. If $A_n(\omega) := A(\omega)A(T\omega)\cdots A(T^{n-1}\omega)$ then the limit $\Lambda := \lim_{n\to\infty}(A_nA_n^t)^{1/2n}$ exists a.e., and $\frac{1}{n}\log\|\Lambda^{-n}A_n\|, \frac{1}{n}\log\|A_n^{-1}\Lambda^n\| \xrightarrow[n\to\infty]{} 0$ a.e.*

Proof. The idea is to apply the Karlsson-Margulis ergodic theorem to the space $X := \mathrm{Pos}_d(\mathbb{R})$, the origin $x_0 := I$, and the isometric action $A(\omega)\cdot P := A(\omega)PA(\omega)^t$. First we need to check the integrability condition:

Step 1. $\int d(x_0, A(\omega)\cdot x_0)d\mu < \infty$.

This follows from by corollary 2.5.

Step 2. For a.e. ω the following limit exists: $s(\omega) := \lim_{n\to\infty}\frac{1}{n}d(I, A_n(\omega)\cdot I)$.

Proof. It is easy to see using the triangle inequality and the fact that $\mathrm{GL}(d,\mathbb{R})$ acts isometrically on $\mathrm{Pos}_d(\mathbb{R})$ that $g^{(n)}(\omega) := d(I, A_n(\omega)\cdot I)$ is a sub-additive cocycle. The step follows from the sub-additive ergodic theorem.

Step 3. Construction of $\Lambda(\omega)$ s.t. $\frac{1}{n}\log\|A_n(\omega)^{-1}\Lambda(\omega)^n\| \xrightarrow[n\to\infty]{} 0$ a.e.

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[6] This can be easily deduced from the singular decomposition: $A = O_1\mathrm{diag}(\lambda_1,\dots,\lambda_d)O_2^t$ where $O_1, O_2 \in O_d(\mathbb{R})$.

The Karlsson-Margulis ergodic theorem provides, for a.e. ω, a geodesic ray $\gamma_\omega(t)$ emanating from I such that

$$d(A_n(\omega)\cdot I,\gamma_\omega(ns(\omega)))=o(n).$$

By theorem 2.15, $\gamma_\omega(t)=\exp(tS(\omega))$, for $S(\omega)\in\mathrm{Sym}_d(\mathbb{R})$ s.t. $\mathrm{Tr}[S^2]=1$. Let

$$\Lambda(\omega):=\exp\left(\frac{1}{2}s(\omega)S(\omega)\right)$$

Then $\Lambda\in\mathrm{Pos}_d(\mathbb{R})$, $d(A_n\cdot I,\Lambda^{2n})=o(n)$, and by Corollary 2.5

$$\begin{aligned}&\max\{|\log\|A_n^{-1}\Lambda^n\||,|\log\|\Lambda^{-n}A_n\||\}\asymp d(I,A_n^{-1}\Lambda^n\cdot I)=d(A_n\cdot I,\Lambda^n\cdot I)\\&=d(A_n\cdot I,\Lambda^{2n})\equiv d(A_n\cdot I,\gamma_\omega(s(\omega)n))=o(n)\end{aligned}$$

Step 4. For a.e. ω, $(A_n(\omega)A_n(\omega)^t)^{1/2n}\xrightarrow[n\to\infty]{}\Lambda(\omega)$.

Proof. Fix ω such that $d(A_n\cdot I,\Lambda^{2n})=o(n)$. Given n, consider the geodesic triangle with vertices $I,A_n\cdot I$, $\Lambda^{2n}=\gamma_\omega(ns)$.

- The geodesic connecting $I,A_n\cdot I$ equals $\gamma_1(t):=\exp[c_nt\log A_nA_n^t]$, $0\le t\le d(I,A_nA_n^t)$, where $c_n:=1/d(I,A_nA_n^t)$. It contains the point

$$A_n':=(A_nA_n^t)^{1/2n}=\exp[\frac{1}{2n}\log A_nA_n^t]=\gamma_1(\frac{1}{2nc_n})$$

  So $d(I,A_n')=d(\gamma_1(0),\gamma_1(\frac{1}{2nc_n}))=\frac{1}{2nc_n}=\frac{1}{2n}d(I,A_nA_n^t)=\frac{1}{2n}d(I,A_n\cdot I)$.
- The geodesic connecting I,Λ^{2n} is $\gamma_\omega(t)=\exp(tS)$, $0\le t\le 2n$. It contains the point $\Lambda=\gamma_\omega(s/2)$. Again, $d(I,\Lambda^2)=s/2=\frac{1}{2n}d(I,\Lambda^{2n})$.
- The geodesic connecting $A_n\cdot I,\Lambda^{2n}$ has length $d(A_n\cdot I,\gamma_\omega(ns))=o(n)$.

We now consider the Euclidean comparison triangle $\overline{I},\overline{A_n\cdot I},\overline{\Lambda^{2n}}$. The Euclidean triangle generated by $\overline{I},\overline{A_n'}$, $\overline{\Lambda}$ is similar to the euclidean triangle $\overline{I}$, $\overline{A_n'}$, $\overline{\Lambda}$, and has sides $\frac{1}{2n}\times$ the lengths of the sides of that triangle. So

$$d_{\mathbb{R}^2}(\overline{A_n'},\overline{\Lambda})=\frac{1}{2n}d_{\mathbb{R}^2}(\overline{A_n\cdot I},\overline{\Lambda^{2n}})\equiv\frac{1}{2n}d(A_n\cdot I,\gamma_\omega(ns))=o(1).$$

By the CAT(0) property $d(A_n',\Lambda)\le d_{\mathbb{R}^2}(\overline{A_n'},\overline{\Lambda})=o(1)$.

Equivalently, $d((A_nA_n^t)^{1/2n},\Lambda)\to 0$. By Lemma 2.4, $\|(A_nA_n^t)^{1/2n}-\Lambda\|\to 0$. □

Problems

2.1. The Mean Ergodic Theorem for Contractions

Suppose H is a Hilbert space, and $U:H\to H$ is a bounded linear operator such that

$\|U\| \leq 1$. Prove that $\frac{1}{N}\sum_{k=0}^{N-1} U^k f$ converges in norm for all $f \in H$, and the limit is the projection of f on the space $\{f : Uf = f\}$.

2.2. Ergodicity as a mixing property
Prove that a ppt $(X, \mathscr{B}, \mu, T)$ is ergodic, iff for every $A, B \in \mathscr{B}$, $\frac{1}{N}\sum_{k=0}^{N-1} \mu(A \cap T^{-k}B) \xrightarrow[N\to\infty]{} \mu(A)\mu(B)$.

2.3. Use the pointwise ergodic theorem to show that any two different ergodic invariant probability measures for the same transformation are mutually singular.

2.4. Ergodicity and extremality
An invariant probability measure μ is called *extremal*, if it cannot be written in the form $\mu = t\mu_1 + (1-t)\mu_2$ where μ_1, μ_2 are different invariant probability measures, and $0 < t < 1$. Prove that an invariant measure is extremal iff it is ergodic, using the following steps.

1. Show that if E is a T–invariant set of non-zero measure, then $\mu(\cdot|E)$ is a T–invariant measure. Deduce that if μ is not ergodic, then it is not extremal.
2. Show that if μ is ergodic, and $\mu = t\mu_1 + (1-t)\mu_2$ where μ_i are invariant, and $0 < t < 1$, then
   a. For every $E \in \mathscr{B}$, $\frac{1}{N}\sum_{k=0}^{N-1} 1_E \circ T^k \xrightarrow[N\to\infty]{} \mu(E)$ μ_i–a.e. $(i = 1, 2)$.
   b. Conclude that $\mu_i(E) = \mu(E)$ for all $E \in \mathscr{B}$ $(i = 1, 2)$. This shows that ergodicity implies extremality.

2.5. Prove that the Bernoulli $(\frac{1}{2}, \frac{1}{2})$–measure is the invariant probability measure for the adding machine (Problem 1.10), by showing that all cylinders of length n must have the same measure as $[0^n]$. Deduce from the previous problem that the adic machine is ergodic.

2.6. Explain why when $f \in L^2$, $\mathbb{E}(f|\mathscr{F})$ is the projection of f on $L^2(\mathscr{F})$. Prove:

1. If $\mathscr{F} = \{\varnothing, A, X \setminus A\}$, then $\mathbb{E}(1_B|\mathscr{F}) = \mu(B|A)1_A + \mu(B|A^c)1_{A^c}$
2. If $\mathscr{F} = \{\varnothing, X\}$, then $\mathbb{E}(f|\mathscr{F}) = \int f d\mu$
3. If $X = [-1, 1]$ with Lebesgue measure, and $\mathscr{F} = \{A : A \text{ is Borel and } -A = A\}$, then $\mathbb{E}(f|\mathscr{F}) = \frac{1}{2}[f(x) + f(-x)]$

2.7. Prove:

1. $f :\mapsto \mathbb{E}(f|\mathscr{F})$ is linear, and a contraction in the L^1–metric
2. $f \geq 0 \Rightarrow \mathbb{E}(f|\mathscr{F}) \geq 0$ a.e.
3. if φ is convex, then $\mathbb{E}(\varphi \circ f|\mathscr{F}) \leq \varphi(\mathbb{E}(f|\mathscr{F}))$
4. if h is $\mathscr{F}$–measurable, then $\mathbb{E}(hf|\mathscr{F}) = h\mathbb{E}(f|\mathscr{F})$
5. If $\mathscr{F}_1 \subset \mathscr{F}_2$, then $\mathbb{E}[\mathbb{E}(f|\mathscr{F}_2)|\mathscr{F}_1] = \mathbb{E}(f|\mathscr{F}_1)$

2.8. A $\mathbb{Z}^d$–PET along more general sequences of boxes (Tempelman). A family of boxes $\{I_r\}_{r\geq 1}$ is called *regular (with constant C)* if there exists an increasing sequence of boxes $\{I'_r\}_{r'\geq 1}$ which tends to $\mathbb{Z}^d_+$ s.t. $|I_r| \leq C|I'_r|$ for all r

1. Show that the following sequences are regular:

   a. Any increasing sequence of boxes which tends to $\mathbb{Z}_+^d$
   b. $I_r := [r, r+r^2]$ (in dimension one)
   c. $I_r := [\underline{0}, \underline{n}(r))$ where $\underline{n}(k) \in \mathbb{Z}_+^d$ is a sequence of vectors which tends to infinity "in a sector" in the sense that (a) $\min\{n_1(k), \ldots, n_d(k)\} \xrightarrow[k\to\infty]{} \infty$, and (b) for some constant K $\max_{1\le i,j\le d}\left(\frac{n_i(k)}{n_j(k)}\right) \le K$ for all k.

2. Suppose $T_1, \ldots, T_d$ are commuting measure preserving maps on a probability space $(\Omega, \mathscr{F}, \mu)$, and $\{I_r\}_{r\ge 1}$ is a regular sequence of boxes with constant C.

   a. Prove the following maximal inequality: If $\varphi \in L^1$ is non-negative, then for all $\alpha > 0$, $\mu[\sup_r \frac{1}{|I_r|} S_{I_r}\varphi > \alpha] < C2^d \|\varphi\|_1/\alpha$.
   b. Deduce that if $f \in L^1$, then $\frac{1}{|I_r|} S_{I_r} f \xrightarrow[r\to\infty]{} \mathbb{E}(f|\mathfrak{Inv}(T_1)\cap\cdots\cap\mathfrak{Inv}(T_d))$ a.e.

2.9. The Martingale Convergence Theorem (Doob)
Suppose $(X, \mathscr{B}, \mu)$ is a probability space, and $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ are σ–algebras all of which are contained in $\mathscr{B}$. Let $\mathscr{F} := \sigma(\bigcup_{n\ge 1}\mathscr{F}_n)$ (the smallest σ–algebra containing the union). If $f \in L^1$, then $\mathbb{E}(f|\mathscr{F}_n) \xrightarrow[n\to\infty]{} \mathbb{E}(f|\mathscr{F})$ a.e. and in L^1

Prove this theorem, using the following steps (W. Parry). It is enough to consider *non-negative* $f \in L^1$.

1. Prove that $\mathbb{E}(f|\mathscr{F}_n) \xrightarrow[n\to\infty]{L^1} \mathbb{E}(f|\mathscr{F})$ using the following observations:

   a. The convergence holds for all elements of $\bigcup_{n\ge 1} L^1(X, \mathscr{F}_n, \mu)$;
   b. $\bigcup_{n\ge 1} L^1(X, \mathscr{F}_n, \mu)$ is dense in $L^1(X, \mathscr{F}, \mu)$.

2. Set $E_a := \{x : \max_{1\le n\le N} \mathbb{E}(f|\mathscr{F}_n)(x) > a\}$. Show that $\mu(E_a) \le \frac{1}{a}\int f d\mu$. (Hint: $E = \biguplus_{n\ge 1}\{x : \mathbb{E}(f|\mathscr{F}_n)(x) > \lambda,$ and $\mathbb{E}(f|\mathscr{F}_k)(x) \le \lambda$ for $k = 1, \ldots, n-1\}$.)
3. Prove that $\mathbb{E}(f|\mathscr{F}_n) \xrightarrow[n\to\infty]{} \mathbb{E}(f|\mathscr{F})$ a.e. for every non-negative $f \in L^1$, using the following steps. Fix $f \in L^1$. For every $\varepsilon > 0$, choose n_0 and $g \in L^1(X, \mathscr{F}_{n_0}, \mu)$ such that $\|\mathbb{E}(f|\mathscr{F}) - g\|_1 < \varepsilon$.

   a. Show that $|\mathbb{E}(f|\mathscr{F}_n) - \mathbb{E}(f|\mathscr{F})| \le \mathbb{E}(|f-g|\,|\mathscr{F}_n)| + |\mathbb{E}(f|\mathscr{F}) - g|$ for all $n \ge n_0$. Deduce that
   $$\mu\left[\limsup_{n\to\infty}|\mathbb{E}(f|\mathscr{F}_n) - \mathbb{E}(f|\mathscr{F})| > \sqrt{\varepsilon}\right] \le \mu\left[\sup_n|\mathbb{E}(|f-g||\mathscr{F}_n)| > \frac{1}{2}\sqrt{\varepsilon}\right] + \mu\left[|\mathbb{E}(f|\mathscr{F}) - g| > \frac{1}{2}\sqrt{\varepsilon}\right]$$
   b. Show that $\mu\left[\limsup_{n\to\infty}|\mathbb{E}(f|\mathscr{F}_n) - \mathbb{E}(f|\mathscr{F})| > \sqrt{\varepsilon}\right] \xrightarrow[\varepsilon\to 0^+]{} 0$. (Hint: Prove first that for every L^1 function F, $\mu[|F| > a] \le \frac{1}{a}\|F\|_1$.)
   c. Finish the proof.

2.10. Hopf's ratio ergodic theorem
Let $(X,\mathscr{B},\mu,T)$ be a conservative ergodic mpt on a σ–finite measure space. If $f,g\in L^1$ and $\int g\neq 0$, then $\frac{\sum_{k=0}^{n-1} f\circ T^k}{\sum_{k=0}^{n-1} g\circ T^k}\xrightarrow[n\to\infty]{}\frac{\int f d\mu}{\int g d\mu}$ almost everywhere.
Prove this theorem using the following steps (R. Zweimüller). Fix a set $A\in\mathscr{B}$ s.t. $0<\mu(A)<\infty$, and let $(A,\mathscr{B}_A,T_A,\mu_A)$ denote the induced system on A (problem 1.14). For every function F, set

$$S_nF := F+F\circ T+\cdots+F\circ T^{n-1}$$
$$S_n^AF := F+F\circ T_A+\cdots+F\circ T_A^{n-1}$$

1. Read problem 1.14, and show that a.e. x has an orbit which enters A infinitely many times. Let $0<\tau_1(x)<\tau_2(x)<\cdots$ be the times when $T^{\tau}(x)\in A$.
2. Suppose $f\geq 0$. Prove that for every $n\in(\tau_{k-1}(x),\tau_k(x)]$ and a.e. $x\in A$,
$$\frac{(S_{k-1}^A f)(x)}{(S_k^A 1_A)(x)}\leq\frac{(S_n f)(x)}{(S_n 1_A)(x)}\leq\frac{(S_k^A f)(x)}{(S_{k-1}^A 1_A)(x)}.$$
3. Verify that $S_j^A 1_A=j$ a.e. on A, and show that $(S_nf)(x)/(S_n1_A)(x)\xrightarrow[n\to\infty]{}\frac{1}{\mu(A)}\int f d\mu$ a.e. on A.
4. Finish the proof.

Notes for chapter 2

For a comprehensive reference to ergodic theorems, see [7]. The mean ergodic theorem was proved by von Neumann, and the pointwise ergodic theorem was proved by Birkhoff. By now there are many proofs of Birkhoff's theorem; the one we use is taken from [6], where it is attributed to Kamae — who found it using ideas from nonstandard analysis. The subadditive ergodic theorem was first proved by Kingman. The proof we give is due to Steele [11]. The proof of Tempelman's pointwise ergodic theorem for $\mathbb{Z}^d$ is taken from [7]. For ergodic theorems for the actions of other amenable groups see [8]. The multiplicative ergodic theorem is due to Oseledets. The "linear algebra" proof is due to Raghunathan and Ruelle, and is taken from [10]. The geometric approach to the multiplicative ergodic theorem is due to Kaimanovich [3] who used it to generalize that theorem to homogeneous spaces other than $\mathrm{Pos}_d(\mathbb{R})$. The ergodic theorem for isometric actions on CAT(0) spaces is due to Karlsson & Margulis [5]. We proof we give is due to Karlsson & Ledrappier [4]. The Martingale convergence theorem (problem 2.9) is due to Doob. The proof sketched in problem 2.9 is taken from [2]. The proof of Hopf's ratio ergodic theorem sketched in problem 2.10 is due to R. Zweimüller [12].

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Chapter 3
Spectral Theory

3.1 The spectral approach to ergodic theory

A basic problem in ergodic theory is to determine whether two ppt are measure theoretically isomorphic. This is done by studying *invariants*: properties, quantities, or objects which are equal for any two isomorphic systems. The idea is that if two ppt have different invariants, then they cannot be isomorphic. Ergodicity and mixing are examples of invariants for measure theoretic isomorphism.

An effective method for inventing invariants is to look for a *weaker* equivalence relation, which is better understood. Any invariant for the weaker equivalence relation is automatically an invariant for measure theoretic isomorphism. The spectral approach to ergodic theory is an example of this method.

The idea is to associate to the ppt $(X,\mathscr{B},\mu,T)$ the operator $U_T : L^2(X,\mathscr{B},\mu) \to L^2(X,\mathscr{B},\mu)$, $U_T f = f \circ T$. This is an isometry of L^2 (i.e. $\|U_T f\|_2 = \|f\|_2$ and $\langle U_T f, U_T g\rangle = \langle f, g\rangle$).

It is useful here to think of L^2 as a Hilbert space over $\mathbb{C}$, with inner product $\langle f, g\rangle := \int f\overline{g}d\mu$. The need to consider complex numbers arises from the need to consider complex eigenvalues, see below.

Definition 3.1. Two ppt $(X,\mathscr{B},\mu,T)$, $(Y,\mathscr{C},\nu,S)$ are called *spectrally isomorphic*, if their associated L^2–isometries U_T and U_S are *unitarily equivalent*: There exists a linear operator $W : L^2(X,\mathscr{B},\mu) \to L^2(Y,\mathscr{C},\nu)$ s.t.

1. W is invertible;
2. $\langle Wf, Wg\rangle = \langle f, g\rangle$ for all $f, g \in L^2(X,\mathscr{B},\mu)$;
3. $WU_T = U_S W$.

It is easy to see that any two measure theoretically isomorphic ppt are spectrally isomorphic, but we will see later that there are Bernoulli schemes which are spectrally isomorphic but not measure theoretically isomorphic.

Definition 3.2. A property of ppt is called a *spectral invariant*, if whenever it holds for $(X,\mathscr{B},\mu,T)$, it holds for all ppt which are spectrally isomorphic to $(X,\mathscr{B},\mu,T)$.

Proposition 3.1. *Ergodicity and mixing are spectral invariants.*

Proof. Suppose $(X,\mathscr{B},\mu,T)$ is a ppt, and let U_T be as above. The trick is to phrase ergodicity and mixing in terms of U_T.

Ergodicity is equivalent to the statement "all invariant L^2–functions are constant", which is the same as saying that $\dim\{f : U_T f = f\} = 1$. Obviously, this is a spectral invariant.

Mixing is equivalent to the following statement:

$$\langle f, U_T^n g\rangle \xrightarrow[n\to\infty]{} \langle f,1\rangle\overline{\langle g,1\rangle} \text{ for all } f,g \in L^2. \tag{3.1}$$

To see that (3.1) is a spectral invariant, we first note that (3.1) implies:

1. $\dim\{g : U_T g = g\} = 1$. One way to see this is to note that mixing implies ergodicity. Here is a more direct proof: If $U_T g = g$, then $\frac{1}{N}\sum_{n=0}^{N-1} U_T^n g = g$. By (3.1), $\frac{1}{N}\sum_{n=0}^{N-1} U_T^n g \xrightarrow[N\to\infty]{w} \langle g,1\rangle 1$. Necessarily $g = \langle g,1\rangle 1 = const$.
2. For unitary equivalences W, $W1 = c$ with $|c| = 1$. Proof: $W1$ is an eigenfunction with eigenvalue one, so $W1 = const$ by 1. Since $\|W1\|_1 = \|1\|_2 = 1$, $|c| = 1$.

Suppose now that $W : L^2(X,\mathscr{F},\mu) \to L^2(Y,\mathscr{F},\nu)$ is a unitary equivalence between $(X,\mathscr{F},\mu,T)$ and $(Y,\mathscr{G},\nu,S)$. Suppose T satisfies (3.1). For every $F,G \in L^2(Y)$ we can write $F = Wf, G = Wg$ with $f,g \in L^2(Y)$, whence

$$\langle F, U_S^n G\rangle = \langle Wf, U_S^n Wg\rangle = \langle Wf, WU_T^n g\rangle = \langle f, U_T^n g\rangle \to \langle f,1\rangle\overline{\langle g,1\rangle}.$$

Since $W1 = c$ with $|c|^2 = 1$, $\langle f,1\rangle\overline{\langle g,1\rangle} = \langle Wf,W1\rangle\overline{\langle Wg,W1\rangle} = \langle F,c1\rangle\overline{\langle G,c1\rangle} = \overline{c}\langle F,1\rangle\cdot c\overline{\langle G,1\rangle} = \langle F,1\rangle\overline{\langle G,1\rangle}$, and we see that U_S satisfies (3.1). ☐

The spectral point of view immediately suggests the following invariant.

Definition 3.3. Suppose $(X,\mathscr{B},\mu,T)$ is a ppt. If $f : X \to \mathbb{C}$, $0 \neq f \in L^2$ satisfies $f\circ T = \lambda f$, then we say that f is an *eigenfunction* and that λ is an *eigenvalue*. The *point spectrum* T is the set $H(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue}\}$.

$H(T)$ is a countable subgroup of the unit circle (problem 3.1). Evidently $H(T)$ is a spectral invariant of T.

It is easy to see using Fourier expansions that for the irrational rotation R_α, $H(R_\alpha) = \{e^{ik\alpha} : k \in \mathbb{Z}\}$ (problem 3.2), thus irrational rotations by different angles are non-isomorphic.

Here are other related invariants:

Definition 3.4. Given a ppt $(X,\mathscr{B},\mu,T)$, let $V_d := \overline{\text{span}}\{\text{eigenfunctions}\}$. We say that $(X,\mathscr{B},\mu,T)$ has

1. *discrete spectrum* (sometime called *pure point spectrum*), if $V_d = L^2$,
2. *continuous spectrum*, if $V_d = \{\text{constants}\}$ (i.e. is smallest possible),
3. *mixed spectrum*, if $V_d \neq L^2, \{\text{constants}\}$.

Any irrational rotation has discrete spectrum (problem 3.2). Any mixing transformation has continuous spectrum, because a non-constant eigenfunction $f \circ T = \lambda f$ satisfies
$$\langle f, f \circ T^{n_k} \rangle \xrightarrow[n \to \infty]{} \|f\|_2^2 \neq \left| \int f \right|^2$$
along any $n_k \to \infty$ s.t. $\lambda^{n_k} \to 1$. (To see that $\|f\|_2 \neq |\int f d\mu|^2$ for all non-constant functions, apply Cauchy-Schwarz to $f - \int f$, or note that non-constant L^2 functions have positive variance.)

The invariant $H(T)$ is tremendously successful for transformations with discrete spectrum:

Theorem 3.1 (Discrete Spectrum Theorem). *Two ppt with discrete spectrum are measure theoretically isomorphic iff they have the same group of eigenvalues.*

But this invariant cannot distinguish transformations with continuous spectrum. In particular - it is unsuitable for the study of mixing transformations.

3.2 Weak mixing

3.2.1 Definition and characterization

We saw that if a transformation is mixing, then it does not have non-constant eigenfunctions. But the absence of non-constant eigenfunctions is not equivalent to mixing (see problems 3.8–3.10 for an example). Here we study the dynamical significance of this property. First we give it a name.

Definition 3.5. A ppt is called *weak mixing*, if every $f \in L^2$ s.t. $f \circ T = \lambda f$ a.e. is constant almost everywhere.

Theorem 3.2. *The following are equivalent for a ppt $(X, \mathscr{B}, \mu, T)$ on a Lebesgue space:*

1. *weak mixing;*
2. *for all $E, F \in \mathscr{B}$, $\frac{1}{N}\sum_{n=0}^{N-1} |\mu(E \cap T^{-n}F) - \mu(E)\mu(F)| \xrightarrow[N\to\infty]{} 0$;*
3. *for every $E, F \in \mathscr{B}$, $\exists \mathscr{N} \subset \mathbb{N}$ of density zero (i.e. $|\mathscr{N} \cap [1,N]|/N \xrightarrow[N\to\infty]{} 0$) s.t. $\mu(E \cap T^{-n}F) \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} \mu(E)\mu(F)$;*
4. *$T \times T$ is ergodic.*

Proof. We prove (2) ⇒ (3) ⇒ (4) ⇒ (1). The remaining implication (1) ⇒ (2) requires additional preparation, and will be shown later.

(2) ⇒ (3) is a general fact from calculus (Koopman–von Neumann Lemma): *If a_n is a bounded sequence of non-negative numbers, then $\frac{1}{N}\sum_{n=1}^{N} a_n \to 0$ iff there is a set of zero density $\mathscr{N} \subset \mathbb{N}$ s.t. $a_n \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} 0$* (Problem 3.3).

We show that (3) $\Rightarrow$ (4). Let $\mathscr{S}$ be the semi-algebra $\{E \times F : E, F \in \mathscr{B}\}$ which generates $\mathscr{B} \otimes \mathscr{B}$, and fix $E_i \times F_i \in \mathscr{S}$. By (3), $\exists \mathscr{N}_i \subset \mathbb{N}$ of density zero s.t.

$$\mu(E_i \cap T^{-n} F_i) \xrightarrow[\mathscr{N}_i \not\ni n \to \infty]{} \mu(E_i)\mu(F_i) \quad (i = 1, 2).$$

The set $\mathscr{N} = \mathscr{N}_1 \cup \mathscr{N}_2$ also has zero density, and

$$\mu(E_i \cap T^{-n} F_i) \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} \mu(E_i)\mu(F_i) \quad (i = 1, 2).$$

Writing $m = \mu \times \mu$ and $S = T \times T$, we see that this implies that

$$m[(E_1 \times E_2) \cap S^{-n}(F_1 \times F_2)] \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} m(E_1 \times F_1)m(E_2 \times F_2),$$

whence $\frac{1}{N}\sum_{k=0}^{N-1} m[(E_1 \times F_1) \cap S^{-n}(E_2 \times F_2)] \xrightarrow[N \to \infty]{} m(E_1 \times F_1)m(E_2 \times F_2)$. In summary, $\frac{1}{N}\sum_{k=0}^{N-1} m[A \cap S^{-n} B] \xrightarrow[N \to \infty]{} m(A)m(B)$ for all $A, B \in \mathscr{S}$.

Since $\mathscr{S}$ generates $\mathscr{B} \otimes \mathscr{B}$, every $A, B \in \mathscr{B} \otimes \mathscr{B}$ can be approximated up to measure ε by finite disjoint unions of elements of $\mathscr{S}$. A standard approximation argument shows that $\frac{1}{N}\sum_{k=0}^{N-1} m[A \cap S^{-n} B] \xrightarrow[N \to \infty]{} m(A)m(B)$ for all $A, B \in \mathscr{B} \otimes \mathscr{B}$. and this implies that $T \times T$ is ergodic.

Proof that (4)$\Rightarrow$ (1): Suppose T were not weak mixing, then T has an non-constant eigenfunction f with eigenvalue λ. The eigenvalue λ has absolute value equal to one, because $|\lambda|\|f\|_2 = \||f| \circ T\|_2 = \|f\|_2$. Thus

$$F(x, y) = f(x)\overline{f(y)}$$

is $T \times T$–invariant. Since f is non-constant, F is non-constant, and we get a contradiction to the ergodicity of $T \times T$.

The proof that (1)$\Rightarrow$ (2) is presented in the next section. $\square$

3.2.2 Spectral measures and weak mixing

Suppose $(X, \mathscr{B}, \mu, T)$ is an *invertible* ppt. We are interested in the behavior of $U_T^n f$ as $n \to \pm\infty$. Clearly, all the action takes place in the closed invariant subspace

$$H_f := \overline{\text{span}}\{U_T^n f : n \in \mathbb{Z}\}.$$

It turns out that $U_T : H_f \to H_f$ is unitarily equivalent to the operator $M : g(z) \mapsto zg(z)$ on $L^2(S^1, \mathscr{B}(S^1), \nu_f)$ where ν_f a special finite measure on S^1. This measure is called the *spectral measure* of f, and it contains all the information on $U_T : H_f \to H_f$.

To construct it, we need the following important tool from harmonic analysis. Recall that The n-th *Fourier coefficient* of μ is the number $\widehat{\mu}(n) = \int_{S^1} z^n d\mu$.

Theorem 3.3 (Herglotz). *A sequence $\{r_n\}_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of a positive Borel measure on S^1 iff $r_{-n}=\overline{r_n}$ and $\{r_n\}$ is* positive definite*: $\sum_{n,m=-N}^{N} r_{n-m}a_m\overline{a_n}\geq 0$ for all sequences $\{a_n\}$ and N. This measure is unique.*

It is easy to check that $r_n=\langle U_T^n f,f\rangle$ is positive definite (to see this expand $\langle \Sigma_{n=-N}^{N}a_nU_T^n f,\Sigma_{m=-N}^{N}a_mU_T^m f\rangle$ noting that $\langle U_T^n f,U_T^m f\rangle=\langle U_T^{n-m}f,f\rangle$).

Definition 3.6. Suppose $(X,\mathscr{B},\mu,T)$ is an invertible ppt, and $f\in L^2\setminus\{0\}$. The *spectral measure* of f is the unique measure ν_f on S^1 s.t. $\langle f\circ T^n,f\rangle=\int_{S^1}z^n d\nu_f$ for $n\in\mathbb{Z}$.

Proposition 3.2. *Suppose $(X,\mathscr{B},\mu,T)$ is an invertible ppt, $f\in L^2\setminus\{0\}$, and $H_f:=\overline{\mathrm{span}}\{U_T^n f:n\in\mathbb{Z}\}$. Then $U_T:H_f\to H_f$ is unitarily equivalent to the operator $g(z)\mapsto zg(z)$ on $L^2(S^1,\mathscr{B}(S^1),\nu_f)$.*

Proof. By the definition of the spectral measure,

$$\left\|\sum_{n=-N}^{N}a_nz^n\right\|^2_{L^2(\nu_f)}=\left\langle\sum_{n=-N}^{N}a_nz^n,\sum_{m=-N}^{N}a_mz^m\right\rangle=\sum_{n,m=-N}^{N}a_n\overline{a}_m\int_{S^1}z^{n-m}d\nu_f(z)$$
$$=\sum_{n,m=-N}^{N}a_n\overline{a}_m\langle U_T^{n-m}f,f\rangle=\sum_{n,m=-N}^{N}a_n\overline{a}_m\langle U_T^n f,U_T^m f\rangle=\left\|\sum_{n=-N}^{N}a_nU_T^n f\right\|^2_{L^2(\mu)}$$

In particular, if $\Sigma_{n=-N}^{N}a_nU_T^n f=0$ in $L^2(\mu)$, then $\Sigma_{n=-N}^{N}a_nz^n=0$ in $L^2(\nu_f)$. It follows that $W:U_T^n f\mapsto z^n$ extends to a linear map from $\mathrm{span}\{U_T^n f:n\in\mathbb{Z}\}$ to $L^2(\nu_f)$.

This map is an isometry, and it is bounded. It follows that W extends to an linear isometry $W:H_f\to L^2(\nu_f)$. The image of W contains all the trigonometric polynomials, therefore $W(H_f)$ is dense in $L^2(\nu_f)$. Since W is an isometry, its image is closed (exercise). It follows that W is an isometric bijection from H_f onto $L^2(\nu_f)$.

Since $(WU_t)[g(z)]=z[Wg(z)]$ on $\mathrm{span}\{U_T^n f:n\in\mathbb{Z}\}$, $WU_Tg(z)=zg(z)$ on H_f, and so W is the required unitary equivalence. $\square$

Proposition 3.3. *Suppose T is weak mixing invertible ppt on a Lebesgue space, then all the spectral measures of $f\in L^2$ s.t. $\int f=0$ are non-atomic (this explains the terminology "continuous spectrum").*

Proof. Suppose $f\in L^2$ has integral zero and that ν_f has an atom $\lambda\in S^1$. We construct an eigenfunction (with eigenvalue λ). Consider the sequence $\frac{1}{N}\sum_{n=0}^{N-1}\lambda^{-n}U_T^n f$. This sequence is bounded in norm, therefore has a weakly convergent subsequence (here we use the fact that L^2 is separable — a consequence of the fact that $(X,\mathscr{B},\mu)$ is a Lebesgue space):

$$\frac{1}{N_k}\sum_{n=0}^{N_k-1}\lambda^{-n}U_T^n f\xrightarrow[N\to\infty]{w}g.$$

The limit g must satisfy $\langle U_Tg,h\rangle=\langle\lambda g,h\rangle$ for all $h\in L^2$ (check!), therefore it must be an eigenfunction with eigenvalue λ.

Next we claim that g is not constant, and obtain a contradiction to weak mixing:

$$\begin{aligned}\langle g,f\rangle &= \lim_{k\to\infty}\frac{1}{N_k}\sum_{n=0}^{N_k-1}\lambda^{-n}\langle U_T^n f,f\rangle = \lim_{k\to\infty}\frac{1}{N_k}\sum_{n=0}^{N_k-1}\int \lambda^{-n}z^n d\nu_f(z)\\ &= \nu_f\{\lambda\}+\lim_{k\to\infty}\frac{1}{N_k}\sum_{n=0}^{N_k-1}\int_{S^1\setminus\{\lambda\}}\lambda^{-n}z^n d\nu_f(z)\\ &= \nu_f\{\lambda\}+\lim_{k\to\infty}\int_{S^1\setminus\{\lambda\}}\frac{1}{N_k}\frac{1-\lambda^{-N_k}z^{N_k}}{1-\lambda^{-1}z}d\nu_f(z).\end{aligned}$$

The limit is equal to zero, because the integrand tends to zero and is uniformly bounded (by one). Thus $\langle g,f\rangle=\nu_f\{\lambda\}\neq 0$, whence $g\neq const$. ☐

Lemma 3.1. *Suppose T is an invertible ppt on a Lebesgue space. If T is weak mixing, then for every real-valued $f\in L^2$, $\frac{1}{N}\sum\limits_{n=0}^{N-1}\left|\int f\cdot f\circ T^n d\mu-(\int f d\mu)^2\right|\xrightarrow[N\to\infty]{}0$.*

Proof. It is enough to treat the case when $\int f d\mu=0$. Let ν_f denote the spectral measure of f, then

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1}\left|\int f\cdot f\circ T^n d\mu\right|^2 &= \frac{1}{N}\sum_{k=0}^{N-1}|\langle U_T^n f,f\rangle|^2=\frac{1}{N}\sum_{k=0}^{N-1}\left|\int_{S^1}z^n d\nu_f(z)\right|^2\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\int_{S^1}z^n d\nu_f(z)\right)\overline{\left(\int_{S^1}z^n d\nu_f(z)\right)}\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\int_{S^1}\int_{S^1}z^n\overline{w}^n d\nu_f(z)d\nu_f(w)\\ &= \int_{S^1}\int_{S^1}\frac{1}{N}\left(\sum_{k=0}^{N-1}z^n\overline{w}^n\right)d\nu_f(z)d\nu_f(w)\end{aligned}$$

The integrand tends to zero and is bounded outside $\Delta:=\{(z,w):z=w\}$. If we can show that $(\nu_f\times\nu_f)(\Delta)=0$, then it will follow that $\frac{1}{N}\sum\limits_{k=0}^{N-1}|\int f\cdot f\circ T^n d\mu|^2\xrightarrow[N\to\infty]{}0$. This is indeed the case: T is weak mixing, so by the previous proposition ν_f is non-atomic, whence $(\nu_f\times\nu_f)(\Delta)=\int_{S^1}\nu_f\{w\}d\nu_f(w)=0$ by Fubini-Tonelli.

It remains to note that by the Koopman - von Neumann theorem, for every bounded non-negative sequence a_n, $\frac{1}{N}\sum_{k=1}^N a_n^2\to 0$ iff $\frac{1}{N}\sum_{k=1}^N a_n\to 0$, because both conditions are equivalent to saying that a_n converges to zero outside a set of indices of density zero. ☐

We can now complete the proof of the theorem in the previous section:

Proposition 3.4. *If T is weak mixing (possibly non-invertible) ppt, then for all real-valued $f,g\in L^2$,*

$$\frac{1}{N}\sum_{n=0}^{N-1}\left|\int g\cdot f\circ T^n d\mu-\left(\int f d\mu\right)\left(\int g d\mu\right)\right|\xrightarrow[N\to\infty]{}0. \tag{3.2}$$

Proof. Suppose first that T is invertible. It is enough to consider $f\in L^2$ real valued such that $\int f d\mu=0$.

Set $S(f):=\overline{\mathrm{span}\{U_T^k f: k\geq 0\}\cup\{1\}}$. Then $L^2=S(f)\oplus S(f)^\perp$, and

1. Every $g\in S(f)$ satisfies (3.2). To see this note that if $g_1,\dots,g_m$ satisfy (3.2) then so does $g=\sum\alpha_k g_k$ for any $\alpha_k\in\mathbb{R}$. Therefore it is enough to check (3.2) for $g:=U_T^k f$ and $g=const$. Constant functions satisfy (3.2) because for such functions, $\int g f\circ T^n d\mu=0$ for g for all $n\geq 0$. Functions of the form $g=U_T^k f$ satisfy (3.2) because for such functions for all $n>k$,
$$\int g f\circ T^n d\mu=\int f\circ T^k f\circ T^n d\mu=\int f\cdot f\circ T^{n-k}d\mu\xrightarrow[\mathscr{N}\not\ni n\to\infty]{}0$$
for some $\mathscr{N}\subset\mathbb{N}$ of density zero, by Lemma 3.1.
2. Every $g\perp S(f)\oplus\{\text{constants}\}$ satisfies (3.2) because $\langle g,f\circ T^n\rangle$ is eventually zero and $\int g=\langle g,1\rangle=0$.

It follows that every $g\in L^2$ satisfies (3.2).

This proves the proposition for invertible ppt. Now consider the case of a non-invertible ppt. Let $(\widetilde{X},\widetilde{\mathscr{B}},\widetilde{\mu},\widetilde{T})$ be the natural extension. A close look at the definition of $\widetilde{\mathscr{B}}$ shows that if $\widetilde{f}:\widetilde{X}\to\mathbb{R}$ is a $\widetilde{\mathscr{B}}$–measurable eigenfunction, then $\widetilde{f}(\dots,x_{-1},x_0,x_1,\dots)$ can be approximated in L^2 by functions which depend only on x_0. Therefore $\widetilde{f}(\dots,x_{-1},x_0,x_1,\dots)$ is a.e. completely determined by x_0. Let $\pi:\widetilde{X}\to X$ be the natural projection $\pi(\dots,x_{-1},x_0,x_1,\dots)=x_0$, then $\widetilde{f}:\widetilde{X}\to\mathbb{C}$ is of the form $f\circ\widetilde{\pi}$ where f is $\mathscr{B}$–measurable. Thus every eigenfunction for $\widetilde{T}$ is a lift of an eigenfunction for T. It follows that if T is weak mixing, then $\widetilde{T}$ is weak mixing. By the first part of the proof, $\widetilde{T}$ satisfies (3.2). Since T is a factor of T, it also satisfies (3.2).

For a direct proof of the proposition for non-invertible maps, which does not rely on the natural extension, see Exercise 3.4. $\square$

3.3 The Koopman operator of a Bernoulli scheme

In this section we analyze the Koopman operator of an invertible Bernoulli scheme. The idea is to produce an orthonormal basis for L^2 which makes the action of U_T transparent.

We cannot expect to diagonalize U_T: Bernoulli schemes are mixing, so they have no non-constant eigenfunctions. But we shall we see that we can get the following nice structure:

Definition 3.7. An invertible ppt is said to have *countable Lebesgue spectrum* if L^2 has an orthonormal basis of the form $\{1\} \cup \{f_{\lambda,j} : \lambda \in \Lambda, j \in \mathbb{Z}\}$ where Λ is countable, and $U_T f_{\lambda,j} = f_{\lambda,j+1}$ for all i, j.

The reason for the terminology is that the spectral measure of each $f_{\lambda,j}$ is proportional to the Lebesgue measure on S^1 (problem 3.6).

Example. The invertible Bernoulli scheme with probability vector $(\frac{1}{2}, \frac{1}{2})$ has countable Lebesgue spectrum.

Proof. The phase space is $X = \{0,1\}^{\mathbb{Z}}$. Define for every finite non-empty $A \subset \mathbb{Z}$ the function $\varphi_A(\underline{x}) := \prod_{j \in A}(-1)^{x_j}$. Define $\varphi_\varnothing := 1$. Then,

1. if $A \neq \varnothing$ then $\int \varphi_A = 0$ (exercise)
2. if $A \neq B$, then $\varphi_A \perp \varphi_B$ because $\langle \varphi_A, \varphi_B \rangle = \int \varphi_{A \triangle B} = 0$;
3. $\mathrm{span}\{\varphi_A : A \subset \mathbb{Z} \text{ finite}\}$ is algebra of functions which separates points, and contains the constants.

By the Stone-Weierstrass theorem, $\overline{\mathrm{span}}\{\varphi_A : A \subset \mathbb{Z} \text{ finite}\} = L^2$, so $\{\varphi_A\}$ is an orthonormal basis of L^2. This is called the *Fourier–Walsh* system.

Note that $U_T \varphi_A = \varphi_{A+1}$, where $A+1 := \{a+1 : a \in A\}$. Take Λ the set of equivalence classes of the relation $A \sim B \Leftrightarrow \exists c$ s.t. $A = c + B$. Let A_λ be a representative of $\lambda \in \Lambda$. The basis is $\{1\} \cup \{\varphi_{A_\lambda + n} : \lambda \in \Lambda, n \in \mathbb{Z}\} = \{\text{Fourier Walsh functions}\}$. $\square$

It is not easy to produce such bases for other Bernoulli schemes. But they exist. To prove this we introduce the following sufficient condition for countable Lebesgue spectrum, which turns out to hold for all Bernoulli schemes as well as many smooth dynamical systems:

Definition 3.8. An invertible ppt $(X, \mathscr{B}, \mu, T)$ is called a *K automorphism* if there is a σ–algebra $\mathscr{A} \subset \mathscr{B}$ s.t.

1. $T^{-1}\mathscr{A} \subset \mathscr{A}$;
2. $\mathscr{A}$ generates $\mathscr{B}$: $\sigma(\bigcup_{n \in \mathbb{Z}} T^{-n}\mathscr{A}) = \mathscr{B} \mod \mu$;[1]
3. the tail of $\mathscr{A}$ is trivial: $\bigcap_{n=0}^{\infty} T^{-n}\mathscr{A} = \{\varnothing, X\} \mod \mu$.

Proposition 3.5. *Every invertible Bernoulli scheme has the K property.*

Proof. Let $(S^{\mathbb{Z}}, \mathscr{B}(S^{\mathbb{Z}}), \mu, T)$ be a Bernoulli scheme, i.e. $\mathscr{B}(S^{\mathbb{Z}})$ is the sigma algebra generated by *cylinders* ${}_{-k}[a_{-k}, \dots, a_\ell] := \{x \in S^{\mathbb{Z}} : x_i = a_i \ (-k \le i \le \ell)\}$, T is the left shift map, and $\mu({}_k[a_{-k}, \dots, a_\ell]) = p_{a_{-k}} \cdots p_{a_\ell}$.

Call a cylinder *non-negative*, if it is of the form ${}_0[a_0, \dots, a_n]$. Let $\mathscr{A}$ be the sigma algebra generated by all non-negative cylinders. It is clear that $T^{-1}\mathscr{A} \subset \mathscr{A}$ and that $\bigcup_{n \in \mathbb{Z}} T^{-n}\mathscr{A}$ generates $\mathscr{B}(S^{\mathbb{Z}})$. We show that the measure of every element of $\bigcap_{n=0}^{\infty} T^{-n}\mathscr{A}$ is either zero or one.[2]

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[1] $\mathscr{F}_1 \subset \mathscr{F}_2 \mod \mu$ is for all $F_1 \in \mathscr{F}_2$ there is a set $F_2 \in \mathscr{F}_2$ s.t. $\mu(F_1 \triangle F_2) = 0$, and $\mathscr{F}_1 = \mathscr{F}_2 \mod \mu$ iff $\mathscr{F}_1 \subset \mathscr{F}_2 \mod \mu$ and $\mathscr{F}_2 \subset \mathscr{F}_1 \mod \mu$.

[2] Probabilists call the elements of this intersection *tail events*. The fact that every tail event for a sequence of independent identically distributed random variables has probability zero or one is called "Kolmogorov's zero–one law."

Two measurable sets A,B are called *independent*, if $\mu(A\cap B)=\mu(A)\mu(B)$. For Bernoulli schemes, any two cylinders with non-overlapping set of indices is independent (check). Thus for every cylinder B of length $|B|$,

$$B \text{ is independent of } T^{-|B|}A \text{ for all non-negative cylinders } A.$$

It follows that B is independent of every element of $T^{-|B|}\mathscr{A}$ (the set of B's like that is a sigma-algebra). Thus every cylinder B is independent of every element of $\bigcap_{n\geq 1}T^{-n}\mathscr{A}$. Thus every element of $\mathscr{B}$ is independent of every element of $\bigcap_{n\geq 1}T^{-n}\mathscr{A}$ (another sigma-algebra argument).

This means that every $E\in\bigcap_{n\geq 1}T^{-n}\mathscr{A}$ is independent of itself. Thus $\mu(E)=\mu(E\cap E)=\mu(E)^2$, whence $\mu(E)=0$ or 1. $\square$

Proposition 3.6. *Every K automorphism on a non-atomic standard probability space has countable Lebesgue spectrum.*

Proof. Let $(X,\mathscr{B},\mu,T)$ be a K automorphism of a non-atomic standard probability space. Since $(X,\mathscr{B},\mu)$ is a non-atomic standard space, $L^2(X,\mathscr{B},\mu)$ is (i) infinite dimensional, and (ii) separable.

Let $\mathscr{A}$ be a sigma algebra in the definition of the K property. Set $V:=L^2(X,\mathscr{A},\mu)$. This is a closed subspace of $L^2(X,\mathscr{B},\mu)$, and

1. $U_T(V)\subseteq V$, because $T^{-1}\mathscr{A}\subset\mathscr{A}$;
2. $\bigcup_{n\in\mathbb{Z}}U_T^n(V)$ is dense in $L^2(X,\mathscr{B},\mu)$, because $\bigcup_{n\in\mathbb{Z}}T^{-n}\mathscr{A}$ generates $\mathscr{B}$, so every $B\in\mathscr{B}$ can be approximated by a finite disjoint union of elements of $\bigcup_{n\in\mathbb{Z}}T^{-n}\mathscr{A}$;
3. $\bigcap_{n=1}^{\infty}U_T^n(V)=\{\text{constant functions}\}$, because $\bigcap_{n\geq 1}T^{-n}\mathscr{A}=\{\varnothing,X\} \mod \mu$.

Now let $W:=V\ominus U_T(V)$ (the orthogonal complement of $U_T(V)$ in V). For all $n>0$, $U_T^n(W)\subset U_T^n(V)\subset U_T(V)\perp W$. Thus $W\perp U_T^n(W)$ for all $n>0$. Since U_T^{-1} is an isometry, $W\perp U_T^n(W)$ for all $n<0$. It follows that

$$L^2(X,\mathscr{B},\mu)=\{\text{constants}\}\oplus\bigoplus_{n\in\mathbb{Z}}U_T^n(W) \quad \text{(orthogonal sum)}.$$

If $\{f_\lambda:\lambda\in\Lambda\}$ is an orthonormal basis for W, then the above implies that

$$\{1\}\cup\{U_T^n f_\lambda:\lambda\in\Lambda\}$$

is an orthonormal basis of $L^2(X,\mathscr{B},\mu)$ (check!).

This is almost the full countable Lebesgue spectrum property. It remains to show that $|\Lambda|=\aleph_0$. $|\Lambda|\leq\aleph_0$ because $L^2(X,\mathscr{B},\mu)$ is separable. We show that Λ is infinite by proving $\dim(W)=\infty$. We use the following fact (to be proved later):

$$\forall N\ \exists A_1,\ldots,A_N\in\mathscr{A} \text{ pairwise disjoint sets, with positive measure.} \tag{3.3}$$

Suppose we know this. Pick $f\in W\setminus\{0\}$ ($W\neq\{0\}$, otherwise $L^2=\{\text{constants}\}$ and $(X,\mathscr{B},\mu)$ is atomic). Set $w_i:=f1_{A_i}\circ T$ with $A_1,\ldots,A_N$ as above, then (i) w_i are linearly independent (because they have disjoint supports); (ii) $w_i\in V$ (because

$T^{-1}A_i \in T^{-1}\mathscr{A} \subset \mathscr{A}$, so w_i is $\mathscr{A}$–measurable); and (iii) $w_i \perp U_T(V)$ (check, using $f \in W$). It follows that $\dim(W) \geq N$. Since N was arbitrary, $\dim(W) = \infty$.

Here is the proof of (3.3). Since $(X, \mathscr{B}, \mu)$ is non-atomic, $\exists B_1, \ldots, B_N \in \mathscr{B}$ pairwise disjoint with positive measure. By assumption, $\bigcup_{n\in\mathbb{Z}} T^n\mathscr{A}$ generates $\mathscr{B}$, thus we can approximate B_i arbitrarily well by elements of $\bigcup_{n\in\mathbb{Z}} T^n\mathscr{A}$. By assumption, $\mathscr{A} \subseteq T\mathscr{A}$. This means that we can approximate B_i arbitrarily well by sets from $T^n\mathscr{A}$ by choosing n sufficiently large. It follows that $L^2(X, T^n\mathscr{A}, \mu)$ has dimension at least N. This forces $T^n\mathscr{A}$ to contain at least N pairwise disjoint sets of positive measure. It follows that $\mathscr{A}$ contains at least N pairwise disjoint sets of positive measure. ☐

Corollary 3.1. *All systems with countable Lebesgue spectrum, whence all invertible Bernoulli schemes, are spectrally isomorphic.*

Proof. Problem 3.7. ☐

We see that all Bernoulli schemes are spectrally isomorphic. But it is not true that all Bernoulli schemes are measure theoretically isomorphic. To prove this one needs new (non-spectral) invariants. Enter the *measure theoretic entropy*, a new invariant which we discuss in the next chapter.

Problems

3.1. Suppose $(X, \mathscr{B}, \mu, T)$ is an ergodic ppt on a Lebesgue space, and let $H(T)$ be its group of eigenvalues.

1. show that if f is an eigenfunction, then $|f| = \text{const.}$ a.e., and that if $\lambda, \mu \in H(T)$, then so do $1, \lambda\mu, \lambda/\mu$.
2. Show that eigenfunctions of different eigenvalue are orthogonal. Deduce that $H(T)$ is a countable subgroup of the unit circle.

3.2. Prove that the irrational rotation R_α has discrete spectrum, and calculate $H(R_\alpha)$.

3.3. Koopman - von Neumann Lemma
Suppose a_n is a bounded sequence of non-negative numbers. Prove that $\frac{1}{N}\sum_{n=1}^N a_n \to 0$ iff there is a set of zero density $\mathscr{N} \subset \mathbb{N}$ s.t. $a_n \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} 0$. Guidance: Fill in the details in the following argument.

1. Suppose $\mathscr{N} \subset \mathbb{N}$ has density zero and $a_n \xrightarrow[\mathscr{N} \not\ni n \to \infty]{} 0$, then $\frac{1}{N}\sum_{n=1}^N a_n \to 0$.
2. Now assume that $\frac{1}{N}\sum_{n=1}^N a_n \to 0$.
   a. Show that $\mathscr{N}_m := \{k : a_k > 1/m\}$ form an increasing sequence of sets of density zero.
   b. Fix $\varepsilon_i \downarrow 0$, and choose $k_i \uparrow \infty$ such that if $n > k_i$, then $(1/n)|\mathscr{N}_i \cap [1, n]| < \varepsilon_i$. Show that $\mathscr{N} := \bigcup_i \mathscr{N}_i \cap (k_i, k_{i+1}]$ has density zero.

c. Show that $a_n \xrightarrow[\mathcal{N}\not\ni n\to\infty]{} 0$.

3.4. Here is a sketch of an alternative proof of proposition 3.4, which avoids natural extensions (B. Parry). Fill in the details.

1. Set $H := L^2$, $V := \bigcap_{n\geq 0} U_T^n(H)$, and $W := H \ominus U_T H := \{g \in H, g \perp U_T H\}$.
   a. $H = V \oplus [(U_T H)^\perp + (U_T^2)^\perp + \cdots]$
   b. $\{U_T^k H\}$ is decreasing, $\{(U_T^k H)^\perp\}$ us increasing.
   c. $H = V \oplus \bigoplus_{k=1}^\infty U_T^k W$ (orthogonal space decomposition).
2. $U_T : V \to V$ has a bounded inverse (hint: use the fact from Banach space theory that any bounded linear operator between mapping one Banach space *onto* another Banach space which is one-to-one, has a bounded inverse).
3. (3.2) holds for any $f, g \in V$.
4. if $g \in U_T^k W$ for some k, then (3.2) holds for all $f \in L^2$.
5. if $g \in V$, but $f \in U_T^k W$ for some k, then (3.2) holds for f, g.
6. (3.2) holds for all $f, g \in L^2$.

3.5. Show that every invertible ppt with countable Lebesgue spectrum is mixing, whence ergodic.

3.6. Suppose $(X, \mathscr{B}, \mu, T)$ has countable Lebesgue spectrum. Show that $\{f \in L^2 : \int f = 0\}$ is spanned by functions f whose spectral measures ν_f are equal to the Lebesgue measure on S^1.

3.7. Show that any two ppt with countable Lebesgue spectrum are spectrally isomorphic.

3.8. Cutting and Stacking and Chacon's Example
This is an example of a ppt which is weak mixing but not mixing. The example is a certain map of the unit interval, which preserves Lebesgue's measure. It is constructed using the method of "cutting and stacking" which we now explain.

Let $A_0 = [1, \frac{2}{3})$ and $R_0 := [\frac{2}{3}, 1]$ (thought of as reservoir).

Step 1: Divide A_0 into three equal subintervals of length $\frac{2}{9}$. Cut a subinterval B_0 of length $\frac{2}{9}$ from the left end of the reservoir.

- Stack the three thirds of A_0 one on top of the other, starting from the left and moving to the right.
- Stick B_0 between the second and third interval.
- Define a partial map f_1 by moving points vertically in the stack. The map is defined everywhere except on $R \setminus B_0$ and the top floor of the stack. It can be viewed as a partially defined map of the unit interval.

Update the reservoir: $R_1 := R \setminus B_0$. Let A_1 be the base of the new stack (equal to the rightmost third of A_0).

Step 2: Cut the stack vertically into three equal stacks. The base of each of these thirds has length $\frac{1}{3} \times \frac{2}{9}$. Cut an interval B_1 of length $\frac{1}{3} \times \frac{2}{9}$ from the left side of the reservoir R_1.

- Stack the three stacks one on top of the other, starting from the left and moving to the right.
- Stick B_1 between the second stack and the third stack.
- Define a partial map f_2 by moving points vertically in the stack. This map is defined everywhere except the union of the top floor floor and $R_1 \setminus B_1$.

Update the reservoir: $R_2 := R_1 \setminus B_1$. Let A_2 be the base of the new stack (equal to the rightmost third of A_1).

Step 3: Cut the stack vertically into three equal stacks. The base of each of these thirds has length $\frac{1}{3^2} \times \frac{2}{9}$. Cut an interval B_2 of length $\frac{1}{3^2} \times \frac{2}{9}$ from the left side of the reservoir R_2.

- Stack the three stacks one on top of the other, starting from the left and moving to the right.
- Stick B_2 between the second stack and the third stack.
- Define a partial map f_3 by moving points vertically in the stack. This map is defined everywhere except the union of the top floor floor and $R_2 \setminus B_2$.

Update the reservoir: $R_3 := R_2 \setminus B_2$. Let A_3 be the base of the new stack (equal to the rightmost third of A_2)

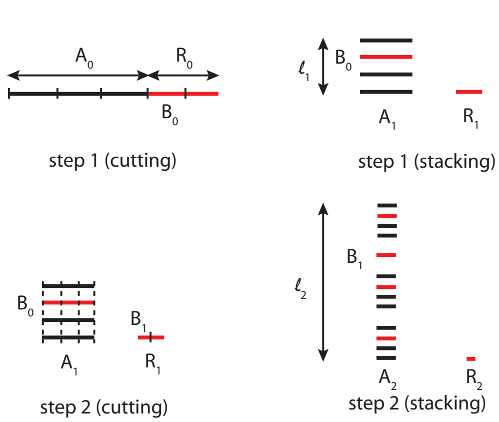

Fig. 3.1 The construction of Chacon's example

Continue in this manner, to obtain a sequence of partially defined maps f_n. There is a canonical way of viewing the intervals composing the stacks as of subintervals of the unit interval. Using this identification, we may view f_n as partially defined maps of the unit interval.

1. Show that f_n is measure preserving where it is defined (the measure is Lebesgue's measure). Calculate the Lebesgue measure of the domain of f_n.
2. Show that f_{n+1} extends f_n (i.e. the maps agree on the intersection of their domains). Deduce that the common extension of f_n defines an invertible probability preserving map of the open unit interval. This is Chacon's example. Denote it by $(I, \mathscr{B}, m, T)$.
3. Let ℓ_n denote the height of the stack at step n. Show that the sets $\{T^i(A_n) : i = 0, \ldots, \ell_n,\ n \geq 1\}$ generate the Borel σ–algebra of the unit interval.

3.9. (Continuation) Prove that Chacon's example is weak mixing using the following steps. Suppose f is an eigenfunction with eigenvalue λ.

1. We first show that if f is constant on A_n for some n, then f is constant everywhere. (A_n is the base of the stack at step n.)

   a. Let ℓ_k denote the height of the stack at step k. Show that $A_{n+1} \subset A_n$, and $T^{\ell_n}(A_{n+1}) \subset A_n$. Deduce that $\lambda^{\ell_n} = 1$.
   b. Prove that $\lambda^{\ell_{n+1}} = 1$. Find a recursive formula for ℓ_n. Deduce that $\lambda = 1$.
   c. The previous steps show that f is an invariant function. Show that any invariant function which constant on A_n is constant almost everywhere.

2. We now consider the case of a general L^2– eigenfunction.

   a. Show, using Lusin's theorem, that there exists an n such that f is nearly constant on most of A_n. (Hint: part 3 of the previous question).
   b. Modify the argument done above to show that any L^2–eigenfunction is constant almost everywhere.

3.10. (Continuation) Prove that Chacon's example is not mixing, using the following steps.

1. Inspect the image of the top floor of the stack at step n, and show that for every n and $0 \leq k \leq \ell_{n-1}$, $m(T^k A_n \cap T^{k+\ell_n} A_n) \geq \frac{1}{3} m(T^k A_n)$.
2. Use problem 3.8 part 3 and an approximation argument to show that for every Borel set E and $\varepsilon > 0$, $m(E \cap T^{\ell_n} E) \geq \frac{1}{3} m(E) - \varepsilon$ for all n. Deduce that T cannot be mixing.

Notes to chapter 3

The spectral approach to ergodic theory is due to von Neumann. For a thorough modern introduction to the theory, see Nadkarni's book [1]. Our exposition follows in parts the books by Parry [2] and Petersen [1]. A proof of the discrete spectrum theorem mentioned in the text can be found in Walters' book [5]. A proof of Herglotz's theorem is given in [2].

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Chapter 4
Entropy

We saw at the end of the last chapter that every two Bernoulli schemes are spectrally isomorphic (because they have countable Lebesgue spectrum). The question whether any two Bernoulli schemes are measure theoretically isomorphic was a major open problem until it was answered negatively by Kolmogorov and Sinai, who introduced for this purpose a new invariant: *metric entropy*. Later, Ornstein proved that this invariant is complete within the class of Bernoulli schemes: Two Bernoulli schemes are measure theoretically isomorphic iff their metric entropies are equal.

4.1 Information content and entropy

Let $(X,\mathscr{B},\mu)$ be a probability space. Suppose $x\in X$ is unknown. How to quantify the "information content" $I(A)$ of the statement "x belongs to A?"

1. $I(A)$ should be non-negative, and if $\mu(A)=1$ then $I(A)=0$;
2. If A,B are independent, i.e. $\mu(A\cap B)=\mu(A)\mu(B)$, then $I(A\cap B)=I(A)+I(B)$;
3. $I(A)$ should be a Borel decreasing function of the probability of A.

Proposition 4.1. *The only functions $\varphi:[0,1]\to\mathbb{R}^+$ such that $I(A)=\varphi[\mu(A)]$ satisfies the above axioms for all probability spaces $(X,\mathscr{B},\mu)$ are $c\ln t$ with $c<0$.*

We leave the proof as an exercise. This leads to the following definition.

Definition 4.1 (Shannon). Let $(X,\mathscr{B},\mu)$ be a probability space.

1. The *Information Content* of a set $A\in\mathscr{B}$ is $I_\mu(A):=-\log_2\mu[A]$
2. The *Information Function* of a countable measurable partition α of X is

$$I_\mu(\alpha)(x):=\sum_{A\in\alpha}I_\mu(A)1_A(x)=-\sum_{A\in\alpha}\log_2\mu(A)1_A(x)$$

3. The *Entropy* of a countable measurable partition is the average of the information content of its elements: $H_\mu(\alpha):=\int_X I_\mu(\alpha)d\mu=-\sum_{A\in\alpha}\mu(A)\log_2\mu(A)$.

Conventions: Henceforth, $\log = \log_2, \ln = \log_e$ and $0\log 0 = 0$.

The are important conditional versions of these notions:

Definition 4.2. Let $(X,\mathscr{B},\mu)$ be a probability space, and suppose $\mathscr{F}$ is a sub-σ-algebra of $\mathscr{B}$. We use the notation $\mu(A|\mathscr{F})(x) := \mathbb{E}(1_A|\mathscr{F})(x)$ (as L^1–elements).

1. The *information content* of A *given* $\mathscr{F}$ is $I_\mu(A|\mathscr{F})(x) := -\log_2 \mu(A|\mathscr{F})(x)$
2. The *information function* of a finite measurable partition α *given* $\mathscr{F}$ is $I_\mu(\alpha|\mathscr{F}) := \sum_{A\in\alpha} I_\mu(A|\mathscr{F})1_A$
3. The *conditional entropy* of α *given* $\mathscr{F}$ is $H_\mu(\alpha|\mathscr{F}) := \int I_\mu(\alpha|\mathscr{F})d\mu$.

Convention: Let α,β be partitions; We write $H_\mu(\alpha|\beta)$ for $H_\mu(\alpha|\sigma(\beta))$, where $\sigma(\beta)$:=smallest σ–algebra which contains β.

The following formulæ are immediate:

$$H_\mu(\alpha|\mathscr{F}) = -\int_X \sum_{A\in\alpha} \mu(A|\mathscr{F})(x)\log\mu(A|\mathscr{F})(x)d\mu(x)$$

$$H_\mu(\alpha|\beta) = -\sum_{B\in\beta}\mu(B)\sum_{A\in\alpha}\mu(A|B)\log\mu(A|B), \text{ where } \mu(A|B) = \frac{\mu(A\cap B)}{\mu(B)}.$$

4.2 Properties of the entropy of a partition

We need some notation and terminology. Let α,β be two countable partitions.

1. $\sigma(\alpha)$ is the smallest σ–algebra which contains α;
2. $\alpha \leq \beta$ means that $\alpha \subseteq \sigma(\beta) \mod \mu$, i.e. every element of α is equal up to a set of measure zero to an element of $\sigma(\beta)$. Equivalently, $\alpha \leq \beta$ if every element of α is equal up to a set of measure zero to a union of elements of β. We say that β is *finer* than α, and that α is *coarser* than β.
3. $\alpha = \beta \mod \mu$ iff $\alpha \leq \beta$ and $\beta \leq \alpha$.
4. $\alpha \vee \beta$ is the smallest partition which is finer than both α and β. Equivalently, $\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}$.

If $\mathscr{F}_1,\mathscr{F}_2$ are two σ–algebras, then $\mathscr{F}_1 \vee \mathscr{F}_2$ is the smallest σ–algebra which contains $\mathscr{F}_1,\mathscr{F}_2$.

4.2.1 The entropy of $\alpha \vee \beta$

Suppose $\alpha = \{A_1,\dots,A_n\}$ is a finite measurable partition of a probability space $(X,\mathscr{B},\mu)$. Suppose $x \in X$ is unknown. It is useful to think of the information content of α as of the partial information on x "which element of α contains an x."

We state and prove a formula which says that the information content of α *and* β is the information content of α plus the information content of β *given* the knowledge α.

Theorem 4.1 (The Basic Identity). *Suppose α, β are measurable countable partitions, and assume $H_\mu(\alpha), H_\mu(\beta) < \infty$, then*

1. $I_\mu(\alpha \vee \beta|\mathscr{F}) = I_\mu(\alpha|\mathscr{F}) + I_\mu(\beta|\mathscr{F} \vee \sigma(\alpha))$;
2. $H_\mu(\alpha \vee \beta) = H_\mu(\alpha) + H_\mu(\beta|\alpha)$.

Proof. We calculate $I_\mu(\beta|\mathscr{F} \vee \sigma(\alpha))$:

$$I_\mu(\beta|\mathscr{F} \vee \sigma(\alpha)) = -\sum_{B\in\beta} 1_B \log \mu(B|\mathscr{F} \vee \sigma(\alpha))$$

Claim: $\mu(B|\mathscr{F} \vee \sigma(\alpha)) = \sum_{A\in\alpha} 1_A \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})}$:

1. This expression is $\mathscr{F} \vee \sigma(\alpha)$–measurable
2. Observe that $\mathscr{F} \vee \sigma(\alpha) = \{\biguplus_{A\in\alpha} A \cap F_A : F_A \in \mathscr{F}\}$ (this is a σ–algebra which contains α and $\mathscr{F}$). Thus every $\mathscr{F} \vee \sigma(\alpha)$–measurable function is of the form $\sum_{A\in\alpha} 1_A \varphi_A$ with φ_A $\mathscr{F}$–measurable. It is therefore enough to check test functions of the form $1_A\varphi$ with $\varphi \in L^\infty(\mathscr{F})$. For such functions

$$\int 1_A \varphi \sum_{A'\in\alpha} 1_{A'} \frac{\mu(B\cap A'|\mathscr{F})}{\mu(A'|\mathscr{F})} d\mu = \int 1_A \varphi \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})} d\mu =$$
$$= \int \mathbb{E}(1_A|\mathscr{F}) \varphi \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})} d\mu = \int \varphi \mu(B\cap A|\mathscr{F}) d\mu = \int 1_A \varphi \cdot 1_B d\mu.$$

So $\mu(B|\mathscr{F} \vee \sigma(\alpha)) = \sum_{A\in\alpha} 1_A \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})}$ as claimed.

Using the claim, we see that

$$\begin{aligned}
I_\mu(\beta|\mathscr{F} \vee \sigma(\alpha)) &= -\sum_{B\in\beta} 1_B \log \sum_{A\in\alpha} 1_A \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})} \\
&= -\sum_{B\in\beta}\sum_{A\in\alpha} 1_{A\cap B} \log \frac{\mu(B\cap A|\mathscr{F})}{\mu(A|\mathscr{F})} \\
&= -\sum_{B\in\beta}\sum_{A\in\alpha} 1_{A\cap B} \log \mu(B\cap A|\mathscr{F}) + \sum_{A\in\alpha}\sum_{B\in\beta} 1_{A\cap B} \log \mu(A|\mathscr{F}) \\
&= I_\mu(\alpha \vee \beta|\mathscr{F}) - I_\mu(\alpha|\mathscr{F}).
\end{aligned}$$

This proves the first part of the theorem.

Integrating, we get $H_\mu(\alpha \vee \beta|\mathscr{F}) = H_\mu(\alpha|\mathscr{F}) + H_\mu(\beta|\mathscr{F} \vee \alpha)$. If $\mathscr{F} = \{\varnothing, X\}$, then $H_\mu(\alpha \vee \beta) = H_\mu(\alpha) + H_\mu(\beta|\alpha)$. ☐

4.2.2 Convexity properties

Lemma 4.1. *Let $\varphi(t) := -t\log t$, then for every probability vector $(p_1,\dots,p_n)$ and $x_1,\dots,x_n \in [0,1]$ $\varphi(p_1x_1+\cdots+p_nx_n) \ge p_1\varphi(x_1)+\cdots+p_n\varphi(x_n)$, with equality iff all the x_i with i s.t. $p_i \neq 0$ are equal.*

Proof. This is because $\varphi(\cdot)$ is strictly concave. Let $m := \sum p_i x_i$. If $m = 0$ then the lemma is obvious, so suppose $m > 0$. It is an exercise in calculus to see that $\varphi(t) \le \varphi(m)+\varphi'(m)(t-m)$ for $t \in [0,1]$, with equality iff $t = m$. In the particular case $m = \sum p_i x_i$ and $t = x_i$ we get

$$p_i\varphi(x_i) \le p_i\varphi(m)+\varphi'(m)(p_ix_i - p_im) \text{ with equality iff } p_i = 0 \text{ or } x_i = m.$$

Summing over i, we get $\sum p_i\varphi(x_i) \le \varphi(m)+\varphi'(m)(\sum p_ix_i - m) = \varphi(m)$. There is an equality iff for every i $p_i = 0$ or $x_i = m$. ☐

Proposition 4.2 (Convexity properties). *Let α,β,γ be countable measurable partitions with finite entropies, then*

1. $\alpha \le \beta \Rightarrow H_\mu(\alpha|\gamma) \le H_\mu(\beta|\gamma)$
2. $\alpha \le \beta \Rightarrow H_\mu(\gamma|\alpha) \ge H_\mu(\gamma|\beta)$

Proof. The basic identity shows that $\beta\vee\gamma$ has finite entropy, and so $H_\mu(\beta|\gamma) = H_\mu(\alpha\vee\beta|\gamma) = H_\mu(\alpha|\gamma)+H_\mu(\beta|\gamma\vee\alpha) \ge H_\mu(\alpha|\gamma)$.

For the second inequality, note that $\varphi(t) = -t\log t$ is strictly concave (i.e. its negative is convex), therefore by Jensen's inequality

$$H_\mu(\gamma|\alpha) = \int \sum_{C\in\gamma} \varphi[\mathbb{E}(C|\sigma(\alpha))]d\mu = \int \sum_{C\in\gamma} \varphi[\mathbb{E}(\mathbb{E}(C|\sigma(\beta))|\sigma(\alpha))]d\mu \ge$$

$$\ge \int \sum_{C\in\gamma} \mathbb{E}(\varphi[\mathbb{E}(1_C|\sigma(\beta))]|\sigma(\alpha))]d\mu = \sum_{C\in\gamma}\int \varphi[\mathbb{E}(1_C|\sigma(\beta))]d\mu \equiv H_\mu(\gamma|\beta),$$

proving the inequality. ☐

4.2.3 Information and independence

We say that two partitions are *independent*, if $\forall A\in\alpha, B\in\beta$, $\mu(A\cap B) = \mu(A)\mu(B)$. This the same as saying that the random variables $\alpha(x),\beta(x)$ are independent.

Proposition 4.3 (Information and Independence). $H_\mu(\alpha\vee\beta) \le H_\mu(\alpha)+H_\mu(\beta)$ *with equality iff α,β are independent.*

Proof. $H_\mu(\alpha\vee\beta) = H_\mu(\alpha)+H_\mu(\beta)$ iff $H_\mu(\alpha|\beta) = H_\mu(\alpha)$. But

$$H_\mu(\alpha|\beta) = -\sum_{B\in\beta}\mu(B)\sum_{A\in\alpha}\mu(A|B)\log\mu(A|B).$$

Let $\varphi(t) = -t\log t$. We have:

$$\sum_{A\in\alpha}\sum_{B\in\beta}\mu(B)\varphi[\mu(A|B)] = \sum_{A\in\alpha}\varphi[\mu(A)].$$

But φ is strictly concave, so $\sum_{B\in\beta}\mu(B)\varphi[\mu(A|B)] \leq \varphi[\mu(A)]$, with equality iff $\mu(A|B)$ are equal for all $B\in\beta$ s.t. $\mu(B)\neq 0$.

We conclude that $\mu(A|B) = c(A)$ for all $B\in\beta$ s.t. $\mu(B)\neq 0$. For such B, $\mu(A\cap B) = c(A)\mu(B)$. Summing over B, gives $c(A) = \mu(A)$ and we obtain the independence condition. $\square$

4.3 The Metric Entropy

4.3.1 Definition and meaning

Definition 4.3 (Kolmogorov, Sinai). The *metric entropy* of a ppt $(X,\mathscr{B},\mu,T)$ is defined to be

$$h_\mu(T) := \sup\{h_\mu(T,\alpha) : \alpha \text{ is a countable measurable partition s.t. } H_\mu(\alpha)<\infty\},$$

where $h_\mu(T,\alpha) := \lim\limits_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee\limits_{i=0}^{n-1}T^{-i}\alpha\right)$.

It can be shown that the supremum is attained by *finite* measurable partitions (problem 4.9).

Proposition 4.4. *The limit which defines $h_\mu(T,\alpha)$ exists.*

Proof. Write $\alpha_n := \bigvee_{i=0}^{n-1}T^{-i}\alpha$. Then $a_n := H_\mu(\alpha_n)$ is subadditive, because $a_{n+m} := H_\mu(\alpha_{n+m}) \leq H_\mu(\alpha_n) + H_\mu(T^{-n}\alpha_m) = a_n + a_m$.

We claim that any sequence of numbers $\{a_n\}_{n\geq 1}$ which satisfies $a_{n+m}\leq a_n + a_m$ converges to a limit (possibly equal to minus infinity), and that this limit is $\inf[a_n/n]$. Fix n. Then for every m, $m = kn+r$, $0\leq r\leq n-1$, so

$$a_m \leq ka_n + a_r.$$

Dividing by m, we get that for all $m>n$

$$\frac{a_m}{m} \leq \frac{ka_n + a_r}{kn+r} \leq \frac{a_n}{n} + \frac{a_r}{m},$$

whence $\limsup(a_m/m)\leq a_n/n$. Since this is true for all n, $\limsup a_m/m \leq \inf a_n/n$. But it is obvious that $\liminf a_m/m \geq \inf a_n/n$, so the limsup and liminf are equal, and their common value is $\inf a_n/n$.

We remark that in our case the limit is not minus infinity, because $H_\mu(\bigvee_{i=0}^{n-1}T^{-i}\alpha)$ are all non-negative. $\square$

$H_\mu(\alpha_n)$ is the average information content in the first n–digits of the α–itinerary. Dividing by n gives the average "information per unit time." Thus the entropy measure the maximal rate of information production the system is capable of generating.

It is also possible to think of entropy as a measure of unpredictability. Let's think of T as moving *backward* in time. Then $\alpha_1^\infty := \sigma(\bigcup_{n=1}^\infty T^{-n}\alpha)$ contains the information on the past of the itinerary. Given the future, how unpredictable is the present, on average? This is measured by $H_\mu(\alpha|\alpha_1^\infty)$.

Theorem 4.2. *If* $H_\mu(\alpha) < \infty$*, then* $h_\mu(T,\alpha) = H_\mu(\alpha|\alpha_1^\infty)$*, where* $\alpha_1^\infty = \sigma\left(\bigcup_{i=1}^\infty T^{-i}\alpha\right)$.

Proof. We show that $h_\mu(T,\alpha) = H_\mu(\alpha|\alpha_1^\infty)$. Let $\alpha_m^n := \bigvee_{i=m}^n T^{-i}\alpha$, then

$$H_\mu(\alpha|\alpha_0^n) = H_\mu(\alpha_0^n) - H_\mu(T^{-1}\alpha_0^{n-1}) = H_\mu(\alpha_0^n) - H_\mu(\alpha_0^{n-1}).$$

Summing over n, we obtain

$$H_\mu(\alpha_n) - H_\mu(\alpha) = \sum_{k=1}^n H_\mu(\alpha|\alpha_1^k)$$

Dividing by n and passing to the limit we get

$$h_\mu(T,\alpha) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n H_\mu(\alpha|\alpha_1^k)$$

It is therefore enough to show that $H_\mu(\alpha|\alpha_1^k) \xrightarrow[k\to\infty]{} H_\mu(\alpha|\alpha_1^\infty)$.

This is dangerous! It is true that $H_\mu(\alpha|\alpha_1^k) = \int I_\mu(\alpha|\alpha_1^k)d\mu$ and that by the martingale convergence theorem

$$I_\mu(\alpha|\alpha_1^k) \xrightarrow[k\to\infty]{} I_\mu(\alpha|\alpha_1^\infty) \text{ a.e. .}$$

But the claim that the integral of the limit is equal to the limit of the integrals requires justification.

If $|\alpha| < \infty$, then we can bypass the problem by writing

$$H_\mu(\alpha|\alpha_1^k) = \int \sum_{A\in\alpha} \varphi[\mu(A|\alpha_1^k)]d\mu, \text{ with } \varphi(t) = -t\log t,$$

and noting that *this* function is bounded (by $|\alpha|\max\varphi$). This allows us to apply the bounded convergence theorem, and deduce that $H_\mu(\alpha|\alpha_1^k) \xrightarrow[k\to\infty]{} H_\mu(\alpha|\alpha_1^\infty)$.

If $|\alpha| = \infty$ (but $H_\mu(\alpha) < \infty$) then we need to be more clever, and appeal to the following lemma (proved below):

Lemma 4.2 (Chung–Neveu). *Suppose* α *is a countable measurable partition with finite entropy, then the function* $f^* := \sup_{n\geq 1} I_\mu(\alpha|\alpha_1^n)$ *is absolutely integrable.*

The result now follows from the dominated convergence theorem. □

Here is the proof of the Chung Neveu Lemma. Fix $A \in \alpha$, then we may decompose $A \cap [f^* > t] = \biguplus_{m \geq 1} A \cap B_m(t;A)$, where

$$B_m(t;A) := \{x \in X : \ m \text{ is the minimal natural number s.t. } -\log_2 \mu(A|\alpha_1^m) > t\}.$$

We have

$$\begin{aligned}
\mu[A \cap B_m(t;A)] &= \mathbb{E}_\mu\left(1_A 1_{B_m(t;A)}\right) = \mathbb{E}_\mu\left(\mathbb{E}_\mu\left(1_A 1_{B_m(t;A)} | \sigma(\alpha_1^m)\right)\right) \\
&= \mathbb{E}_\mu\left(1_{B_m(t;A)} \mathbb{E}_\mu\left(1_A | \sigma(\alpha_1^m)\right)\right), \text{ because } B_m(t;A) \in \sigma(\alpha_1^m) \\
&\equiv \mathbb{E}_\mu\left(1_{B_m(t;A)} 2^{\log_2 \mu\left(A|\sigma(\alpha_1^m)\right)}\right) \\
&\leq \mathbb{E}_\mu\left(1_{B_m(t;A)} 2^{-t}\right) = 2^{-t} \mu[B_m(t;A)].
\end{aligned}$$

Summing over m we see that $\mu(A \cap [f^* > t]) \leq 2^{-t}$. Of course we also have $\mu(A \cap [f^* > t]) \leq \mu(A)$. Thus $\mu(A \cap [f^* > t]) \leq \min\{\mu(A), 2^{-t}\}$.

We now use the following fact from measure theory: If $g \geq 0$, then $\int g d\mu = \int_0^\infty \mu[g > t] dt$:[1]

$$\begin{aligned}
\int_A f^* d\mu &= \int_0^\infty \mu(A \cap [f^* > t]) dt \leq \int_0^\infty \min\{\mu(A), 2^{-t}\} dt \\
&\leq \int_0^{-\log_2 \mu(A)} \mu(A) dt + \int_{-\log_2 \mu(A)}^\infty 2^{-t} dt = -\mu(A) \log_2 \mu(A) - \left.\frac{2^{-t}}{\ln 2}\right]_{-\log_2 \mu(A)}^\infty \\
&= -\mu(A) \log_2 \mu(A) + \mu(A)/\ln 2.
\end{aligned}$$

Summing over $A \in \alpha$ we get that $\int f^* d\mu \leq H_\mu(\alpha) + (\ln 2)^{-1} < \infty$. ☐

4.3.2 The Shannon–McMillan–Breiman Theorem

Theorem 4.3 (Shannon–McMillan–Breiman). *Let $(X, \mathscr{B}, \mu, T)$ be an ergodic ppt, and α a countable measurable partition of finite entropy, then*

$$\frac{1}{n} I_\mu(\alpha_0^{n-1}) \xrightarrow[n\to\infty]{} h_\mu(T, \alpha) \text{ a.e.}$$

In particular, if $\alpha_n(x)$:=element of α_n which contains x, then

$$-\frac{1}{n} \log \mu(\alpha_n(x)) \xrightarrow[n\to\infty]{} h_\mu(T, \alpha) \text{ a.e.}$$

Proof. Let $\alpha_0^n := \bigvee_{i=0}^n T^{-i}\alpha$. Recall the basic identity $I_\mu(\alpha_0^{n-1}) \equiv I_\mu(\alpha_1^{n-1} \vee \alpha) = I_\mu(\alpha_1^{n-1}) + I_\mu(\alpha|\alpha_1^{n-1})$. This gives

[1] Proof: $\int_X g d\mu = \int_X \int_0^\infty 1_{[0 \leq t < g(x)]}(x,t) dt d\mu(x) = \int_0^\infty \int_X 1_{[g>t]}(x,t) d\mu(x) dt = \int_0^\infty \mu[g > t] dt$.

$$\begin{aligned}I_\mu(\alpha_0^n) &= I_\mu(\alpha|\alpha_1^n) + I_\mu(\alpha_0^{n-1})\circ T\\ &= I_\mu(\alpha|\alpha_1^n) + [I_\mu(\alpha|\alpha_1^{n-1}) + I_\mu(\alpha_0^{n-2})\circ T]\circ T\\ &= \cdots = \sum_{k=0}^{n-1} I_\mu(\alpha|\alpha_1^{n-k})\circ T^k + I_\mu(\alpha)\circ T^n\\ &= \sum_{k=1}^{n} I_\mu(\alpha|\alpha_1^{k})\circ T^{n-k} + \left(\sum_{k=0}^{n} I_\mu(\alpha)\circ T^k - \sum_{k=0}^{n-1} I_\mu(\alpha)\circ T^k\right)\\ &= \sum_{k=1}^{n} I_\mu(\alpha|\alpha_1^{k})\circ T^{n-k} + o(n) \text{ a.e.,}\end{aligned}$$

because $\frac{1}{n}\times$brackets tends to zero a.e. by the pointwise ergodic theorem.

By the Martingale Convergence Theorem, $I_\mu(\alpha|\alpha_1^k) \xrightarrow[k\to\infty]{} I_\mu(\alpha|\alpha_1^\infty)$. The idea of the proof is to use this to say

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}I_\mu(\alpha_0^n) &\equiv \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I_\mu(\alpha|\alpha_1^k)\circ T^{n-k}\\ &\stackrel{?}{=} \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I_\mu(\alpha|\alpha_1^\infty)\circ T^{n-k} \equiv \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} I_\mu(\alpha|\alpha_1^\infty)\circ T^{k}\\ &= \int I_\mu(\alpha|\alpha_1^\infty)d\mu \qquad \text{(Ergodic Theorem)}\\ &= H_\mu(\alpha|\alpha_1^\infty) = h_\mu(T,\alpha).\end{aligned}$$

The point is to justify the question mark. Write $f_n := I_\mu(\alpha|\alpha_1^n)$ and $f_\infty = I_\mu(\alpha|\alpha_1^\infty)$. It is enough to show

$$\int \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|f_k - f_\infty|\circ T^{n-k}d\mu = 0. \tag{4.1}$$

(This implies that the limsup is zero almost everywhere.) Set $F_n := \sup_{k>n}|f_k - f_\infty|$. Then $F_n \to 0$ almost everywhere. We claim that $F_n \to 0$ in L^1. This is because of the dominated convergence theorem and the fact that $F_n \le 2f^* := 2\sup_m f_m \in L^1$ (Chung–Neveu Lemma). Fix some large N, then

$$\begin{aligned}&\int \limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|f_{n-k} - f_\infty|\circ T^k d\mu =\\ &= \int \limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-N-1}|f_{n-k} - f_\infty|\circ T^k d\mu + \int \limsup_{n\to\infty}\frac{1}{n}\sum_{k=n-N}^{n-1}|f_{n-k} - f_\infty|\circ T^k d\mu\\ &\le \int \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-N-1}F_N\circ T^k d\mu + \int\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{N-1}2f^*\circ T^k\right)\circ T^{n-N}d\mu\end{aligned}$$

$$= \int F_N d\mu + \int \left(\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{N-1} 2f^* \circ T^k \right) d\mu = \int F_N d\mu.$$

Since $F_N \to 0$ in L^1, $\int F_N d\mu \to 0$. So (4.1) holds. $\square$

4.3.3 Sinai's Generator theorem

Let $\mathscr{F}_1, \mathscr{F}_2$ be two sub-σ–algebras of a probability space $(X, \mathscr{B}, \mu)$. We write $\mathscr{F}_1 \subseteq \mathscr{F}_2 \mod \mu$, if $\forall F_1 \in \mathscr{F}_1$, $\exists F_2 \in \mathscr{F}_2$ s.t. $\mu(F_1 \triangle F_2) = 0$. We write $\mathscr{F}_1 = \mathscr{F}_2 \mod \mu$, if both inclusions hold $\mod \mu$. For example, $\mathscr{B}(\mathbb{R}) = \mathscr{B}_0(\mathbb{R}) \mod$ Lebesgue's measure. For every partition α, let $\alpha_{-\infty}^{\infty} = \bigvee_{i=-\infty}^{\infty} T^{-i}\alpha$, $\alpha_0^{\infty} := \bigvee_{i=0}^{\infty} T^{-i}\alpha$ denote the smallest σ–algebras generated by, respectively, $\bigcup_{i=-\infty}^{\infty} T^{-i}\alpha$ and $\bigcup_{i=0}^{\infty} T^{-i}\alpha$.

Definition 4.4. A countable measurable partition α is called a *generator* for an invertible $(X, \mathscr{B}, \mu, T)$ if $\bigvee_{i=-\infty}^{\infty} T^{-i}\alpha = \mathscr{B} \mod \mu$, and a *strong generator*, if $\bigvee_{i=0}^{\infty} T^{-i}\alpha = \mathscr{B} \mod \mu$.

(This latter definition makes sense in the non-invertible case as well)

Example: $\alpha = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$ is a strong generator for $Tx = 2x \mod 1$, because $\bigvee_{i=0}^{\infty} T^{-i}\alpha = \sigma(\bigcup_n \alpha_0^{n-1})$ is the Borel σ–algebra (it contains all dyadic intervals, whence all open sets).

Theorem 4.4 (Sinai's Generator Theorem). *Let $(X, \mathscr{B}, \mu, T)$ be an invertible ppt. If α is a generator of finite entropy, then $h_\mu(T) = h_\mu(T, \alpha)$. A similar statement holds for non-invertible ppt assuming that α is a strong generator.*

Proof. We treat the inverible case only, and leave the non-invertible case as an exercise. Fix a *finite* measurable partition β; Must show that $h_\mu(T, \beta) \le h_\mu(T, \alpha)$.

Step 1. $h_\mu(T, \beta) \le h_\mu(T, \alpha) + H_\mu(\beta|\alpha)$

$$\frac{1}{n} H_\mu(\beta_0^{n-1}) \le \frac{1}{n} H_\mu(\beta_0^{n-1} \vee \alpha_0^{n-1}) = \frac{1}{n} \left[H_\mu(\alpha_0^{n-1}) + H_\mu(\beta_0^{n-1}|\alpha_0^{n-1}) \right]$$

$$\le \frac{1}{n} \left[H_\mu(\alpha_0^{n-1}) + \sum_{k=0}^{n-1} H_\mu(T^{-k}\beta|\alpha_0^{n-1}) \right] \quad \because H_\mu(\xi \vee \eta|\mathscr{G}) \le H_\mu(\xi|\mathscr{G}) + H_\mu(\eta|\mathscr{G})$$

$$\le \frac{1}{n} \left[H_\mu(\alpha_0^{n-1}) + \sum_{k=0}^{n-1} H_\mu(T^{-k}\beta|T^{-k}\alpha) \right]$$

$$= \frac{1}{n} H_\mu(\alpha_0^{n-1}) + H_\mu(\beta|\alpha).$$

Now pass to the limit.

Step 2. For every N, $h_\mu(T,\beta) \le h_\mu(T,\alpha) + H_\mu(\beta|\alpha_{-N}^N)$

Repeat the previous step with α_{-N}^N instead of α, and check that $h_\mu(T,\alpha_{-N}^N) = h_\mu(T,\alpha)$.

Step 3. $H_\mu(\beta|\alpha_{-N}^N) \xrightarrow[N\to\infty]{} H_\mu(\beta|\mathscr{B}) = 0$.

$$\begin{aligned}
H_\mu(\beta|\alpha_{-N}^N) &= \int I_\mu(\beta|\alpha_{-N}^N)d\mu = -\sum_{B\in\beta}\int 1_B \log\mu(B|\alpha_{-N}^N)d\mu \\
&= -\sum_{B\in\beta}\int \mu(B|\alpha_{-N}^N)\log\mu(B|\alpha_{-N}^N)d\mu \\
&= \sum_{B\in\beta}\int \varphi[\log\mu(B|\alpha_{-N}^N)]d\mu \xrightarrow[N\to\infty]{} \sum_{B\in\beta}\int \varphi[\log\mu(B|\mathscr{B})]d\mu = 0,
\end{aligned}$$

because $\mu(B|\mathscr{B}) = 1_B$, $\varphi[1_B] = 0$, and $|\beta| < \infty$

This proves that $h_\mu(T,\alpha) \ge \sup\{h_\mu(T,\beta) : |\beta| < \infty\}$. Problem 4.9 says that this supremum is equal to $h_\mu(T)$, so we are done. $\square$

The following variation on Sinai's Generator Theorem is very useful. Suppose (X,d) is a compact metric space. A *finite Borel partition* of X is finite collection $\alpha = \{A_1,\dots,A_N\}$ of pairwise disjoint Borel sets whose union covers X. Let

$$\operatorname{diam}(\alpha) := \max\{\operatorname{diam}(A) : A \in \alpha\},$$

where $\operatorname{diam}(A) := \sup\{d(x,y) : x,y \in A\}$.

Proposition 4.5. *Let T be a Borel map on a compact metric space (X,d), and suppose α_n is a sequence of finite Borel partitions such that $\operatorname{diam}(\alpha_n) \xrightarrow[n\to\infty]{} 0$. Then for every T-invariant Borel probability measure μ, $h_\mu(T) = \lim_{n\to\infty} h_\mu(T,\alpha_n)$.*

Proof. We begin, as in the proof of Sinai's generator theorem, with the inequality

$$h_\mu(T,\beta) \le h_\mu(T,\alpha) + H_\mu(\beta|\alpha).$$

Step 1. For every ε and m there is a δ such that if $\beta = \{B_1,\dots,B_m\}$ and $\gamma = \{C_1,\dots,C_m\}$ are partitions such that $\sum\mu(B_i \triangle C_i) < \delta$, then $H_\mu(\beta|\gamma) < \varepsilon$:

Proof. We prepare a partition ξ with small entropy such that $\beta\vee\gamma = \xi\vee\gamma$ and then write $H_\mu(\beta|\gamma) = H_\mu(\beta\vee\gamma) - H_\mu(\gamma) = H_\mu(\xi\vee\gamma) - H_\mu(\gamma) \le H_\mu(\xi)$.

Here is the partition ξ: $\xi := \{B_i \cap C_j : i \ne j\} \cup \{\bigcup B_i \cap C_i\}$. Then:

1. $\beta\vee\xi = \beta\vee\gamma$: $\beta\vee\xi \le \beta\vee\gamma$ by construction, and $\beta\vee\xi \ge \beta\vee\gamma$ because $B_i\cap C_j \in \beta\vee\xi$ for $i \ne j$ and $B_i \cap C_i = B_i \setminus \bigcup_{k\ne\ell} B_k \cap C_\ell$
2. $\gamma\vee\xi = \beta\vee\gamma$ for the same reason
3. ξ contains $m(m-1)$ elements with total measure $< \delta$ and one element with total measure $> 1-\delta$. So $H_\mu(\xi) \le m(m-1)\delta|\log\delta| + (1-\delta)|\log(1-\delta)|$.

Taking δ sufficiently small gives $H_\mu(\xi) < \varepsilon$. So $H_\mu(\beta|\gamma) \leq H_\mu(\xi) < \varepsilon$.

Step 2. Suppose $\operatorname{diam}(\alpha_n) \xrightarrow[n\to\infty]{} 0$*, then* $H_\mu(\beta|\alpha_n) \xrightarrow[n\to\infty]{} 0$.

Proof. Fix δ, to be determined later. Write $\beta = \{B_1,\dots,B_m\}$ and find compact sets K_i and open sets U_i such that $K_i \subset B_i \subset U_i$ and $\sum \mu(U_i \setminus K_i) < \delta$. Find $\delta' > 0$ so small that $\operatorname{dist}(K_i,K_j) > \delta'$ for all $i \neq j$, and $N_{\delta'}(K_i) \subset U_i$.

Fix N so that $n > N \Rightarrow \operatorname{diam}(\alpha_n) < \delta'/2$. By choice of δ', every $A \in \alpha_n$ can intersect at most one K_i. This gives rise to a partition $\gamma = \{C_1,\dots,C_{m+1}\}$ where

$$C_i := \bigcup\{A \in \alpha_n : A \cap K_i \neq \varnothing\} \ \ (i = 1,\dots,m)\ ,\ C_{m+1} := X \setminus \bigcup_{i=1}^{m} C_i.$$

Note that $K_i \subset C_i \subset U_i$, whence $\mu(B_i \triangle C_i) \leq \mu(U_i \setminus K_i)$. Also

$$\mu(C_{m+1}) \leq 1 - \sum_{i=1}^{m} \mu(K_i) \leq 1 - \sum_{i=1}^{m} (\mu(B_i) - \mu(U_i \setminus K_i)) = \sum_{i=1}^{m} \mu(U_i \setminus K_i) < \delta$$

If we extend β to the partition $\{B_1,\dots,B_{m+1}\}$ where $B_{m+1} = \varnothing$, then we get

$$\sum_{i=1}^{m+1} \mu(B_i \triangle C_i) \leq \sum_{i=1}^{m} \mu(U_i \setminus K_i) + \mu(C_{m+1}) < 2\delta.$$

We now use Claim 2 to choose δ so that $\sum_{i=1}^{m+1} \mu(B_i \triangle C_i) < 2\delta \Rightarrow H_\mu(\beta|\gamma) < \varepsilon$. Since $\gamma \geq \alpha_n$, $H_\mu(\beta|\alpha_n) < \varepsilon$ for all $n > N$.

Step 3. Proof of the proposition: Pick $n_k \to \infty$ such that $h_\mu(T,\alpha_{n_k}) \to \liminf h_\mu(T,\alpha_n)$. For every finite partition β, $h_\mu(T,\beta) \leq h_\mu(T,\alpha_{n_k}) + H_\mu(\beta|\alpha_{n_k})$.

By step 1, $H_\mu(\beta|\alpha_{n_k}) \xrightarrow[k\to\infty]{} 0$, so $h_\mu(T,\beta) \leq \liminf_{n\to\infty} h_\mu(T,\alpha_n)$. Passing to the supremum over all finite partitions β, we find that

$$h_\mu(T) \leq \liminf_{n\to\infty} h_\mu(T,\alpha_n) \leq \limsup_{n\to\infty} h_\mu(T,\alpha_n) \leq h_\mu(T),$$

which proves the proposition. ☐

4.4 Examples

4.4.1 Bernoulli schemes

Proposition 4.6. *The entropy of the Bernoulli shift with probability vector* $\underline{p}$ *is* $-\sum p_i \log p_i$*. Thus the* $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$*–Bernoulli scheme and the* $(\frac{1}{2},\frac{1}{2})$*–Bernoulli scheme are not isomorphic.*

Proof. The partition into 1-cylinders $\alpha = \{[i]\}$ is a strong generator, and $H_\mu(\alpha_0^{n-1}) = -\sum_{x_0,\dots,x_{n-1}} p_{x_0} \cdots p_{x_{n-1}} \left(\log p_{x_0} + \cdots \log p_{x_{n-1}}\right) = -n \sum p_i \log p_i$. $\square$

4.4.2 Irrational rotations

Proposition 4.7. *The irrational rotation has entropy zero w.r.t. the Haar measure.*

Proof. The reason is that it is an invertible transformation with a strong generator. We first explain why any invertible map with a strong generator must have zero entropy. Suppose α is such a strong generator. Then

$$\begin{aligned} h_\mu(T,\alpha) = H_\mu(\alpha|\alpha_1^\infty) = H_\mu(T\alpha|T(\alpha_1^\infty)) = \\ = H_\mu(T\alpha|\alpha_0^\infty) = H_\mu(T\alpha|\mathscr{B}) = 0, \text{ because } T\alpha \subset \mathscr{B}. \end{aligned}$$

We now claim that $\alpha := \{A_0, A_1\}$ (the two halves of the circle) is a strong generator. It is enough to show that for every ε, $\bigcup_{n\geq 1} \alpha_0^{n-1}$ contains open covers of the circle by open arcs of diameter $< \varepsilon$ (because this forces α_0^∞ to contain all open sets).

It is enough to manufacture one arc of diameter less than ε, because the translations of this arc by $k\alpha$ will eventually cover the circle.

But such an arc necessarily exits: Choose some n s.t. $n\alpha \mod 1 \in (0,\varepsilon)$. Then $A_1 \setminus T^{-n}A_1 = (A_1 \setminus [A_1 - n\alpha]$ is an arc of diameter less than ε.

4.4.3 Markov measures

Proposition 4.8. *Suppose μ is a shift invariant Markov measure with transition matrix $P = (p_{ij})_{i,j\in S}$ and probability vector $(p_i)_{i\in S}$. Then $h_\mu(\sigma) = -\sum p_i p_{ij} \log p_{ij}$.*

Proof. The natural partition $\alpha = \{[a] : a \in S\}$ is a strong generator.

$$\begin{aligned} H_\mu(\alpha_0^n) &= -\sum_{\xi_0,\dots,\xi_n \in S} \mu[\underline{\xi}] \log \mu[\underline{\xi}] \\ &= -\sum_{\xi_0,\dots,\xi_n \in S} p_{\xi_0} p_{\xi_0\xi_1} \cdots p_{\xi_{n-1}\xi_n} \left(\log p_{\xi_0} + \log p_{\xi_0\xi_1} + \cdots + \log p_{\xi_{n-1}\xi_n}\right) \\ &= -\sum_{j=0}^{n-1} \sum_{\xi_0,\dots,\xi_n \in S} p_{\xi_0} p_{\xi_0\xi_1} \cdots p_{\xi_{n-1}\xi_n} \log p_{\xi_j\xi_{j+1}} \\ &\qquad\qquad - \sum_{\xi_0,\dots,\xi_n \in S} p_{\xi_0} p_{\xi_0\xi_1} \cdots p_{\xi_{n-1}\xi_n} \log p_{\xi_0} \end{aligned}$$

$$= -\sum_{j=0}^{n-1} \sum_{\xi_0,\dots,\xi_n \in S} p_{\xi_0} p_{\xi_0\xi_1} \cdots p_{\xi_{j-1}\xi_j} \cdot p_{\xi_{j+1}\xi_{j+2}} \cdots p_{\xi_{n-1}\xi_n} p_{\xi_j\xi_{j+1}} \log p_{\xi_j\xi_{j+1}} - \sum_{\xi_0 \in S} p_{\xi_0} \log p_{\xi_0}$$

$$= -\sum_{j=0}^{n-1} \sum_{\xi_j,\dots,\xi_n \in S} \mu(\sigma^{-j}[\xi_j]) \cdot p_{\xi_{j+1}\xi_{j+2}} \cdots p_{\xi_{n-1}\xi_n} p_{\xi_j\xi_{j+1}} \log p_{\xi_j\xi_{j+1}} - \sum_{\xi_0 \in S} p_{\xi_0} \log p_{\xi_0}$$

$$= -\sum_{j=0}^{n-1} \sum_{\xi_j,\xi_{j+1} \in S} p_{\xi_j} p_{\xi_j\xi_{j+1}} \log p_{\xi_j\xi_{j+1}} \sum_{\xi_{j+2},\dots,\xi_{n-1} \in S} p_{\xi_{j+1}\xi_{j+2}} \cdots p_{\xi_{n-1}\xi_n} - \sum_{\xi_0 \in S} p_{\xi_0} \log p_{\xi_0}$$

$$= -\sum_{j=0}^{n-1} \sum_{\xi_j,\xi_{j+1} \in S} p_{\xi_j} p_{\xi_j\xi_{j+1}} \log p_{\xi_j\xi_{j+1}} - \sum_{\xi_0 \in S} p_{\xi_0} \log p_{\xi_0}$$

$$= n\left(-\sum_{i,j} p_i p_{ij} \log p_{ij}\right) - \sum_i p_i \log p_i$$

Now divide by $n+1$ and pass to the limit. □

4.4.4 Expanding Markov Maps of the Interval

Theorem 4.5 (Rokhlin formula). *Suppose $T : [0,1] \to [0,1]$ and $\alpha = \{I_1, \dots, I_N\}$ is a partition into intervals s.t.*

1. *α is a Markov partition*
2. *The restriction of T to α is C^1, monotonic, and $|T'| > \lambda > 1$*
3. *T has an invariant measure μ.*

Then $h_\mu(T) = -\int \log \frac{d\mu}{d\mu\circ T} d\mu$, where $(\mu \circ T)(E) = \sum_{A\in\alpha} \mu[T(A \cap E)]$.

Proof. One checks that the elements of α_0^{n-1} are all intervals of length $O(\lambda^{-n})$. Therefore α is a strong generator, whence

$$h_\mu(T) = h_\mu(T,\alpha) = H_\mu(\alpha|\alpha_1^\infty) = \int I_\mu(\alpha|\alpha_1^\infty) d\mu.$$

We calculate $I_\mu(\alpha|\alpha_1^\infty)$.

First note that $\alpha_1^\infty = T^{-1}(\alpha_0^\infty) = T^{-1}\mathscr{B}$, thus $I_\mu(\alpha|\alpha_1^\infty) = -\sum_{A\in\alpha} 1_A \log \mu(A|T^{-1}\mathscr{B})$.

We need to calculate $\mathbb{E}(\cdot|T^{-1}\mathscr{B})$. For this purpose, introduce the operator $\widehat{T} : L^1 \to L^1$ given by $(\widehat{T} f)(x) = \sum_{Ty=x} \frac{d\mu}{d\mu\circ T}(y) f(y)$.

Exercise: Verify: $\forall\varphi\in L^\infty$ and $f\in L^1$, $\int\varphi\widehat{T}f d\mu=\int\varphi\circ T\cdot f d\mu$.

We claim that $\mathbb{E}(f|T^{-1}\mathscr{B})=(\widehat{T}f)\circ T$. Indeed, the $T^{-1}\mathscr{B}$–measurable functions are exactly the functions of the form $\varphi\circ T$ with φ $\mathscr{B}$–measurable; Therefore $(\widehat{T}f)\circ T$ is $T^{-1}\mathscr{B}$–measurable, and

$$\int\varphi\circ T\cdot\widehat{T}f\circ T d\mu=\int\varphi\cdot\widehat{T}f d\mu=\int\varphi\circ T\cdot f d\mu,$$

proving the identity.

We can now calculate and see that

$$\begin{aligned}
I_\mu(\alpha|\alpha_1^\infty)&=-\sum_{A\in\alpha}1_A(x)\log\mathbb{E}(1_A|T^{-1}\mathscr{B})(x)\\
&=-\sum_{A\in\alpha}1_A(x)\log\sum_{Ty=Tx}\frac{d\mu}{d\mu\circ T}(y)1_A(y)\equiv-\sum_{A\in\alpha}1_A(x)\log\frac{d\mu}{d\mu\circ T}(x)\\
&=-\log\frac{d\mu}{d\mu\circ T}(x).
\end{aligned}$$

We conclude that $h_\mu(T)=-\int\log\frac{d\mu}{d\mu\circ T}(x)d\mu(x)$. $\square$

4.5 Abramov's Formula

Suppose $(X,\mathscr{B},\mu,T)$ is a ppt. A set A is called *spanning*, if $X=\bigcup_{n=0}^\infty T^{-n}A \mod \mu$. If T is ergodic, then every set of positive measure is spanning.

Theorem 4.6 (Abramov). *Suppose $(X,\mathscr{B},\mu,T)$ is a ppt on a Lebesgue space, let A be a spanning measurable set, and let $(A,\mathscr{B}_A,\mu_A,T_A)$ be the induced system, then $h_{\mu_A}(T_A)=\frac{1}{\mu(A)}h_\mu(T)$.*

Proof. (Scheller) We prove the theorem in the case when T is invertible. The non-invertible case is handled by passing to the natural extension.

The idea is to show, for as many partitions α as possible, that $h_\mu(T,\alpha)=\mu(A)h_{\mu_A}(T_A,\alpha\cap A)$, where $\alpha\cap A:=\{E\cap A:E\in\alpha\}$. As it turns out, this is the case for all partitions s.t. (a) $H_\mu(\alpha)<\infty$; (b) $A^c\in\alpha$; and (c) $\forall n,\ T_A[\varphi_A=n]\in\sigma(\alpha)$. Here, as always, $\varphi_A(x):=1_A(x)\inf\{n\geq 1:T^n x\in A\}$ (the first return time).

To see that there are such partitions, we let

$$\xi_A:=\{A^c\}\cup T_A\eta_A,\text{ where }\eta_A:=\{[\varphi_A=n]:n\in\mathbb{N}\}$$

(the coarsest possible) and show that $H_\mu(\xi_A)<\infty$. A routine calculation shows that $H_\mu(\xi_A)=H_\mu(\{A,A^c\})+\mu(A)H_{\mu_A}(T_A\eta_A)\leq 1+H_{\mu_A}(\eta_A)$. It is thus enough to show that $-\sum p_n\log_2 p_n<\infty$, where $p_n:=\mu_A[\varphi_A=n]$. This is because $\sum np_n=1/\mu(A)$

(Kac formula) and the following fact from calculus: probability vectors with finite expectations have finite entropy.[2]

Assume now that α is a partition which satisfies (a)–(c) above. We will use throughout the following fact:

$$A, A^c, [\varphi_A = n] \in \alpha_1^\infty. \tag{4.2}$$

Here is why: $[\varphi_A = n] = T^{-n}T_A[\varphi_A = n] \in T^{-n}\alpha \subset \alpha_1^\infty$. Since $AA = \bigcup_{n\geq 1}[\varphi_A = n]$, we automatically have $A, A^c \in \alpha_1^\infty$.

Let α be a finite entropy countable measurable partition of X such that A^c is an atom of α and such that $\alpha \geq \xi_A$. In what follows we use the notation $A \cap \alpha := \{B \cap A : B \in \alpha\}$, $\alpha_1^\infty \cap A := \{B \cap A : B \in \alpha_1^\infty\}$. Since $H_\mu(\alpha) < \infty$,

$$\begin{aligned}
h_\mu(T,\alpha) &= H_\mu(\alpha|\alpha_1^\infty) \\
&= \int \sum_{B\in A\cap\alpha} 1_B \log \mu(B|\alpha_1^\infty) d\mu + \int 1_{A^c} \log \mu(A^c|\alpha_1^\infty) d\mu \\
&= \int \sum_{B\in A\cap\alpha} 1_B \log \mu(B|\alpha_1^\infty) d\mu, \text{ because } A^c \in (\xi_A)_1^\infty \subseteq \alpha_1^\infty \\
&= \mu(A) \int_A \sum_{B\in A\cap\alpha} 1_B \log \mu_A(B|\alpha_1^\infty) d\mu_A,
\end{aligned}$$

because $A \in \alpha_1^\infty$ and $B \subset A$ imply $\mathbb{E}_\mu(1_B|\mathscr{F}) = 1_A \mathbb{E}_{\mu_A}(1_B|A \cap \mathscr{F})$.

It follows that $h_\mu(T,\alpha) = \mu(A) H_{\mu_A}(A \cap \alpha | A \cap \alpha_1^\infty)$. We will show later that

$$A \cap \alpha_1^\infty = \bigvee_{i=1}^{\infty} T_A^{-i}(A \cap \alpha) \tag{4.3}$$

This implies that $h_\mu(T,\alpha) = \mu(A) h_{\mu_A}(T_A, A \cap \alpha)$. Passing to the supremum over all α which contain A^c as an atom, we obtain

$$\begin{aligned}
\mu(A) h_{\mu_A}(T_A) &= \sup\{h_\mu(T,\alpha) : \alpha \geq \xi_A\ ,\ A^c \in \alpha, H_\mu(\alpha) < \infty\} \\
&= \text{Entropy of } (X, \mathscr{B}', \mu, T),\ \mathscr{B}' := \sigma\left(\bigcup\{\alpha_{-\infty}^\infty : A^c \in \alpha, H_\mu(\alpha) < \infty\}\right).
\end{aligned}$$

(See problem 4.11).

Now $\mathscr{B}' = \mathscr{B} \mod \mu$, because A is spanning, so $\forall E \in \mathscr{B}$, $E = \bigcup_{n=0}^\infty T^{-n}(T^n E \cap A) \mod \mu$, whence $E \in \mathscr{B}' \mod \mu$. This shows Abramov's formula, given (4.3).

The proof of (4.3):

Proof of $\subseteq$: Suppose B is an atom of $A \cap \bigvee_{j=1}^n T^{-j}\alpha$, then B has the form $A \cap \bigcap_{j=1}^n T^{-j}A_j$ where $A_j \in \alpha$. Let $j_1 < j_2 < \cdots < j_N$ be an enumeration of the j's s.t. $A_j \subset A$ (possibly an empty list). Since A^c is an atom of α, $A_j = A^c$ for j not in

[2] Proof: Enumerate (p_n) in a decreasing order: $p_{n_1} \geq p_{n_2} \geq \cdots$. If $C = \sum n p_n$, then $C \geq \sum_{i=1}^k n_i p_{n_i} \geq p_{n_k}(1+\cdots+k)$, whence $p_{n_k} = O(k^{-2})$. Since $-x\log x = O(x^{1-\varepsilon})$ as $x \to 0^+$, this means that $-p_{n_k}\log p_{n_k} = O(k^{-(2-\varepsilon)})$, and so $-\sum p_n \log p_n = -\sum p_{n_k} \log p_{n_k} < \infty$.

this list, and so $B = \bigcap_{k=1}^{N-1} T_A^{-k}(A_{j_k} \cap [\varphi_A = j_{k+1} - j_k]) \cap T_A^{-N}[\varphi_A > n - j_N]$. Since $\eta_A \leq \alpha \cap A$, $B \in \bigvee_{i=1}^{\infty} T_A^{-1}(\alpha \cap A)$.

Proof of $\supseteq$: $T_A^{-1}(\alpha \cap A) \leq A \cap \bigvee_{i=1}^{\infty} T^{-i}\alpha$, because if $B \in \alpha \cap A$, then

$$T_A^{-1}B = \bigcup_{n=1}^{\infty} T^{-n}(B \cap T_A[\varphi_A = n]) \in \bigvee_{n=1}^{\infty} T^{-n}\alpha \ \ (\because T_A\eta_A \leq \xi_A \leq A \cap \alpha).$$

The same proof shows that $T_A^{-1}(T^{-n}\alpha \cap A) \leq A \cap \bigvee_{i=1}^{\infty} T^{-i}\alpha$. It follows that

$$T_A^{-2}(\alpha \cap A) \leq T_A^{-1}\left(A \cap \bigvee_{i=1}^{\infty} T^{-i}\alpha\right) \subseteq A \cap \bigvee_{i=1}^{\infty} T_A^{-1}(A \cap T^{-i}\alpha) \subseteq A \cap \bigvee_{i=1}^{\infty} T^{-i}\alpha.$$

Iterating this procedure we see that $T_A^{-n}(\alpha \cap A) \leq A \cap \bigvee_{i=1}^{\infty} T^{-i}\alpha$ for all n, and $\supseteq$ follows. □

4.6 Topological Entropy

Suppose $T : X \to X$ is a continuous mapping of a compact topological space (X, d). Such a map can have many different invariant Borel probability measures. For example, the left shift on $\{0, 1\}^{\mathbb{N}}$ has an abundance of Bernoulli measures, Markov measures, and there are many others.

Different measures may have different entropies. What is the largest value possible? We study this question in the context of continuous maps on topological spaces which are *compact* and *metric*.

4.6.1 The Adler–Konheim–McAndrew definition

Let (X, d) be a compact metric space, and $T : X \to X$ a continuous map. Some terminology and notation:

1. an *open cover* of X is a collection of open sets $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ s.t. $X = \bigcup_{\alpha \in \Lambda} U_\alpha$;
2. if $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover, then $T^{-k}\mathscr{U} := \{T^{-k}U_\alpha : \alpha \in \Lambda\}$. Since T is continuous, this is another open cover.
3. if $\mathscr{U}, \mathscr{V}$ are open covers, then $\mathscr{U} \vee \mathscr{V} := \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$.

Since X is compact, every open cover of X has a finite subcover. Define

$$N(\mathscr{U}) := \min\{\#\mathscr{V} : \mathscr{V} \subseteq \mathscr{U} \text{ is finite, and } X = \bigcup \mathscr{V}\}.$$

It easy to check that $N(\cdot)$ is subadditive in the following sense:

$$N(\mathscr{U} \vee \mathscr{V}) \leq N(\mathscr{U}) + N(\mathscr{V}).$$

Definition 4.5. Suppose $T : X \to X$ is a continuous mapping of a compact metric space (X,d), and let $\mathscr{U}$ be an open cover of X. The *topological entropy of* $\mathscr{U}$ is

$$h_{\text{top}}(T,\mathscr{U}) := \lim_{n\to\infty} \frac{1}{n} \log_2 N(\mathscr{U}_0^{n-1}), \text{ where } \mathscr{U}_0^{n-1} := \bigvee_{i=0}^{n-1} T^{-k}\mathscr{U}.$$

The limit exists because of the subadditivity of $N(\cdot)$: $a_n := \log N(\mathscr{U}_0^{n-1})$ satisfies $a_{m+n} \leq a_m + a_n$, so $\lim a_n/n$ exists.

Definition 4.6. Suppose $T : X \to X$ is a continuous mapping of a compact metric space (X,d), then the *topological entropy* of T is the (possibly infinite)

$$h_{\text{top}}(T) := \sup\{h_{\text{top}}(T,\mathscr{U}) : \mathscr{U} \text{ is an open cover of } X\}.$$

The following theorem was first proved by Goodwyn.

Theorem 4.7. *Suppose T is a continuous mapping of a compact metric space, then every invariant Borel probability measure μ satisfies $h_\mu(T) \leq h_{\text{top}}(T)$.*

Proof. Eventually everything boils down to the following inequality, which can be checked using Lagrange multipliers: For every probability vector $(p_1,\ldots,p_k)$,

$$-\sum_{i=1}^{k} p_i \log_2 p_i \leq \log k, \tag{4.4}$$

with equality iff $p_1 = \cdots = p_k = 1/k$.

Suppose μ is an invariant probability measure, and let $\alpha := \{A_1,\ldots,A_k\}$ be a measurable partition.

We approximate α by a partition into sets with better topological properties. Fix $\varepsilon > 0$ (to be determined later), and construct *compact* sets

$$B_j \subset A_j \text{ s.t. } \mu(A_j \setminus B_j) < \varepsilon \quad (j = 1,\ldots,k).$$

Let $B_0 := X \setminus \bigcup_{j=1}^{k} B_j$ be the remainder (of measure less than $k\varepsilon$), and define $\beta = \{B_0; B_1,\ldots,B_k\}$.

Step 1 in the proof of Sinai's theorem says that $h_\mu(T,\alpha) \leq h_\mu(T,\beta) + H_\mu(\alpha|\beta)$. We claim that $H_\mu(\alpha|\beta)$ can be made uniformly bounded by a suitable choice of $\varepsilon = \varepsilon(\alpha)$:

$$
\begin{aligned}
H_\mu(\alpha|\beta) &= -\sum_{A\in\alpha}\sum_{B\in\beta}\mu(A\cap B)\log_2\mu(A|B)\\
&= -\sum_{B\in\beta\setminus\{B_0\}}\sum_{A\in\alpha}\mu(A\cap B)\log_2\mu(A|B) - \sum_{A\in\alpha}\mu(A\cap B_0)\log_2\mu(A|B_0)\\
&= \sum_{i=1}^{k}\mu(B_i)\log_2 1 - \sum_{A\in\alpha}\mu(A\cap B_0)\log_2\mu(A|B_0)\\
&= -\mu(B_0)\sum_{A\in\alpha}\mu(A|B_0)\log_2\mu(A|B_0) \le -\mu(B_0)\log(\#\alpha) \le k\varepsilon\cdot\log_2 k.
\end{aligned}
$$

If we choose $\varepsilon < 1/(k\log_2 k)$, then we get $H_\mu(\alpha|\beta) \le 1$, and

$$h_\mu(T,\alpha) \le h_\mu(T,\beta) + 1. \tag{4.5}$$

We now create an open cover from β by setting $\mathscr{U} := \{B_0\cup B_1,\dots,B_0\cup B_k\}$. This is a cover. To see that it is open note that

$$
\begin{aligned}
B_0\cup B_j &= B_0\cup(A_j\setminus B_0) \ \ (\because A_j\cap B_0 = A_j\setminus B_j)\\
&= B_0\cup A_j = B_0\cup\left(X\setminus\bigcup_{i\neq j}A_i\right) = B_0\cup\left(X\setminus\bigcup_{i\neq j}B_i\right).
\end{aligned}
$$

We compare the number of elements in $\mathscr{U}_0^{n-1}$ to the number of elements in β_0^{n-1}. Every element of $\mathscr{U}_0^{n-1}$ is of the form

$$(B_0\uplus B_{i_0})\cap T^{-1}(B_0\uplus B_{i_1})\cap\cdots\cap T^{-(n-1)}(B_0\uplus B_{i_{n-1}}).$$

This can be written as a pairwise disjoint union of 2^n elements of β_0^{n-1} (some of which may be empty sets). Thus every element of $\mathscr{U}_0^{n-1}$ contains at most 2^n elements of β_0^{n-1}. Forming the union over a sub cover of $\mathscr{U}_0^{n-1}$ with cardinality $N(\mathscr{U}_0^{n-1})$, we get that $\#\beta_0^{n-1} \le 2^n N(\mathscr{U}_0^{n-1})$.

We no appeal to (4.4): $H_\mu(\beta_0^{n-1}) \le \log_2(\#\beta_0^{n-1}) \le H(\mathscr{U}_0^{n-1}) + n$. Dividing by n and passing to the limit as $n\to\infty$, we see that $h_\mu(T,\beta) \le h_{\text{top}}(\mathscr{U}) + 1$. By (4.5), $h_\mu(T,\alpha) \le h_{\text{top}}(\mathscr{U}) + 2 \le h_{\text{top}}(T) + 2$.

Passing to the supremum over all α, we get that $h_\mu(T) \le h_{\text{top}}(T) + 2$, and this holds for all continuous mappings T and invariant Borel measures μ. In particular, this holds for T^n (note that $\mu\circ(T^n)^{-1} = \mu$): $h_\mu(T^n) \le h_{\text{top}}(T^n) + 2$. But $h_\mu(T^n) = nh_\mu(T)$ and $h_{\text{top}}(T^n) = nh_{\text{top}}(T)$ (problems 4.4 and 4.13). Thus we get upon division by n that $h_\mu(T) \le h_{\text{top}}(T) + (2/n) \xrightarrow[n\to\infty]{} 0$, which proves the theorem. $\square$

In fact, $h_{\text{top}}(T) = \sup\{h_\mu(T) : \mu \text{ is an invariant Borel probability measure}\}$. But to prove this we need some more preparations. These are done in the next section.

4.6.2 Bowen's definition

We assume as usual that (X,d) is a compact metric space, and that $T : X \to X$ is continuous. For every n we define a new metric d_n on X as follows:

$$d_n(x,y) := \max_{0 \le i \le n-1} d(T^i x, T^i y).$$

This is called *Bowen's metric*. It depends on T. A set $F \subset X$ is called (n,ε)–*separated*, if for every $x,y \in F$ s.t. $x \neq y$, $d_n(x,y) > \varepsilon$.

Definition 4.7.

1. $s_n(\varepsilon,T) := \max\{\#(F) : F \text{ is } (n,\varepsilon)\text{–separated}\}$.
2. $s(\varepsilon,T) := \limsup\limits_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon,T)$
3. $\overline{h}_{\text{top}}(T) := \lim\limits_{\varepsilon\to 0^+} s(\varepsilon,T)$.

Theorem 4.8 (Bowen). *Suppose T is a continuous mapping of a compact metric space X, then $h_{\text{top}}(T) = \overline{h}_{\text{top}}(T)$.*

Proof. Suppose $\mathscr{U}$ is an open cover all of whose elements have diameters less than ε. We claim that $N(\mathscr{U}_0^{n-1}) \geq s_n(\varepsilon,T)$ for all n. To see this suppose F is an (n,ε)–separated set of maximal cardinality. Each $x \in F$ is contained in some $U_x \in \mathscr{U}_0^{n-1}$. Since the d–diameter of every element of $\mathscr{U}$ is less than δ, the d_n–diameter of every element of $\mathscr{U}_0^{n-1}$ is less than δ. Thus the assignment $x \mapsto U_x$ is one-to-one, whence

$$N(\mathscr{U}_0^{n-1}) \geq s_n(\varepsilon,T).$$

It follows that $s(\varepsilon,T) \leq h_{\text{top}}(T,\mathscr{U}) \leq h_{\text{top}}(T)$, whence $\overline{h}_{\text{top}}(T) \leq h_{\text{top}}(T)$.

To see the other inequality we use *Lebesgue numbers*: a number δ is called a Lebesgue number for an open cover $\mathscr{U}$, if for every $x \in X$, the ball with radius δ and center x is contained in some element of $\mathscr{U}$. (Lebesgue numbers exist because of compactness.)

Fix ε and let $\mathscr{U}$ be an open cover with Lebesgue number bigger than or equal to ε. It is easy to check that for every n, ε is a Lebesgue number for $\mathscr{U}_0^{n-1}$ w.r.t. d_n.

Let F be an $(n,\varepsilon/2)$–separated set of *maximal* cardinality, i.e. $\#F = s_n(\varepsilon)$. Then any point y we add to F will break its (n,ε)–separation property, and so

$$\forall y \in X\ \exists x \in F \text{ s.t. } d_n(x,y) \leq \varepsilon/2.$$

It follows that the sets $\overline{B_n(x;\varepsilon/2)} := \{y : d_n(x,y) \leq \varepsilon/2\}$ $(x \in F)$ cover X.

Every $\overline{B_n(x,\varepsilon/2)}$ $(x \in F)$ is contained in some element of $\mathscr{U}_0^{n-1}$, because $\mathscr{U}_0^{n-1}$ has Lebesgue number ε w.r.t d_n. The union of these elements covers X. We found a sub cover of $\mathscr{U}_0^{n-1}$ of cardinality at most $\#F = s_n(\varepsilon)$. This shows that

$$N(\mathscr{U}_0^{n-1}) \leq s_n(\varepsilon).$$

We just proved that for every open cover $\mathscr{U}$ with Lebesgue number at least ε, $h_{\text{top}}(T,\mathscr{U}) \leq s(\varepsilon)$. It follows that

$$\sup\{h_{\text{top}}(T,\mathscr{U}) : \mathscr{U} \text{ has Lebesgue number at least } \varepsilon\} \leq s(\varepsilon).$$

We now pass to the limit $\varepsilon \to 0^+$. The left hand side tends to the supremum over all open covers, which is equal to $h_{\text{top}}(T)$. We obtain $h_{\text{top}}(T) \leq \lim\limits_{\varepsilon\to 0^+} s(\varepsilon)$. ☐

Corollary 4.1. *Suppose T is an isometry, then all its invariant probability measures have entropy zero.*

Proof. If T is an isometry, then $d_n = d$ for all n, therefore $s(\varepsilon,T) = 0$ for all $\varepsilon > 0$, so $\overline{h}_{\text{top}}(T) = 0$. The theorem says that $h_{\text{top}}(T) = 0$. The corollary follows from Goodwyn's theorem. ☐

4.6.3 The variational principle

The following theorem was first proved under additional assumptions by Dinaburg, and then in the general case by Goodman. The proof below is due to Misiurewicz.

Theorem 4.9 (Variational principle). *Suppose $T : X \to X$ is a continuous map of a compact metric space, then $h_{\text{top}}(T) = \sup\{h_\mu(T) : \mu$ is an invariant Borel measure$\}$.*

Proof. We have already seen that the topological entropy is an upper bound for the metric entropies. We just need to show that this is the least upper bound.

Fix ε, and let F_n be a sequence of (n,ε)–separated sets of maximal cardinality (so $\#F_n = s_n(\varepsilon,T)$). Let

$$\nu_n := \frac{1}{\#F_n}\sum_{x\in F_n}\delta_x,$$

where δ_x denotes the Dirac measure at x (i.e. $\delta_x(E) = 1_E(x)$). These measure are not invariant, so we set

$$\mu_n := \frac{1}{n}\sum_{k=0}^{n-1}\nu_n \circ T^{-k}.$$

Any weak star limit of μ_n will be T–invariant (check).

Fix some sequence $n_k \to \infty$ s.t. $\mu_{n_k} \xrightarrow[k\to\infty]{w^*} \mu$ and s.t. $\frac{1}{n_k}\log s_{n_k}(\varepsilon,T) \xrightarrow[n\to\infty]{} s(\varepsilon,T)$. We show that the entropy of μ is at least $s(\varepsilon,T)$. Since $s(\varepsilon,T) \xrightarrow[\varepsilon\to 0^+]{} h_{\text{top}}(T)$, this will prove the theorem.

Let $\alpha = \{A_1,\ldots,A_N\}$ be a measurable partition of X s.t. (1) $\text{diam}(A_i) < \varepsilon$; and (2) $\mu(\partial A_i) = 0$. (Such a partition can be generated from a cover of X by balls of radius less than $\varepsilon/2$ and boundary of zero measure.) It is easy to see that the d_n–diameter of α_0^{n-1} is also less than ε. It is an exercise to see that every element of α_0^{n-1} has boundary with measure μ equal to zero.

We calculate $H_{\nu_n}(\alpha_0^{n-1})$. Since F_n is (n,ε)–separated and every atom of α has d_n–diameter less than ε, α_0^{n-1} has $\#F_n$ elements whose ν_n measure is $1/\#F_n$, and the other elements of α_0^{n-1} have measure zero. Thus

$$H_{\nu_n}(\alpha_0^{n-1}) = \log_2 \#F_n = \log_2 s_n(\varepsilon, T).$$

We now "play" with $H_{\nu_n}(\alpha_0^{n-1})$ with the aim of bounding it by something involving a sum of the form $\sum_{i=0}^{n-1} H_{\nu_n \circ T^{-i}}(\alpha_0^{q-1})$. Fix q, and $j \in \{0, \dots, q-1\}$, then

$$\begin{aligned}\log_2 s_n(\varepsilon,T) = H_{\nu_n}(\alpha_0^{n-1}) &\le H_{\nu_n}(\alpha_0^{j-1} \vee \bigvee_{i=0}^{[n/q]-1} T^{-(qi+j)}\alpha_0^{q-1} \vee \alpha_{q([n/q]-1)+j+1}^{n-1}) \\ &\le \sum_{i=0}^{[n/q]-1} H_{\nu_n \circ T^{-(qi+j)}}(\alpha_0^{q-1}) + 2q \log_2 \#\alpha.\end{aligned}$$

Summing over $j = 0, \dots, q-1$, we get

$$\begin{aligned}q \log_2 s_n(\varepsilon, T) &\le n \cdot \frac{1}{n} \sum_{k=0}^{n-1} H_{\nu_n \circ T^{-k}}(\alpha_0^{q-1}) + 2q \log_2 \#\alpha \\ &\le n H_{\mu_n}(\alpha_0^{q-1}) + 2q \log_2 \#\alpha,\end{aligned}$$

because $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \nu_n \circ T^{-i}$ and $\phi(t) = -t\log_2 t$ is concave. Thus

$$\frac{1}{n_k} \log_2 s_{n_k}(\varepsilon, T) \le \frac{1}{q} H_{\mu_{n_k}}(\alpha_0^{q-1}) + \frac{2}{n_k} \log_2 \#\alpha, \tag{4.6}$$

where $n_k \to \infty$ is the subsequence chosen above.

Since every $A \in \alpha_0^{n-1}$ satisfies $\mu(\partial A) = 0$, $\mu_{n_k}(A) \xrightarrow[k\to\infty]{w^*} \mu(A)$ for all $A \in \alpha_0^{n-1}$.[3] It follows that $H_{\mu_{n_k}}(\alpha_0^{q-1}) \xrightarrow[k\to\infty]{} H_\mu(\alpha_0^{q-1})$. Passing to the limit $k \to \infty$ in (4.6), we have $s(\varepsilon, T) \le \frac{1}{q} H_\mu(\alpha) \xrightarrow[q\to\infty]{} h_\mu(T, \alpha) \le h_\mu(T)$. Thus $h_\mu(T) \ge s(\varepsilon, T)$. Since $s(\varepsilon, T) \xrightarrow[\varepsilon\to 0^+]{} h_{\mathrm{top}}(T)$ the theorem is proved. $\square$

4.7 Ruelle's inequality

Theorem 4.10 (Ruelle). *Let f be a C^1 diffeomorphism of a compact C^2 Riemannian manifold M without boundary. Suppose μ is an f-invariant Borel probability*

---

[3] Fix δ and sandwich $u \le 1_{A \setminus \partial A} \le 1_A \le 1_{A \cup \partial A} \le v$ with u, v continuous s.t. $|\int u d\mu - \mu(E)| < \delta$ and $|\int v d\mu - \mu(E)| < \delta$. This is possible because $A \setminus \partial A$ is open, $A \cup \partial A$ is open, and $\mu(\partial A) = 0$. Then $\mu(A) - \delta \le \int u d\mu = \lim \int u d\mu_{n_k} \le \liminf \mu_{n_k}(E) \le \limsup \mu_{n_k}(E) \le \lim \int v d\mu_{n_k} = \int v d\mu \le \mu(A) + \delta$. Since δ is arbitrary, $\mu_{n_k}(E) \xrightarrow[k\to\infty]{} \mu(E)$.

measure with Lyapunov exponents $\chi_1(x) \geq \cdots \geq \chi_d(x)$*. Then*[4]

$$\int \sum_{\chi_i(x)>0} \chi_i(x) d\mu \geq \ln 2 \cdot h_\mu(f).$$

Corollary 4.2. *Let* μ *be an ergodic invariant probability measure for a* C^1 *diffeomorphism* f *on a compact smooth manifold without boundary. If* $h_\mu(f) > 0$*, then* μ *has at least one positive Lyapunov exponent.*

In other words, positive entropy implies exponential sensitivity to initial conditions almost everywhere!

4.7.1 Preliminaries on singular values

The *singular values* of an invertible $d \times d$ matrix A are the eigenvalues $e^{\chi_1} \geq \cdots \geq e^{\chi_d}$ of the matrix $\sqrt{A^t A}$.

Proposition 4.9. *Let* $e^{\chi_1} \geq \cdots \geq e^{\chi_d}$ *denote the singular values of an invertible matrix A listed with multiplicity. The ellipsoid* $\{A\underline{v} : \underline{v} \in \mathbb{R}^d \, , \, \|\underline{v}\|_2 \leq 1\}$ *can be inscribed in an orthogonal box with sides* $2e^{\chi_1}, \ldots, 2e^{\chi_d}$*.*

Proof. The matrix $\sqrt{A^t A}$ is symmetric and positive definite, therefore it can be diagonalized orthogonally. Let $\underline{u}_1, \ldots, \underline{u}_d$ be an orthonormal basis of eigenvectors so that $A\underline{u}_i = e^{\chi_i} \underline{u}_i$. Using this basis we can represent

$$\left\{A\underline{v} : \underline{v} \in \mathbb{R}^d \, , \, \|\underline{v}\|_2 \leq 1\right\} \equiv \left\{\sum_{i=1}^{d} \alpha_i A\underline{u}_i : \sum \alpha_i^2 \leq 1\right\} \subset \left\{\sum_{i=1}^{d} \alpha_i A\underline{u}_i : |\alpha_i| \leq 1\right\}.$$

The last set is an orthogonal box as requested, because $A\underline{u}_i$ are orthogonal vectors with lengths e^{χ_i}: $\langle A\underline{u}_i, A\underline{u}_j\rangle = \langle A^t A\underline{u}_i, \underline{u}_j\rangle = e^{2\chi_i}\langle \underline{u}_i, \underline{u}_j\rangle = e^{2\chi_i}\delta_{ij}$.

Proposition 4.10 (Minimax Principle). $e^{\chi_i(A)} = \max\limits_{V \subset \mathbb{R}^d, \dim V = i} \; \min\limits_{\underline{v} \in V, \|\underline{v}\|=1} \|A\underline{v}\|.$

Proof. The matrix $\sqrt{A^t A}$ is symmetric and positive definite. Let $\underline{u}_1, \ldots, \underline{u}_d$ be an orthonomal basis of eigenvectors with eigenvalues $e^{\chi_1} \geq \cdots \geq e^{\chi_d}$.

Fix i. For every subspace V of dimension i, $\dim(V \cap \mathrm{span}\{\underline{u}_i, \ldots, \underline{u}_d\}) \geq 1$ by dimension count. Therefore $\exists \underline{v} \in V$ s.t. $\underline{v} = \sum\limits_{j=i}^{d} \alpha_j \underline{u}_j \in V$, and $\|\underline{v}\| = \sum\limits_{j=i}^{d} \alpha_j^2 = 1$. We have $\|A\underline{v}\|^2 = \langle A\underline{v}, A\underline{v}\rangle = \langle A^t A\underline{v}, \underline{v}\rangle = \sum_{j \geq i} e^{2\chi_j}\alpha_j^2 \leq e^{2\chi_i}$. This shows that

[4] The $\ln 2$ is because we followed the tradition in ergodic theory to use base two logarithms to define the entropy, and we followed the tradition is dynamical systems to use natural base logarithms to define Lyapunov exponents. In literature on dynamical systems, it is customary to use natural logarithms also for the definition of the entropy, and then no such factor is necessary.

$$\max_{V\subset\mathbb{R}^d,\dim V=i}\ \min_{\underline{v}\in V,\|\underline{v}\|=1}\|A\underline{v}\|\leq\exp\chi_i.$$

For the other inequality, consider $V_0:=\text{span}\{\underline{u}_1,\dots,\underline{u}_i\}$. Every unit vector $\underline{v}\in V_0$ can be put in the form $\underline{v}=\sum_{j=1}^i\beta_j\underline{u}_j$ with $\sum_{j=1}^i\beta_j^2=1$, whence

$$\|A\underline{v}\|^2=\langle A^tA\underline{v},\underline{v}\rangle=\sum_{j=1}^i e^{2\chi_j}\beta_j^2\geq e^{2\chi_i}\sum_{j=1}^i\beta_j^2=e^{2\chi_i}.$$

The lower bound is achieved for $\underline{v}=\underline{u}_i$, so $\min\limits_{\underline{v}\in V_0,\|\underline{v}\|=1}\|A\underline{v}\|=e^{\chi_i}$, proving that $\max\limits_{V\subset\mathbb{R}^d,\dim V=i}\ \min\limits_{\underline{v}\in V,\|\underline{v}\|=1}\|A\underline{v}\|\geq\min\limits_{\underline{v}\in V_0,\|\underline{v}\|=1}\|A\underline{v}\|=\exp\chi_i.$ □

4.7.2 *Proof of Ruelle's inequality*

Fix $k,n\in\mathbb{N}$ and $\varepsilon>0$, to be determined later.

Step 1. A generating sequence of "nice" partitions.

Proof. Let $E_k:=\{y_1,\dots,y_{\eta_k}\}$ denote a *maximal* ε/k*-separated set*: "ε/k-separation" means that $i\neq j\Rightarrow d(y_i,y_j)>\varepsilon/k$; "maximality" means that $\forall x\in M\exists i$ s.t. $d(x,y_i)\leq\varepsilon/k$. Such a set exists by the compactness of M.

Each $y_i\in E_k$ defines a *Dirichlet domain*

$$A_i':=\{x\in M: d(x,y_i)\leq d(x,y_j)\text{ for all } j\}.$$

The following properties are easy to verify:

1. A_i' cover M
2. A_i' have disjoint interiors and contain the pairwise disjoint balls $B(y_i,\varepsilon/2k)$
3. $A_i'\subset\overline{B(y_i,\varepsilon/k)}$, otherwise we can find $x\in A_i'$ such that for all j, $d(x,y_j)\geq d(x,y_i)>\varepsilon/k$, in contradiction to the maximality of E_k
4. $diam(A_i')\leq 2\varepsilon/k$

Let $A_1:=A_1'$, $A_i':=A_i\setminus\bigcup_{j=1}^{i-1}A_j'$. Then $\alpha_k:=\{A_1,\dots,A_{\eta_k}\}$ is a partition of M such that $\text{diam}(\alpha_k)\xrightarrow[k\to\infty]{}0$. So $h_\mu(T,\alpha_k)\xrightarrow[k\to\infty]{}h_\mu(T,\alpha)$. Note that it is still the case that $B(y_i,\varepsilon/k)\subset A_i\subset\overline{B(y_i,2\varepsilon/2k)}$.

Step 2. $h_\mu(f,\alpha_k)\leq\sum_{A\in\alpha_k}\mu(A)(\frac{1}{n}\log_2K_n(A,k))$, *where*

$$K_n(A,k):=\#\{A'\in\alpha_k: A'\cap f^n(A)\neq\varnothing\}$$

Proof. $h_\mu(f,\alpha_k)=\frac{1}{n}h_\mu(f^n,\alpha_k)=\frac{1}{n}h_\mu(f^{-n},\alpha_k)$, due to exercises 4.4 and 4.5. By Rokhlin's formula,

$$h_\mu(f^{-n},\alpha_k) = H_\mu(\alpha_k|\bigvee_{j=1}^{\infty} f^{+jn}\alpha_k) \leq H_\mu(\alpha_k|f^n\alpha_k) = H_\mu(f^{-n}\alpha_k|\alpha_k).$$

$H_\mu(f^{-n}\alpha_k|\alpha_k) = -\sum_{A\in A_k}\mu(A)(\sum_{B\in f^{-n}\alpha_k}\mu(B|A)\log_2\mu(B|A))$. The inner sum is bounded by $\log_2\#\{B\in\alpha_k : \mu(f^{-n}(B)\cap A)>0\} \equiv \log K_n(A,k)$.

Step 3. There exists a constant C which depends on M but not on ε, n, k such that for all n large enough, for all k large enough

$$K_n(A,k) \leq C\prod_{i=1}^{d}(1+s_i(DF_y^n))$$

for all $y\in A$, where $s_i(DF_y^n)$ denote the singular values of the derivative matrix of f^n at y, expressed in coordinates.

We first explain the method on the heuristic level, and then develop the rigorous implementation. Suppose n is fixed.

Observe that $f^n(A)$ has diameter at most $\mathrm{Lip}(f^n)\mathrm{diam}\alpha_k \leq \mathrm{Lip}(f^n)2\varepsilon/k$, and therefore $f^n(A)$ and all the A' which intersect it are included in a ball B of radius $(\mathrm{Lip}(f^n)+1)\varepsilon/k)$. If ε is very small, this ball lies inside a single coordinate chart $\psi_1(\mathbb{R}^d)$. Similarly, A lies inside a ball of radius $\mathrm{diam}\alpha_k \leq 2\varepsilon/k$, which for ε sufficiently small lies inside a coordinate chart $\psi_2(\mathbb{R}^d)$. This allows us to express $f^n|_{f^{-n}(B)}$ in coordinates: $F^n := \psi_2^{-1}\circ f^n\circ\psi_1 : \mathbb{R}^d\to\mathbb{R}^d$.

If ε is sufficiently small, F^n is very close to its linearization. Suppose for the purpose of the explanation that F^n is *precisely equal* to its linearization on $f^{-n}(B)$. From now on we abuse notation and identify f^n with its derivative matrix DF^n in coordinates.

Since A is contained in a ball of radius ε/k, $DF^n(A)$ is contained in a box with orthogonal sides whose lengths are $2(\varepsilon/k)\times$the singular values

$$e^{\chi_1(DF^n)} \geq \cdots \geq e^{\chi_d(DF^n)}.$$

The A' which intersect $f^n(A)$ are contained in an ε/k–neighborhood of this box, a set whose (Lebesgue) volume is bounded above by

$$\left(\frac{2\varepsilon}{k}\right)^d\prod_{i=1}^{d}(e^{\chi_i(DF^n)}+1)$$

By construction, A_i' contain *pairwise disjoint* balls of radius $\varepsilon/2k$, and volume const. $(\varepsilon/k)^d$. Thus their number is at most const. $\prod_{\chi_i(DF^n)\geq 0}(e^{\chi_i(DF^n)}+1)$, where the constant comes from the volume of the unit ball in $\mathbb{R}^d$.

We now explain how to implement this idea when f^n is not linear. The details are tedious, but are all routine.

We begin by recalling some facts from differential topology. Recall that M is a compact smooth Riemannian manifold of dimension d. Let T_xM denote the tangent space at x, and let TM denote the tangent bundle. Since M is a Riemannian manifold, every T_xM comes equipped with an inner product $\langle\cdot,\cdot\rangle_x$ and a norm $\|\cdot\|_x$. We denote tangent vectors in T_xM by $\vec{v}$, and vectors in $\mathbb{R}^d$ by $\underline{v}$.

Let $\exp_x : T_xM \to M$ denote the *exponential map*, defined by $\exp_x(\vec{0}) := x$ and

$$\exp_x(\vec{v}) = \gamma_{\vec{v}}(\|\vec{v}\|), \text{where } \begin{array}{l}\gamma_{\vec{v}}(t) \text{ is a geodesic with} \\ \text{beginning } x\text{, direction } \vec{v} \text{ and speed } 1.\end{array}$$

It is a basic fact in differential topology, that since M is compact and C^2-smooth, there are constants $\kappa_1, \kappa_2 > 0$ s.t. for all $x \in M$,

1. $(x, \vec{v}) \mapsto \exp_x(\vec{v})$ is a smooth map $TM \to M$
2. $\exp_x|_{B(\vec{0},\kappa_1)}$ is a C^2 diffeomorphism onto its image, and $\exp_x|_{B(\vec{0},\kappa_1)} \supset B(x,\kappa_2)$
3. the derivative of $\exp_x$ at $\underline{0}$ is $id : T_xM \to T_xM$
4. for all $\vec{v}_1, \vec{v}_2 \in B(\vec{0},\kappa_1)$, $\frac{1}{2}\|\vec{v}_1 - \vec{v}_2\|_x \le d(\exp_x(\vec{v}_1), \exp_x(\vec{v}_2)) \le 2\|\vec{v}_1 - \vec{v}_2\|_x$

It is also a standard procedure in differential topology to cover M by a finite collection balls $B(x_i,\kappa_2) \subset M$ of radius κ_2 together with a continuous map $\Theta : TB(x_i,\kappa_2) \to \mathbb{R}^d$ such that $\Theta|_{T_xM} : T_xM \to \mathbb{R}^d$ is a linear isometry. If we glue these identifications we can obtain a *measurable* (even piecewise continuous) map

$$\vartheta : TM \to \mathbb{R}^d \text{ s.t. } \vartheta_x := \vartheta|_{T_xM} : T_xM \to \mathbb{R}^d \text{ is a linear isometry.}$$

Now fix $n \in \mathbb{N}$, $x \in M$, and choose $\kappa_3 < \kappa_1/(4\mathrm{Lip}(f^n))$ where $\mathrm{Lip}(f^n)$ is the Lipschitz constant of f^n. The following map is well defined on $\{\underline{t} \in \mathbb{R}^d : \|\underline{t}\| < \kappa_3\}$:

$$F_x^n := \vartheta_{f^n(x)} \circ \exp_{f^n(x)}^{-1} \circ f^n \circ \exp_x \circ \vartheta_x^{-1}$$

Notice that F_x^n maps a neighborhood of $\underline{0} \in \mathbb{R}^d$ onto a neighborhood of $\underline{0} \in \mathbb{R}^d$, $F_x^n(\underline{0}) = \underline{0}$, and that the derivative at $\underline{0}$ is

$$(DF_x^n)_{\underline{0}} = \vartheta_{f^n(x)} \circ (Df^n)_x \circ \vartheta_x^{-1} \in \mathrm{GL}(d,\mathbb{R}).$$

Here we identified the linear maps $\vartheta_x : T_xM \to \mathbb{R}^d$ with their derivatives at $\vec{0}$, and $(Df^n)_x : T_xM \to T_{f^n(x)}M$ is the derivative of f^n at x.

Claim. For every $\varepsilon > 0$ there is a $\kappa < \kappa_3$ such that for all $x \in M$ and $\|\underline{t}\| < \kappa$ $d(F_x^n(\underline{t}), (DF_x^n)_{\underline{0}}\underline{t}) < \varepsilon\|\underline{t}\|$.

Proof. Write $G = F_x^n$, fix $\underline{t}$, and let $g_{\underline{t}}(s) = G(s\underline{t}) \in \mathbb{R}^d$. Since $g_{\underline{t}}(\underline{0}) = F_x^n(\underline{0}) = \underline{0}$,

$$\begin{aligned}\|G(\underline{t}) - DG_{\underline{0}}\underline{t}\| &= \|g_{\underline{t}}(1) - g_{\underline{t}}(0) - g'_{\underline{t}}(0)\| \le \left\|\int_0^1 g'_{\underline{t}}(s)ds - g'_{\underline{t}}(0)\right\| \\ &\le \int_0^1 \|g'_{\underline{t}}(s) - g'_{\underline{t}}(0)\|ds \le \|\underline{t}\| \sup_{\|\underline{s}_1 - \underline{s}_2\| \le \|\underline{t}\|} \|(DF_x^n)_{\underline{s}_1} - (DF_x^n)_{\underline{s}_2}\|\end{aligned}$$

Since f is C^1 and $\vartheta_{f^n(x)}, \vartheta_x$ are isometries, the supremum can be made smaller than ε by choosing κ sufficiently small.

Claim. Let $s_i(Df_x^n) :=$ i-th singular value of $\vartheta_{f^n(x)} Df_x^n \vartheta_x^{-1}$, then there exists κ_4 such that $|s_i(Df_y^n) - s_i(Df_z^n)| < 1$ for all $d(y,z) < \kappa_4$.

Proof. By the minimax theorem the fact that $\vartheta_x, \vartheta_{f^n(x)}$ are isometries,

$$s_i(Df_x^n) = \max_{V \subset T_x M, \dim V = i} \min_{\vec{v} \in V, \|\vec{v}\|_x = 1} \|(Df_x^n)\vec{v}\|$$

The C^1–smoothness of f implies that $(x, \vec{v}) \mapsto \|(Df^n)_x \vec{v}\|_{f^n(x)}$ is uniformly continuous. From this it is routine to deduce that $s_i(Df_x^n)$ is uniformly continuous.

We can now begin the estimation of $K_n(A,k)$. Recall that A is contained in some ball $B(y_i, \varepsilon/k)$ of radius ε/k. Therefore

$$f^n(A) \subset B(f^n(y_i), \mathrm{Lip}(f^n)\varepsilon/k).$$

So every $A' \in \alpha_k$ which intersects $f^n(A)$ satisfies

$$A' \subset B(f^n(y_i), (\mathrm{Lip}(f^n) + 1)\varepsilon/k)$$

Assume k is so large that $(\mathrm{Lip}(f^n) + 1)\varepsilon/k < \kappa_3$. Using the bound $\mathrm{Lip}(\exp_x^{-1}) \leq 2$, we can write A' and A in coordinates as follows:

$$A = (\exp_{y_i} \circ \vartheta_{y_i}^{-1})(B), \text{ where } B \subset B(\underline{0}, 2\varepsilon/k) \subset \mathbb{R}^d$$
$$A' = (\exp_{f^n(y_i)} \circ \vartheta_{f^n(y_i)}^{-1})(B'), \text{ where } \mathrm{diam}(B') \leq 2\mathrm{diam}(A') \leq 4\varepsilon/k.$$

Since $A' \cap f^n(A) \neq \varnothing$ and $f^n(A) = (\exp_{f^n(y_i)} \circ \vartheta_{f^n(y_i)}^{-1})(F_{y_i}^n(B))$, it must be the case that $B' \cap F_{y_i}^n(B) \neq \varnothing$, whence since $\mathrm{diam}(B') \leq 4\varepsilon/k$

$$B' \subset N_\rho(F_{y_i}^n(B)) := \{x : d(x, F_{y_i}^n(B)) < \rho\}$$

where $\rho := 4\varepsilon/k$.

Claim. Choose k so large that $10\varepsilon/k < 1$ and $\varepsilon < 1/2$. Then the Lebesgue volume of $N_\rho(F_{y_i}^n(B))$ is less than $100^d(\varepsilon/k)^d \prod_{i=1}^d (1 + s_i(\vartheta_{f^n(y_i)}^{-1} Df_{y_i}^n \vartheta_{y_i}))$.

Proof. $B \subset B(\underline{0}, 2\varepsilon/k)$, therefore

$$(DF_{y_i}^n)_{\underline{0}}(B) \subset \begin{pmatrix} \text{orthogonal box whose sides} \\ \text{have lengths } 2 \cdot (2\varepsilon/k) \cdot s_i[(DF_{y_i}^n)_{\underline{0}}] \end{pmatrix} =: \text{Box}$$

Since $d(F_{y_i}^n(\underline{t}), (DF_{y_i}^n)_{\underline{0}}\underline{t}) \leq \varepsilon\|\underline{t}\| < \frac{\varepsilon}{k}$ on $B(\underline{0}, 2\varepsilon/k)$, $F_{y_i}^n(B) \subset N_{\varepsilon/k}(\text{Box})$. Thus

$$N_\rho(F_{y_i}^n(B)) \subset \begin{pmatrix} \text{orthogonal box whose sides} \\ \text{have lengths } (100\varepsilon/k)(1 + s_i[(DF_{y_i}^n)_{\underline{0}}]) \end{pmatrix}.$$

(100 is an over estimate).

We can now bound $K_n(A,k)$: Each A' contains a ball of radius $\varepsilon/2k$. Since $\exp_{f^n(y_i)}^{-1}$ has Lipschitz constant ≤ 2, each B' contains a ball of radius $\varepsilon/4k$. These balls are disjoint, because the A' are disjoint. We find that $K_n(A,k)$ is bounded by the maximal number of pairwise disjoint balls of radius $\varepsilon/4k$ which can be fit inside a set of volume $100^d(\varepsilon/k)^d\prod_{i=1}^d(1+s_i(\vartheta^{-1}_{f^n(y_i)}Df^n_{y_i}\vartheta_{y_i}))$. Thus

$$K_n(A,k)\leq C\prod_{i=1}^d[1+s_i(Df^n_{y_i})],$$

where C is independent of ε,n,k: The $(\varepsilon/k)^d$ term cancels out!

Proof of Ruelle's inequality. Steps 2 and 3 give us the inequality

$$\begin{aligned}h_\mu(f,\alpha_k)&\leq\frac{1}{n}\sum_{A\in\alpha_k}\mu(A)\log_2 K_n(A,k)\\&\leq\sum_{A\in\alpha_k}\mu(A)\sum_{i=1}^d\frac{1}{n}\log_2[1+s_i(Df^n_{y_i})]+\log C \text{ for all } k \text{ large enough}\\&\leq\sum_{i=1}^d\int\frac{1}{n}\log_2[1+s_i(Df^n_y)]d\mu(y)+\frac{1}{n}\log C,\end{aligned}$$

where the last estimate is because $A\subset B(y_i,\varepsilon/k)$ and

$$|s_i(Df^n_y)-s_i(Df^n_{y_i})|<\frac{1}{2}\text{ for all } y\in A\subset B(y_i,\varepsilon/k).$$

Passing to the limit $k\to\infty$ gives us that

$$h_\mu(f)\leq\sum_{i=1}^d\int\frac{1}{n}\log_2[1+s_i(Df^n_y)]d\mu(y)+\frac{1}{n}\log C,$$

whence $h_\mu(f)\leq\lim\limits_{n\to\infty}\sum\limits_{i=1}^d\int\frac{1}{n}\log_2[1+s_i(Df^n_y)]d\mu(y)$.

We analyze the last limit. Let $A(x)=\vartheta_{f(x)}Df_x\vartheta_x^{-1}$, then

$$A(f^{n-1}(x))\cdots A(f(x))A(x)=\vartheta_{f^n(x)}Df^n_x\vartheta_x^{-1}$$

and therefore $s_i(Df^n_y)=i$-th singular value of $A(f^{n-1}(x))\cdots A(f(x))A(x)$. By the multiplicative ergodic theorem, $s_i(x)^{1/n}\xrightarrow[n\to\infty]{}e^{\chi_i(x)}$, where $\chi_1(x)\geq\cdots\geq\chi_d(x)$ are the Lyapunov exponents of $A(x)$. Necessarily,

$$\frac{1}{n}\log_2[1+s_i(Df^n_y)]\xrightarrow[n\to\infty]{}\begin{cases}\chi_i(x)/\ln 2 & \chi_i(x)>0\\ 0 & \chi_i(x)\leq 0,\end{cases}$$

where the $\ln 2$ factor is due to the identity $\log_2(e^{\chi_i})=\chi_i/\ln 2$.

Since $\frac{1}{n}\log_2[1+s_i(Df_y^n)]$ is uniformly bounded (by $(\ln 2)^{-1}\sup_y \|Df_y\|$), we have by the bounded convergence theorem that

$$h_\mu(f) \leq \sum_{i=1}^{d} \int \lim_{n\to\infty} \frac{1}{n}\log_2[1+s_i(Df_y^n)]d\mu(y) = \frac{1}{\ln 2}\int \sum_{i:\chi_i(x)>0} \chi_i(x)d\mu,$$

and we have Ruelle's inequality. $\square$

Problems

4.1. Prove: $H_\mu(\alpha|\beta) = -\sum_{B\in\beta}\mu(B)\sum_{A\in\alpha}\mu(A|B)\log\mu(A|B)$, where $\mu(A|B) = \frac{\mu(A\cap B)}{\mu(B)}$.

4.2. Prove: if $H_\mu(\alpha|\beta) = 0$, then $\alpha \subseteq \beta \mod \mu$.

4.3. Prove that $h_\mu(T)$ is an invariant of measure theoretic isomorphism.

4.4. Prove that $h_\mu(T^n) = nh_\mu(T)$.

4.5. Prove that if T is invertible, then $h_\mu(T^{-1}) = h_\mu(T)$.

4.6. Entropy is affine
Let T be a measurable map on X, and μ_1, μ_2 be two T–invariant probability measures. Set $\mu = t\mu_1 + (1-t)\mu_2$ $(0 \leq t \leq 1)$. Show: $h_\mu(T) = th_{\mu_1}(T) + (1-t)h_{\mu_2}(T)$.
Guidance: Start by showing that for all $0 \leq x, y, t \leq 1$,

$$0 \leq \varphi(tx+(1-t)y) - [t\varphi(x)+(1-t)\varphi(y)] \leq -tx\log t - (1-t)y\log(1-t)$$

4.7. Let $(X, \mathscr{B}, \mu)$ be a probability space. If α, β are two measurable partitions of X, then we write $\alpha = \beta \mod \mu$ if $\alpha = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ where $\mu(A_i \triangle B_i) = 0$ for all i. Let $\mathfrak{P}$ denote the set of all countable measurable partitions of X, modulo the equivalence relation $\alpha = \beta \mod \mu$. Show that

$$\rho(\alpha, \beta) := H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha)$$

induces a metric on $\mathfrak{P}$.

4.8. Let $(X, \mathscr{B}, \mu, T)$ be a ppt. Show that $|h_\mu(T,\alpha) - h_\mu(T,\beta)| \leq H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha)$.

4.9. Use the previous problem to show that

$$h_\mu(T) = \sup\{h_\mu(T,\alpha) : \alpha \textit{ finite} \text{ measurable partition}\}$$

4.10. Suppose $\alpha = \{A_1, \ldots, A_n\}$ is a finite measurable partition. Show that for every ε, there exists $\delta = \delta(\varepsilon, n)$ such that if $\beta = \{B_1, \ldots, B_n\}$ is measurable partition s.t. $\mu(A_i \triangle B_i) < \delta$, then $\rho(\alpha, \beta) < \varepsilon$.

4.11. Entropy via generating sequences of partitions
Suppose $(X, \mathscr{B}, \mu)$ is a probability space, and $\mathscr{A}$ is an algebra of $\mathscr{F}$–measurable subsets (namely a collection of sets which contains $\varnothing$ and which is closed under finite unions, finite intersection, and forming complements). Suppose $\mathscr{A}$ generates $\mathscr{B}$ (i.e. $\mathscr{B}$ is the smallest σ–algebra which contains $\mathscr{A}$).

1. For every $F \in \mathscr{F}$ and $\varepsilon > 0$, there exists $A \in \mathscr{A}$ s.t. $\mu(A \triangle F) < \varepsilon$.
2. For every $\mathscr{F}$–measurable finite partition β and $\varepsilon > 0$, there exists an $\mathscr{A}$–measurable finite partition α s.t. $\rho(\alpha, \beta) < \varepsilon$.
3. If $T : X \to X$ is probability preserving, then
$$h_\mu(T) = \sup\{h_\mu(T, \alpha) : \alpha \text{ is an } \mathscr{A}\text{–measurable finite partition}\}.$$
4. Suppose $\alpha_1 \leq \alpha_2 \leq \cdots$ is an increasing sequence of finite measurable partitions such that $\sigma(\bigcup_{n\geq 1} \alpha_n) = \mathscr{B} \mod \mu$, then $h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \alpha_n)$.

4.12. Show that the entropy of the product of two ppt is the sum of their two entropies.

4.13. Show that $h_{\text{top}}(T^n) = n h_{\text{top}}(T)$.

Notes to chapter 4

The notion of entropy as a measure of information is due to Shannon, the father information theory. Kolmogorov had the idea to adapt this notion to the ergodic theoretic context for the purposes of inventing an invariant which is able to distinguish Bernoulli schemes. This became possible once Sinai has proved his generator theorem — which enables the calculation of this invariant for Bernoulli schemes. Later, in the 1970's, Ornstein has proved that entropy is a complete invariant for Bernoulli schemes: they are isomorphic iff they have the same entropy. The maximum of the possible entropies for a topological Markov shift was first calculated by Parry, who also found the maximizing measure. The material in this chapter is mostly classical, [5] and [1] are both excellent references. For an introduction to Ornstein's isomorphism theorem, see [4]. Our proof of Ruelle's inequality is based on [3] and [2].

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Appendix A
The isomorphism theorem for standard measure spaces

A.1 Polish spaces

Definition A.1. A *polish space* is a metric space (X,d) which is

1. complete (every Cauchy sequence has a limit);
2. and separable (there is a countable dense subset).

Every compact metric space is polish. But a polish space need not be compact, or even locally compact. For example,

$$\mathbb{N}^{\mathbb{N}} := \{\underline{x} = (x_1, x_2, x_3, \dots) : x_k \in \mathbb{N}\}$$

equipped with the metric $d(\underline{x},\underline{y}) := \sum_{k\geq 1} 2^{-k}|x_k^{-1} - y_k^{-1}|$ is a non locally compact polish metric space.

Notation: $B(x,r) := \{y \in X : d(x,y) < r\}$ (the open ball with center x and radius r).

Proposition A.1. *Suppose (X,d) is a polish space, then*

1. Second axiom of countability: *There exists a countable family of open sets $\mathscr{U}$ such that every open set in X is a union of a subfamily of $\mathscr{U}$.*
2. Lindelöf property: *Every cover of X by open sets has a countable sub-cover.*
3. *The intersection of any decreasing sequence of closed balls whose radii tend to zero is a single point.*

Proof. Since X is separable, it contains a countable dense set $\{x_n\}_{n\geq 1}$. Define

$$\mathscr{U} := \{B(x_n, r) : n \in \mathbb{N}, r \in \mathbb{Q}\}.$$

This is a countable collection, and we claim that it satisfies (1). Take some open set U. For every $x \in U$ there are

1. $R > 0$ such that $B(x,R) \subset U$ (because U is open);
2. $x_n \in B(x,R/2)$ (because $\{x_n\}$ is dense); and

3. $r \in \mathbb{Q}$ such that $d(x,x_n) < r < R/2$.

It is easy to check that $x \in B(x_n,r) \subset B(x,2r) \subset B(x,R) \subset U$. Thus for every $x \in U$ there is $U_x \in \mathscr{U}$ s.t. $x \in U_x \subset U$. It follows that U is a union of elements from $\mathscr{U}$ and (1) is proved. (2) is an immediate consequence.

To see (3), suppose $B_n := \overline{B(z_n,r_n)}$ is a sequence of closed balls such that $B_n \supset B_{n+1}$ and $r_n \to 0$. It is easy to verify that $\{z_n\}$ is a Cauchy sequence. Since X is complete, it converges to a limit z. This limit belongs to B_n because $z_k \in B_k \subset B_n$ for all $k > n$, and B_n is closed. Thus the intersection of B_n contains at least one point. It cannot contain more than one point, because its diameter is zero (because it is bounded by $\mathrm{diam}[B_n] \leq r_n \to 0$). □

A.2 Standard probability spaces

Definition A.2. A *standard probability space* is a probability space $(X,\mathscr{B},\mu)$ where X is polish, $\mathscr{B}$ is the σ–algebra of Borel sets of X, and μ is a Borel probability measure.

Theorem A.1. *Suppose $(X,\mathscr{B},\mu)$ is a standard probability space, then*

1. **Regularity:** *Suppose $E \in \mathscr{B}$. For every $\varepsilon > 0$ there exists an open set U and a closed set F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.*
2. **Separability:** *There exists a countable collection of measurable sets $\{E_n\}_{n\geq 1}$ such that for every $E \in \mathscr{B}$ and $\varepsilon > 0$ there exists some n s.t. $\mu(E \triangle E_n) < \varepsilon$. Equivalently, $L^p(X,\mathscr{B},\mu)$ is separable for some (and then for all) $1 \leq p < \infty$.*

Proof. Say that a set E satisfies the *approximation property* if for every ε there are a closed set F and an open set U s.t. $F \subset E \subset U$ and $\mu(U \setminus E) < \varepsilon$.

Open balls $B(x,r)$ have the approximation property: Take $U = B(x,r)$ and $F = \overline{B(x,r-\frac{1}{n})}$ for n sufficiently large (these sets increase to $B(x,r)$ so their measure tends to that of $B(x,r)$).

Open sets U have the approximation property. The approximating open set is the set itself. To find the approximating closed set use the second axiom of countability to write the open set as the countable union of balls B_n, and approximate each B_n from within by a closed set F_n such that $\mu(B_n \setminus F_n) < \varepsilon/2^{n+1}$. Then $\mu(U \setminus \bigcup_{n\geq 1} F_n) < \varepsilon/2$. Now take $F := \bigcup_{i=1}^N F_i$ for N large enough.

Thus the collection $\mathscr{C} := \{E \in \mathscr{B} : E \text{ has the approximation property}\}$ contains the open sets. Since it is a σ–algebra (check!), it must be equal to $\mathscr{B}$, proving (1).

We prove separability. Polish spaces satisfy the second axiom of countability, so there is a countable family of open balls $\mathscr{U} = \{B_n : n \in \mathbb{N}\}$ such that every open set is the union of a countable subfamily of $\mathscr{U}$. This means that every open set can be approximated by a *finite* union of elements of $\mathscr{U}$ to arbitrary precision. By the regularity property shown above, every measurable set can be approximated by a finite union of elements of $\mathscr{U}$ to arbitrary precision. It remains to observe that $\mathscr{E} = \{\text{finite unions of elements of } \mathscr{U}\}$ is countable.

The separability of L^p for $1 \le p < \infty$ follows from the above and the obvious fact that the collection $\{\sum_{i=1}^N \alpha_i 1_{E_i} : N \in \mathbb{N}, \alpha_i \in \mathbb{Q}, E_i \in \mathscr{E}\}$ is dense in L^p (prove!). The other direction is left to the reader. ☐

The following statement will be used in the proof of the isomorphism theorem.

Lemma A.1. *Suppose $(X, \mathscr{B}, \mu)$ is a standard probability space and E is a measurable set of positive measure, then there is a point $x \in X$ such that $\mu[E \cap B(x,r)] \neq 0$ for all $r > 0$.*

Proof. Fix $\varepsilon_n \downarrow 0$. Write X as a countable union of open balls of radius ε_1 (second axiom of countability). At least one of these, B_1, satisfies $\mu(E \cap B_1) \neq 0$. Write B_1 as a countable union of open balls of radius ε_2. At least one of these, B_2, satisfies $\mu[E \cap B_1 \cap B_2] \neq 0$. Continue in this manner. The result is a decreasing sequence of open balls with shrinking diameters $B_1 \supset B_2 \supset \cdots$ which intersect E at a set of positive measure.

The sequence of centers of these balls is a Cauchy sequence. Since X is polish, it converges to a limit $x \in X$. This x belongs to the closure of each B_n.

For every r find n so large that $\varepsilon_n < r/2$. Since $x \in \overline{B_n}$, $d(x,x_n) \le \varepsilon_n$, and this implies that $B(x,r) \supseteq B(x_n, \varepsilon_n) = B_n$. Since B_n intersects E with positive measure, $B(x,r)$ intersects E with positive measure. ☐

A.3 Atoms

Definition A.3. An *atom* of a measure space $(X, \mathscr{B}, \mu)$ is a measurable set A of non-zero measure with the property that for all other measurable sets B contained in A, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. A measure space is called *non-atomic*, if it has no atoms.

Proposition A.2. *For standard spaces $(X, \mathscr{B}, \mu)$, every atom is of the form $\{x\} \cup$null set for some x s.t. $\mu\{x\} \neq 0$.*

Proof. Suppose A is an atom. Since X can be covered by a countable collection of open balls of radius $r_1 := 1$, $A = \bigcup_{i \ge 1} A_i$ where A_i are measurable subsets of A of diameter at most r_1. One of those sets, A_{i_1}, has non-zero measure. Since A is an atom, $\mu(A_{i_1}) = \mu(A)$. Setting $A^{(1)} := A_{i_1}$, we see that

$$A^{(1)} \subset A \ , \ \mathrm{diam}(A^{(1)}) \le r_1 \ , \mu(A^{(1)}) = \mu(A).$$

Of course $A^{(1)}$ is an atom.

Now repeat this argument with $A^{(1)}$ replacing A and $r_2 := 1/2$ replacing r_1. We obtain an atom $A^{(2)}$ s.t.

$$A^{(2)} \subset A \ , \ \mathrm{diam}(A^{(2)}) \le r_2 \ , \mu(A^{(2)}) = \mu(A).$$

We continue in this manner, to obtain a sequence of atoms $A \supset A^{(1)} \supset A^{(2)} \supset \cdots$ of the same measure, with diameters $r_k = 1/k \to 0$. The intersection $\bigcap A^{(k)}$ is non-empty, because its measure is $\lim \mu(A^{(k)}) = \mu(A) \neq 0$. But its diameter is zero. Therefore it is a single point x, and by construction $x \in A$ and $\mu\{x\} = \mu(A)$. $\square$

Lemma A.2. *Suppose $(X, \mathscr{B}, \mu)$ is a non-atomic standard probability space, and $r > 0$. Every $\mathscr{B}$–measurable set E can be written in the form $E = \biguplus_{i=1}^{\infty} F_i \uplus N$ where $\mu(N) = 0$, F_i are closed,* $\operatorname{diam}(F_i) < r$, *and $\mu(F_i) \neq 0$.*

Proof. Since every measurable set is a finite or countable disjoint union of sets of diameter less than r (prove!), it is enough to treat sets E such that $\operatorname{diam}(E) < r$.

Standard spaces are regular, so we can find a closed set $F_1 \subset E$ such that $\mu(E \setminus F_1) < \frac{1}{2}$. If $\mu(E \setminus F_1) = 0$, then stop. Otherwise apply the argument to $E \setminus F_1$ to find a closed set $F_2 \subset E \setminus F_1$ of positive measure such that $\mu[E \setminus (F_1 \cup F_2)] < \frac{1}{2^2}$. Continuing in this manner we obtain pairwise disjoint closed sets $F_i \subset E$ such that $\mu(E \setminus \bigcup_{i=1}^{n} F_i) < 2^{-n}$ for all n, or until we get to an n such that $\mu(E \setminus \biguplus_{i=1}^{n} F_n) = 0$.

If the procedure did not stop at any stage, then the lemma follows with $N := E \setminus \bigcup_{i \geq 1} F_i$.

We show what to do in case the procedure stops after n steps. Set $F = F_n$, the last closed set. The idea is to split F into countably many disjoint closed sets, plus a set of measure zero.

Find an x such that $\mu[F \cap B(x,r)] \neq 0$ for all $r > 0$ (previous lemma). Since X is non-atomic, $\mu\{x\} = 0$. Since $B(x,r) \downarrow \{x\}$, $\mu[F \cap B(x,r)] \xrightarrow[n\to\infty]{} 0$. Choose $r_n \downarrow 0$ for which $\mu[F \cap B(x,r_n)]$ is strictly decreasing. Define

$$C_1 := F \cap \overline{B(x,r_1)} \setminus B(x,r_2),\ C_2 := F \cap \overline{B(x,r_2)} \setminus B(x,r_3) \text{ and so on.}$$

This is an *infinite* sequence of closed pairwise disjoint sets of positive measure inside F. By the construction of F they are disjoint from $F_1, \ldots, F_{n-1}$ and they are contained in E.

Now consider $E' := E \setminus (\bigcup_{i=0}^{n-1} F_i \cup \bigcup_i C_i)$. Applying the argument in the first paragraph to E', we write it as a finite or countable disjoint union of closed sets plus a null set. Adding these sets to the collection $\{F_1, \ldots, F_{n-1}\} \cup \{C_i : i \geq 1\}$ gives us the required decomposition of E. $\square$

A.4 The isomorphism theorem

Definition A.4. Two measure spaces $(X_i, \mathscr{B}_i, \mu_i)$ $(i = 1,2)$ are called *isomorphic* if there are measurable subsets of full measure $X_i' \subset X_i$ and a measurable bijection $\pi : X_1' \to X_2'$ with measurable inverse such that $\mu_2 = \mu_1 \circ \pi^{-1}$.

Theorem A.2 (Isomorphism theorem). *Every non-atomic standard probability space is isomorphic to the unit interval equipped with the Lebesgue measure.*

Proof. Fix a decreasing sequence of positive numbers ε_n which tend to zero. Using lemma A.2, decompose $X = \biguplus_{j=1}^{\infty} F(j) \uplus N$ where $F(j)$ are pairwise disjoint closed sets of positive measure and diameter less than ε_1, and N_1 is a null set.

Applying lemma A.2 to each $F(j)$, decompose $F(j) = \biguplus_{k=1}^{\infty} F(j,k) \uplus N(j)$ where $F(j,k)$ are pairwise disjoint closed sets of positive measure and diameter less than ε_2, and $N(j)$ is a null set.

Continuing in this way we obtain a family of sets $F(x_1,\dots,x_n), N(x_1,\dots,x_n)$, $(n, x_1,\dots,x_n \in \mathbb{N})$ such that

1. $F(x_1,\dots,x_n)$ are closed, have positive measure, and $\mathrm{diam}[F(x_1,\dots,x_n)] < \varepsilon_n$;
2. $F(x_1,\dots,x_{n-1}) = \biguplus_{y\in\mathbb{N}} F(x_1,\dots,x_{n-1},y) \uplus N(x_1,\dots,x_{n-1})$;
3. $\mu[N(x_1,\dots,x_n)] = 0$.

Set $X' := \bigcap_{n\geq 1} \biguplus_{x_1,\dots,x_n\in\mathbb{N}} F(x_1,\dots,x_n)$. It is a calculation to see that $\mu(X \setminus X') = 0$. The set X' has tree-like structure: every $x \in X'$ determines a unique sequence $(x_1, x_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $x \in F(x_1,\dots,x_n)$ for all n. Define $\pi : X' \to [0,1]$ by

$$\pi(x) = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cdots}}$$

This map is one-to-one on X', because if $\pi(x) = \pi(y)$, then $1/(x_1 + 1/(x_2 + \cdots)) = 1/(y_2 + 1/(y_2 + \cdots))$ whence $x_k = y_k$ for all k;[1] this means that $x, y \in F(x_1,\dots x_n)$ for all n, whence $d(x,y) \leq \varepsilon_n \xrightarrow[n\to\infty]{} 0$.

This map is onto $[0,1]\setminus\mathbb{Q}$, because every irrational $t \in [0,1]$ has an infinite continued fraction expansion $1/(a_1 + 1/(a_2 + \cdots))$, so $t = \pi(x)$ for the unique x in $\bigcap_{n\geq 1} F(a_1,\dots,a_n)$. (This intersection is non-empty because it is the decreasing intersection of closed sets of shrinking diameters in a complete metric space.)

We claim that $\pi : X' \to [0,1]\setminus\mathbb{Q}$ is Borel measurable. Let $[a_1,\dots,a_n]$ denote the collection of all irrationals in $[0,1]$ whose continued fraction expansion starts with $(a_1,\dots,a_n)$. We call such sets "cylinders." We have $\pi^{-1}[a_1,\dots,a_n] = F(a_1,\dots,a_n) \cap X'$, a Borel set. Thus

$$\mathscr{C} := \{E \in \mathscr{B}([0,1]\setminus\mathbb{Q}) : \pi^{-1}(E) \in \mathscr{B}\}$$

contains the cylinders. It is easy to check that $\mathscr{C}$ is a σ–algebra. The cylinders generate $\mathscr{B}([0,1]\setminus\mathbb{Q})$ (these are intervals whose length tends to zero as $n\to\infty$). It follows that $\mathscr{C} = \mathscr{B}([0,1]\setminus\mathbb{Q})$ and the measurability of π is proved.

Next we observe that $\pi[F(a_1,\dots,a_n)\cap X'] = [a_1,\dots,a_n]$, so $\pi^{-1} : [0,1]\setminus\mathbb{Q} \to X'$ is Borel measurable by an argument similar to the one in the previous paragraph.

It follows that $\pi : (X, \mathscr{B}, \mu) \to ([0,1]\setminus\mathbb{Q}, \mathscr{B}([0,1]\setminus\mathbb{Q}), \mu\circ\pi^{-1})$ is an isomorphism of measure spaces. There is an obvious extension of $\mu\circ\pi^{-1}$ to $\mathscr{B}([0,1])$ obtained by declaring $\mu(\mathbb{Q}) := 0$. Let m denote this extension. Then we get an isomorphism between $(X, \mathscr{B}, \mu)$ to $([0,1], \mathscr{B}([0,1]), m)$ where is m is some Borel probability measure on $[0,1]$. Since μ is non-atomic, m is non-atomic.

[1] Hint: apply the transformation $x \mapsto [1/x]$ to both sides.

We now claim that $([0,1],\mathscr{B}([0,1]),m)$ is isomorphic to $([0,1],\mathscr{B}([0,1]),\lambda)$, where λ is the Lebesgue measure.

Consider first the distribution function of m, $s \mapsto m[0,s)$. This is a monotone increasing function (in the weak sense). We claim that it is continuous. Otherwise it has a jump J at some point x_0:

$$m[0,x_0+\varepsilon) - m[0,x_0) > J \text{ for all } \varepsilon > 0.$$

This means that $m\{x_0\} \geq J$, which cannot be the case since m is non-atomic.

Since $F_m(s) = m([0,s))$ is continuous, and $F_m(0) = 0, F_m(1) = m([0,1]) - m\{1\} = 1$, $F_m(s)$ attains any real value $0 \leq t \leq 1$. So the following definition makes sense:

$$\vartheta(t) := \min\{s \geq 0 : m([0,s)) = t\} \quad (0 \leq t \leq 1)$$

Notice that $F_m(\vartheta(t)) = t$. We will show that $\vartheta : ([0,1],\mathscr{B},m) \to ([0,1],\mathscr{B},\lambda)$ is an isomorphism of measure spaces.

Step 1. $m([0,1] \setminus \vartheta[0,1]) = 0$.

Proof. $s \in \vartheta[0,1]$ iff $s = \vartheta(F_m(s))$ iff $\forall s' < s$, $m[s',s] > 0$.

Conversely, $s \notin \vartheta[0,1]$, iff s belongs to an interval with positive length and zero m measure. Let $I(s)$ denote the union of all such intervals. This is a closed interval (because m is non-atomic), $m(I(s)) = 0$, $\lambda(I(s)) > 0$, and for any $s_1, s_2 \in [0,1] \setminus \vartheta[0,1]$, either $I(s_1) = I(s_2)$ or $I(s_1) \cap I(s_2) = \varnothing$.

There can be at most countably many different intervals $I(s)$. It follows that $[0,1] \setminus \vartheta[0,1]$ is a finite or countable union of closed intervals with zero m-measure. In particular, $m([0,1] \setminus \vartheta[0,1]) = 0$.

Step 2. ϑ is one-to-one on $[0,1]$.

Proof: $F_m \circ \vartheta = id$

Step 3. ϑ is measurable, with measurable inverse.

Proof. ϑ is measurable, because for every interval (a,b), $\vartheta^{-1}(a,b) = \{t : t = F_m(\vartheta(t)) \in F_m(a,b)\} = (F_m(a), F_m(b))$, a Borel set.

To see that ϑ^{-1} is measurable, note that ϑ is strictly increasing, therefore $\vartheta(a,b) = (\vartheta(a),\vartheta(b)) \cap \vartheta([0,1])$. This set is Borel, because as we saw in the proof of step 1, $\vartheta([0,1])$ is the complement of a countable collection of intervals.

Step 4. $m \circ \vartheta = \lambda$.

Proof. By construction, $m[0,\vartheta(t)) = t$. So $m[\vartheta(s),\vartheta(t)) = t - s$ for all $0 < s < t < 1$. This the semi-algera of half-open intervals generates the Borel σ-algebra of the interval, $m \circ \vartheta = \lambda$.

Steps 1–4 show that $\vartheta : ([0,1],\mathscr{B}([0,1]),m) \to ([0,1],\mathscr{B}([0,1]),\lambda)$ is an isomorphism. Composing this with π, we get an isomorphism between $(X,\mathscr{B},\mu)$ and the unit interval equipped with Lebesgue's measure. $\square$

We comment on the atomic case. A standard probability space $(X, \mathscr{B}, \mu)$ can have at most countably many atoms (otherwise it will contain an uncountable collection of pairwise disjoint sets of positive measure, which cannot be the case). Let $\{x_i : i \in \Lambda\}$ be a list of the atoms, where $\Lambda \subset \mathbb{N}$. Then

$$\mu = \mu' + \sum_{i \in \Lambda} \mu\{x_i\}\delta_{x_i} \quad (\delta_x = \text{Dirac measure})$$

where μ' is non-atomic.

Suppose w.l.o.g that $X \cap \mathbb{N} = \varnothing$. The map

$$\pi : X \to X \cup \Lambda,\ \pi(x) = \begin{cases} x & x \notin \{x_i : i \in \Lambda\} \\ i & x = x_i \end{cases}$$

is an isomorphism between X and the measure space obtained by adding to $(X, \mathscr{B}, \mu')$ atoms with right mass at points of Λ. The space $(X, \mathscr{B}, \mu')$ is non-atomic, so it is isomorphic to $[0, \mu'(X)]$ equipped with the Lebesgue measure. We obtain the following generalization of the isomorphism theorem: *Every standard probability space is isomorphic to the measure space consisting of a finite interval equipped with Lebesgue's measure, and a finite or countable collection of atoms.*

Definition A.5. A measure space is called a *Lebesgue space*, if it is isomorphic to the measure space consisting of a finite interval equipped with the Lebesgue measurable sets and Lebesgue's measure, and a finite or countable collection of atoms.

Note that the σ–algebra in the definition is the Lebesgue σ–algebra, not the Borel σ–algebra. (The Lebesgue σ–algebra is the completion of the Borel σ–algebra with respect to the Lebesgue measure, see problem 1.2.) The isomorphism theorem and the discussion above say that the *completion* of a standard space is a Lebesgue space. So the class of Lebesgue probability spaces is enormous!

Appendix A
The Monotone Class Theorem

Definition A.1. A sequence of sets $\{A_n\}$ is called *increasing* (resp. *decreasing*) if $A_n \subseteq A_{n+1}$ for all n (resp. $A_n \supseteq A_{n+1}$ for all n).

Notation: $A_n \uparrow A$ means that $\{A_n\}$ is an increasing sequence of sets, and $A = \bigcup A_n$. $A_n \downarrow A$ means that $\{A_n\}$ is a decreasing sequence of sets, and $A = \bigcap A_n$.

Proposition A.1. *Suppose $(X, \mathscr{B}, \mu)$ is a measure space, and $A_n \in \mathscr{B}$.*

1. *if $A_n \uparrow A$, then $\mu(A_n) \xrightarrow[n\to\infty]{} \mu(A)$;*
2. *if $A_n \downarrow A$ <u>and $\mu(A_n) < \infty$ for some n</u>, then $\mu(A_n) \xrightarrow[n\to\infty]{} \mu(A)$.*

Proof. For (1), observe that $A = \biguplus_{n\geq 1} A_{n+1} \setminus A_n$ and use σ–additivity. For (2), fix n_0 s.t. $\mu(A_{n_0}) < \infty$, and observe that $A_n \downarrow A$ implies that $(A_{n_0} \setminus A_n) \uparrow (A_{n_0} \setminus A)$. $\square$

The example $A_n = (n, \infty)$, $\mu =$Lebesgue measure on $\mathbb{R}$, shows that the condition in (2) cannot be removed.

Definition A.2. Let X be a set. A *monotone class* of subsets of X is a collection $\mathscr{M}$ of subsets of X which contains the empty set, and such that if $A_n \in \mathscr{M}$ and $A_n \uparrow A$ or $A_n \downarrow A$, then $A \in \mathscr{M}$.

Recall that an *algebra* of subsets of a set X is a collection of subsets of X which contains the empty set, and which is closed under finite unions, finite intersections, and forming the complement.

Theorem A.1 (Monotone Class Theorem). *A monotone class which contains an algebra, also contains the sigma–algebra generated by this algebra.*

Proof. Let $\mathscr{M}$ be a monotone class which contains an algebra $\mathscr{A}$. Let $\mathscr{M}(\mathscr{A})$ denote the intersection of all the collections $\mathscr{M}' \subset \mathscr{M}$ such that (a) $\mathscr{M}'$ is a monotone class, and (b) $\mathscr{M}' \supseteq \mathscr{A}$. This is a monotone class (check!). In fact it is the *minimal* monotone class which contains $\mathscr{A}$. We prove that it is a σ–algebra. Since $\mathscr{M}(\mathscr{A}) \subset \mathscr{M}$, this completes the proof.

We begin by claiming that $\mathscr{M}(\mathscr{A})$ is closed under forming complements. Suppose $E \in \mathscr{M}(\mathscr{A})$. The set

$$\mathscr{M}' := \{E' \in \mathscr{M}(\mathscr{A}) : (E')^c \in \mathscr{M}(\mathscr{A})\}$$

contains $\mathscr{A}$ (because $\mathscr{A}$ is an algebra), and it is a monotone class (check!). But $\mathscr{M}(\mathscr{A})$ is the minimal monotone class which contains $\mathscr{A}$, so $\mathscr{M}' \supset \mathscr{M}(\mathscr{A})$. It follows that $E \in \mathscr{M}'$, whence $E^c \in \mathscr{M}(\mathscr{A})$.

Next we claim that $\mathscr{M}(\mathscr{A})$ has the following property:

$$E \in \mathscr{M}(\mathscr{A}), A \in \mathscr{A} \Longrightarrow E \cup A \in \mathscr{M}(\mathscr{A}).$$

Again, the reason is that the collection $\mathscr{M}'$ of sets with this property contains $\mathscr{A}$, and is a monotone class.

Now fix $E \in \mathscr{M}(\mathscr{A})$, and consider the collection

$$\mathscr{M}' := \{F \in \mathscr{M}(\mathscr{A}) : E \cup F \in \mathscr{M}(\mathscr{A})\}.$$

By the previous paragraph, $\mathscr{M}'$ contains $\mathscr{A}$. It is clear that $\mathscr{M}'$ is a monotone class. Thus $\mathscr{M}(\mathscr{A}) \subseteq \mathscr{M}'$, and as a result $E \cup F \in \mathscr{M}(\mathscr{A})$ for all $F \in \mathscr{M}(\mathscr{A})$. But $E \in \mathscr{M}(\mathscr{A})$ was arbitrary, so this means that $\mathscr{M}(\mathscr{A})$ is closed under finite unions.

Since $\mathscr{M}(\mathscr{A})$ is closed under finite unions, and countable increasing unions, it is closed under general countable unions.

Since $\mathscr{M}(\mathscr{A})$ is closed under forming complements and taking countable unions, it is a sigma algebra. By definition this sigma algebra contains $\mathscr{A}$ and is contained in $\mathscr{M}$. $\square$

Index