UNIQUENESS OF EQUILIBRIUM MEASURES FOR COUNTABLE MARKOV SHIFTS AND MULTI-DIMENSIONAL PIECEWISE EXPANDING MAPS

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ABSTRACT. We prove that potentials with summable variations on topologically transitive countable Markov shifts have at most one equilibrium measure. We apply this to multi-dimensional piecewise expanding maps using their Markov diagrams.

RÉSUMÉ. Nous montrons que les décalages markoviens topologiquement transitifs admettent au plus une mesure d'équilibre pour les potentiels à variations sommables. Nous appliquons ce résultat aux applications multi-dimensionnelles dilatantes par morceaux.

1. INTRODUCTION

Let S be a countable set and $A = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. The (one-sided) *topological Markov shift* with set of *states* S and *transition matrix* A is

$$\Sigma = \Sigma_A^+ := \{(x_0, x_1, \ldots) \in S^{\mathbb{N}\cup\{0\}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}$$

together with the action of the left shift $\sigma : \Sigma \to \Sigma$. The topology on Σ is assumed to be the relative product topology inside $S^{\mathbb{N}\cup\{0\}}$, S being discrete. A shift invariant probability measure μ is called a *maximal measure* if $h_\mu(\sigma)$ is maximal, and an *equilibrium measure* for $\phi : \Sigma \to \mathbb{R}$ if $h_\mu(\sigma) + \int \phi d\mu$ is well-defined and maximal.

W. Parry proved in [17] that if S is finite and σ is topologically transitive, then there exists exactly one maximal measure, and that this measure is the Markov measure with initial distribution (p_i) and transition matrix (p_{ij}) where $p_i = u_i v_i$, $p_{ij} = \frac{u_j t_{ij}}{\lambda u_i}$ and $u = (u_i)$, $v = (v_i)$ and $\lambda > 0$ are given by $vA = \lambda v, Au = \lambda u$ and $\langle u, v\rangle = 1$.

Ruelle [18] improved this and showed that if S is finite, σ is topologically transitive, and $var_n(\phi) = O(\theta^n)$ for some $\theta \in (0,1)$ where

$$var_n(\phi) := \sup\{\phi(x) - \phi(y) : x_i = y_i \text{ for } i = 0, \ldots, n-1\},$$

then ϕ has exactly one equilibrium measure. Furthermore, he showed that if L_ϕ is the operator $L_\phi f = \sum_{\sigma y = x} e^{\phi(y)} f(y)$, then the unique equilibrium measure is given by $hd\nu$ where h and ν are a positive continuous function and a Borel probability measure such that $\int hd\nu = 1$, $L_\phi h = \lambda h$ and $L_\phi^* \nu = \lambda\nu$ for $\lambda > 0$. The condition $var_n(\phi) = O(\theta^n)$ was relaxed to $\sum var_n(\phi) < \infty$ by Walters [25].

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If S is infinite there may be no maximal measure, but if it exists it must be unique as long as σ is topologically transitive (Gurevich [10],[11]). Indeed, Gurevich showed that in this case there exists a maximal measure if and only if A is R-recurrent and R-positive, and that in this case the unique maximal measure can be determined as in the case $|S| < \infty$ (the R-positivity and R-recurrence conditions are necessary and sufficient conditions for the existence of u and v). The same techniques yield the uniqueness of equilibrium measures of potentials of the form $\phi(x) = \phi(x_0, \ldots, x_N)$ (Gurevich and Savchenko [12]).

Our aim in this paper is to give a Ruelle–type generalization of these results for ϕ which depend on an infinite number of coordinates such that $\sum_{n\geqslant 2} var_n(\phi) < \infty$. We then apply this generalization to certain (non-Markovian) multi-dimensional piecewise expanding maps.

The *Gurevich pressure* of ϕ is defined by $P_G(\phi) := \lim\limits_{n\to\infty} \frac{1}{n} \log \sum\limits_{\sigma^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$ where $a \in S$ is fixed, $[a] := \{x \in \Sigma : x_0 = a\}$ and $\phi_n := \sum_{i=0}^{n-1} \phi \circ T^i$. Let $\mathcal{P}_\sigma(\Sigma)$ denote the collection of σ-invariant Borel probability measures on Σ. The *metric pressure* (or just *pressure*) of $\mu \in \mathcal{P}_\sigma(\Sigma)$ is:

$$P_\mu(\phi) = P_\mu(\phi, \sigma) := h_\mu(\sigma) + \int \phi d\mu.$$

Note that this is not always well-defined (ϕ might not be integrable, or worse still it might happen that $h_\mu(\sigma) = +\infty$ and $\int \phi d\mu = -\infty$). One of us showed [20], [21] that if σ is topologically mixing and $\sup \phi < \infty$, then

$$P_G(\phi) = \sup \left\{ P_\mu(\phi) : \mu \in \mathcal{P}_\sigma(\Sigma), P_\mu(\phi) \text{ is well-defined} \right\}.$$

The condition $\sup \phi < \infty$ guarantees that $\int \phi d\mu$ is well-defined (though possibly infinite), so the 'well-defined' condition reduces to a preclusion of the μ's for which $h_\mu(\sigma) = \infty$ and $\int \phi d\mu = -\infty$.[1] We prove:

THEOREM 1.1. *Let (Σ, σ) be a topologically transitive countable Markov shift, and suppose $\phi : \Sigma \to \mathbb{R}$ satisfies $\sup \phi < \infty, P_G(\phi) < \infty$ and $\sum_{n\geqslant 2} var_n(\phi) < \infty$. There exists at most one invariant probability measure μ such that $h_\mu(\sigma) + \int \phi d\mu$ is well-defined and maximal.*

THEOREM 1.2. *Under the assumptions of the previous theorem, if μ exists, then there exist a positive continuous function h and a Borel measure ν finite on each $[a]$, $a \in S$ and with full support such that for some $\lambda > 0$, $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$ and $\int h d\nu = 1$. Moreover, $d\mu = h d\nu$.*

These two theorems show that for ϕ with summable variations and finite Gurevich pressure, if ϕ has an equilibrium measure then it is positive recurrent in the terminology of [21],[22]. The opposite is *not* true, because it might happen that $d\mu = h d\nu$ where $L_\phi^* \nu = \lambda \nu$, $L_\phi h = \lambda h$ and $\int f d\nu = 1$, satisfies $h_\mu(\sigma) = \infty$ and $\int \phi d\mu = -\infty$ (in which case $h_\mu(\sigma) + \int \phi d\mu$ is meaningless). Nevertheless, even in these situations, the measure $d\mu = h d\nu$ can still be interpreted as some kind of weak equilibrium measure. We refer the reader to [21],[22] for details.

Finally, we remark that if the equilibrium measure of ϕ is a Gibbs measure (see [2]), then its uniqueness can be deduced from the uniqueness of Gibbs measures,

[1] Measures μ with $\int \phi d\mu = -\infty$ cannot contribute to the supremum (which is always larger than $-\infty$ because of the existence of invariant measures supported on periodic points). Therefore, the condition of $h_\mu + \int \phi$ being well-defined can replaced by the condition $-\int \phi < \infty$.

as in [2]. Unfortunately, if $|S| = \infty$ this never happens, unless A satisfied the BIP property (see [16] and [24]), so the general case cannot be treated this way.

Theorem 1.1 can be applied to non-Markov, multi-dimensional piecewise expanding maps by using the connected Markov diagram introduced in [5]. To state our result we need some definitions.

A *piecewise expanding map* (X, P, T) is a locally connected compact metric space X together with a "partition" P which is just a finite collection of non-empty, pairwise disjoint open subsets of X with dense union and a map $T : \bigcup_{A \in P} A \to X$ such that for each $A \in P$, each restriction $T|_A$ can be extended to an expanding homeomorphism between a neighborhood of $\bar{A}$ and one of $\overline{TA}$.

The *boundary* of such a system is $\partial P := \bigcup_{A \in P} \partial A$.

P_0^{n-1} denotes the collection of n-cylinders, i.e., the non-empty intersections $A_0 \cap T^{-1}A_1 \cap \cdots \cap T^{-n+1}A_{n-1}$ where the A_i's are elements of P.

A *piecewise Hölder-continuous potential* is $\phi : X \to \mathbb{R}$ such that the restriction of ϕ to any element of P is Hölder-continuous, i.e., for all x, y in the same element of P,

$$|\phi(x) - \phi(y)| \leqslant K d(x, y)^\alpha$$

for some $\alpha > 0$, $K < \infty$ (the values of ϕ on ∂P are irrelevant for our purposes).

The *pressure of a subset* $S \subseteq X$ (non-necessarily invariant) is:

$$P_{\mathrm{top}}(\phi, S, T) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{A \in P_0^{n-1} | S \cap \bar{A} \neq \varnothing} \sup_{x \in A} \exp \phi_n(x)$$

The *topological pressure* of T is $P_{\mathrm{top}}(\phi, T) := P_{\mathrm{top}}(X, T)$.

An *equilibrium measure* is, like in the case of Markov shifts, an invariant probability measure μ for which the metric pressure $P_\mu(\phi, T)$ is equal to the topological pressure $P_{\mathrm{top}}(\phi, T)$.

THEOREM 1.3. *Let (X, P, T) be a piecewise expanding map with a piecewise Hölder-continuous potential ϕ. Assume that:*

$$P_{\mathrm{top}}(\phi, \partial P, T) < P_{\mathrm{top}}(\phi, T).$$

Then:

(i) $P_{\mathrm{top}}(\phi, T) = \sup_{\mu \in \mathcal{P}_T(X)} P_\mu(\phi, T)$ and this supremum is realized by at least one measure.

(ii) there exist at most finitely many ergodic equilibrium measures.

(iii) If, additionally, T is strongly topologically transitive *in the sense that for all non-empty open sets U, $\bigcup_{k \geqslant 0} T^k(U) \supseteq T(X)$, then there is a* unique *equilibrium measure.*

Example. Recall that a *multidimensional β-transformation* is a map $T : [0, 1[^d \to [0, 1[^d$, $d \geqslant 1$ of the form $T(x) = \{B(x)\}$ where $B : \mathbb{R}^d \to \mathbb{R}^d$ is an expanding affine map and $\{y\}$ is the unique vector in $[0, 1[^d$ equal to $y \mod \mathbb{Z}^d$. T is obviously a piecewise expanding map on $[0, 1]^d$, the partition P being the maximum connected open sets of continuity.

COROLLARY 1.4. *A multidimensional β-transformation T admits finitely many ergodic equilibrium measures w.r.t. any non negative Hölder-continuous potential ϕ with $\sup \phi < \lambda_d := \log \min\{\|\vec{B}.v\| : \|v\| = 1\}$, $\vec{B}$ being the corresponding linear map.*

Moreover if $\lambda_d > \log(1 + \sqrt{d})$ then the equilibrium measure is unique.

To see that the corollary follows from the theorem observe that, ∂P being included in a finite union of hyper planes, Proposition 4 of [4] gives

$$P_{\mathrm{top}}(\phi, \partial P, T) \leqslant P_{\mathrm{top}}(0, \partial P, T) + \sup \phi \leqslant \lambda_1 + \cdots + \lambda_{d-1} + \sup \phi$$

and

$$P_{\mathrm{top}}(\phi, T) \geqslant \lambda_1 + \cdots + \lambda_d$$

with $\lambda_1 \geqslant \ldots \geqslant \lambda_d$ the logarithms of the moduli of the eigenvalues of $\vec{B}$. It follows that

$$P_{\mathrm{top}}(\phi, \partial P, T) \leqslant P_{\mathrm{top}}(\phi, T) - (\lambda_d - \sup \phi) < P_{\mathrm{top}}(\phi, T)$$

so that one can indeed apply Theorem 1.3 to get the finiteness.

The uniqueness follows from Proposition 1 of [4], according to which the lower-bound on the expansion of the statement of the Corollary implies strong topological mixing.

Theorem 1.3 generalizes a result of [6] which required the existence of a conformal measure with full support — which had been proved only under additional assumptions (especially covering, a strong form of mixing) by [8]). Indeed, it is easy to build examples where the equilibrium measure is supported by a Cantor set and this shows that [6] cannot apply, since in this case the support would contain a non-empty open set (more precisely, the support would be equal to an open set modulo a set with zero conformal measure).

The core of the proof of this statement is the isomorphism theorem, Theorem 3.2, identifying T and the Markov shift defined by its so-called connected Markov extension w.r.t. all measures with metric pressure sufficiently close to $P_{\mathrm{top}}(\phi, T)$.

2. Equilibrium measures on the Markov shift

This section contains the proof of theorems 1.1 and 1.2. We use the cylinder notation $[a_0, \ldots, a_{n-1}] := \{x \in \Sigma : x_i \in [a_i]\}$ and set $\alpha := \{[a] : a \in S\}$.

Lemma 2.1. *Suppose (Σ, σ) is topologically transitive and $\phi : \Sigma \to \mathbb{R}$ is such that $\sup_{n \geqslant 1} var_{n+1}(\phi_n) < \infty$. Let ν be a conservative σ-finite measure which is finite and positive on some partition set. If $L_\phi^* \nu = \lambda \nu$ for some $\lambda > 0$, then ν is ergodic.*

Proof: We indicate the proof, which is well-known (see [1], chapter 4). If ν is such a measure, then its transfer operator is $\lambda^{-1} L_\phi$ and it is easy to see that if $C := \exp \sup_{n \geqslant 1} var_{n+1}(\phi_n)$, then for every $[\underline{a}] = [a_0, \ldots, a_{n-1}] \in \alpha_0^{n-1}, [\underline{b}] \in \alpha_0^{m-1}$ with $[\underline{b}] \subseteq [a_{n-1}]$ and $x \in [\underline{a}]$,

$$\frac{1}{C} \lambda^{-(n-1)} e^{\phi_{n-1}(x)} \nu[\underline{b}] \leqslant \nu\big([\underline{a}] \cap \sigma^{-(n-1)}[\underline{b}]\big) \leqslant C \lambda^{-(n-1)} e^{\phi_{n-1}(x)} \nu[\underline{b}].$$

Summing over $[\underline{b}] \in \alpha_0^{m-1} \cap [a_{n-1}]$ gives

$$\frac{1}{C} \lambda^{-(n-1)} e^{\phi_{n-1}(x)} \nu[a_{n-1}] \leqslant \nu[\underline{a}] \leqslant C \lambda^{-(n-1)} e^{\phi_{n-1}(x)} \nu[a_{n-1}].$$

This shows (by transitivity) that ν is finite positive on all cylinders (although it's total mass may be infinite). Another corollary is that

$$\nu\big([\underline{a}] \cap \sigma^{-(n-1)}[\underline{b}]\big) \geqslant \frac{1}{C^2} \nu[\underline{a}] \nu[\underline{b}] / \nu[a_{n-1}].$$

A monotone class argument shows that for every Borel set E,

$$\nu\big([\underline{a}] \cap \sigma^{-(n-1)} E\big) \geqslant \frac{1}{C^2} \nu[\underline{a}] \nu(E \cap [a_{n-1}]) / \nu[a_{n-1}].$$

In particular, if E is an invariant set of positive measure, then

$$\mathbb{E}_\nu(1_E|\alpha_0^{n-1})(x) = \frac{\nu([x_0,\ldots,x_{n-1}]\cap T^{-(n-1)}E)}{\nu[x_0,\ldots,x_{n-1}]} \geqslant \frac{1}{C^2}\frac{\nu(E\cap[x_{n-1}])}{\nu[x_{n-1}]}.$$

By conservativity, the limit inferior of the right hand side as $n\to\infty$ is positive almost everywhere. This implies by the Martingale Convergence Theorem that $1_E>0$ a.e., whence $E=\Sigma$ modulo ν. ∎

Our next lemma says that it is possible to assume without loss of generality that (Σ,σ) is topologically mixing.

LEMMA 2.2. *If theorems* 1.1 *and* 1.2 *are true for all topologically mixing shifts, they are true for all topologically transitive shifts.*

Proof: We use the well-known spectral decomposition (see e.g. [13]): If Σ is topologically transitive, then there exist pairwise disjoint closed sets $\Sigma_0,\ldots,\Sigma_{p-1}$ such that $\Sigma=\bigcup_{i=0}^{p-1}\Sigma_i$, $\sigma(\Sigma_i)=\Sigma_{(i+1)\bmod p}$, and such that (Σ_i,σ^p) is topologically mixing. Moreover, each Σ_i is the union of partition sets.

Recoding (Σ_i,σ^p) by the partition into cylinders of length p, we may view it as a topologically mixing topological Markov shift.

In order to prove theorem 1.1 for σ, it is enough to prove that there exists a unique ergodic invariant $\mu\in\mathcal{P}_\sigma(\Sigma)$ with $h_\mu(\sigma)+\int\phi d\mu = P_G(\phi)$, because if there were more than one invariant equilibrium measure, there would have been more than one ergodic equilibrium measure. Suppose, then, that μ is an ergodic equilibrium measure.

Note that $\mu(\Sigma_i)=\frac{1}{p}>0$ for every i, because $\Sigma=\bigcup_{k=0}^{p-1}\sigma^{-k}(\Sigma_i)$. Let μ_i be the measure $\mu_i(E):=\mu_i(E\cap\Sigma_i)/\mu(\Sigma_i)$. Since $\mu=\frac{1}{p}(\mu_0+\ldots+\mu_{p-1})$,

$$\begin{aligned}\frac{1}{p}\sum_{i=0}^{p-1}P_{\mu_i}(\phi_p,\sigma^p) &= P_\mu(\phi_p,\sigma^p) = p\cdot P_\mu(\phi,\sigma)\equiv p\cdot P_G(\phi,\sigma)\\ &= P_G(\phi_p,\sigma^p)\geqslant \max_i P_G\big(\phi_p|_{\Sigma_i},\sigma^p|_{\Sigma_i}\big).\end{aligned}$$

This is possible only if for every i, μ_i is an equilibrium measures of ϕ_p with respect to $\sigma^p:\Sigma_i\to\Sigma_i$. This system is topologically mixing, so μ_i are uniquely determined. These determine μ. This proves the topological transitive version of theorem 1.1, given its topological mixing version.

Next, note that the topologically mixing version of theorem 1.2 implies that each μ_i above satisfies: $d\mu_i=h_i d\nu_i$ where h_i is a continuous function supported and positive on Σ_i with $L_\phi^p h_i=\lambda_i^p h_i$, $\lambda>0$, and ν_i is a Borel measure finite and positive on cylinders included in Σ_i with $(L_\phi^p)^*\nu_i=\lambda_i^p\nu_i$. Set

$$h:=\sum_{i=0}^{p-1}\lambda_0^{-i}L_\phi^i h_0,\quad \nu:=\sum_{i=0}^{p-1}\lambda_0^{-i}(L_\phi^i)^*\nu_0\quad\text{and}\quad dm:=\frac{hd\nu}{\int hd\nu}.$$

It is easy to check that $L_\phi h=\lambda_0 h$ and $L_\phi^*\nu=\lambda_0\nu$. It follows from this that m is invariant. It is an ergodic measure, since ν is ergodic by lemma 2.1. Our equilibrium measure μ is also ergodic, because almost all its ergodic components are equilibrium measures, and there is only one such measure. Therefore, μ and m are two ergodic invariant probability measures, and by construction $\mu|_{\Sigma_0}$ and $m|_{\Sigma_0}$ are proportional. Ergodicity now implies that they are equal. This shows that μ has the form $hd\nu$ where h and ν are eigenvectors of L_ϕ. ∎

For the remainder of the section, we only treat topologically mixing Markov shifts (the previous lemma says that's enough).

DEFINITION 1. *Let* (Σ, σ) *be a topologically mixing countable Markov shift. A function* $g : \Sigma \to \mathbb{R}$ *is called a* sub g-function *if* g *is strictly positive,* $P_G(\log g) = 0$ *and* $\forall x \in \Sigma$, $\sum_{\sigma y = x} g(y) \leqslant 1$.

We recall the following known results:

LEMMA 2.3. *If* (Σ, σ) *is topologically mixing, then every* $\phi : \Sigma \to \mathbb{R}$ *such that* $P_G(\phi) < \infty$ *and* $\sum_{n \geqslant 2} var_n(\phi) < \infty$ *is of the form* $\log g + \varphi - \varphi \circ \sigma + P_G(\phi)$ *where* g *is a sub* g*-function and* φ *is continuous with* $var_n(\varphi) \leqslant \sum_{k \geqslant n+1} var_k(\phi)$.

Proof: Lemma 1 in [23]. We remark that this proof uses the generalized Ruelle's Perron-Frobenius theorem of [22, 23]. This is the only place we use this theorem and it would be a significant improvement to have a different argument which does not require it. ■

LEMMA 2.4. *Let* $p_i, x_i (i = 1, 2, 3, \ldots)$ *be real numbers such that* $p_i \geqslant 0$, $x_i > 0$ *and* $\sum p_i = 1$. *If* $\sum_{i=1}^{\infty} p_i \log x_i = \log\left(\sum_{i=1}^{\infty} p_i x_i\right)$ *, then all* x_i *with* $p_i \neq 0$ *are equal.*

Proof: Standard. ■

Proof of theorem 1.1: By lemma 2.2 it is enough to treat the topologically mixing case. Assume without loss of generality that $P_G(\phi) = 0$ (else pass to $\phi - P_G(\phi)$). By lemma 2.3 there exists a sub g-function and a continuous function $\varphi : \Sigma \to \mathbb{R}$ with $var_1(\varphi) < \infty$ such that

$$\phi = \log g + \varphi - \varphi \circ \sigma. \tag{1}$$

Our strategy of proof is to show that every ergodic equilibrium measure μ satisfies

$$L_\phi^*(e^{-\varphi}\mu) = e^{-\varphi}\mu. \tag{2}$$

(Here and throughout an *equilibrium measure* means an invariant probability measure μ for which $h_\mu(\sigma) + \int \phi d\mu$ is well-defined and maximal.)

Once we prove this we can proceed as follows. Assume by way of contradiction that there is more than one invariant equilibrium measure. Every equilibrium measure is a barycenter of the collection of ergodic equilibrium measures, since almost every ergodic component of an equilibrium measure is itself an equilibrium measure. Therefore, if there is more than one equilibrium measure, there must be more than one ergodic equilibrium measure. Let μ_1 and μ_2 be two different ergodic equilibrium measures and set $\mu := \frac{1}{2}(\mu_1 + \mu_2)$. This is a non-ergodic measure which satisfies (2), in contradiction to lemma 2.1.[2] This contradiction proves the theorem.

Recall that an invariant probability measure μ is said to satisfy the *Rokhlin formula*, if

$$h_\mu(\sigma) = -\int \log \frac{d\mu}{d\mu \circ \sigma} d\mu$$

[2]The conditions of lemma 2.1 are satisfied: (1) $e^{-\varphi}\mu$ is conservative since it is equivalent to the invariant probability measure μ; (2) $e^{-\varphi}\mu$ is finite and positive on some partition set, since μ is a probability measure and $\mathrm{var}_1(\varphi) < \infty$.

where $\mu\circ\sigma$ is the σ-finite measure $(\mu\circ\sigma)(E)=\sum_{A\in\alpha}\mu\big(\sigma(E\cap A)\big)$. Ledrappier showed in [15] that[3] $\mathbb{E}_\mu(f|\alpha_1^\infty)(x)=\sum_{\sigma y=\sigma x}\frac{d\mu}{d\mu\circ\sigma}(y)f(y)$ and deduced that $I_\mu(\alpha|\alpha_1^\infty):=-\sum_{A\in\alpha}1_A\log\mathbb{E}_\mu(1_A|\alpha_1^\infty)=-\log\frac{d\mu}{d\mu\circ\sigma}$. It follows from this that $H(\alpha|\alpha_1^\infty)\equiv\int I_\mu(\alpha|\alpha_1^\infty)\,d\mu=-\int\log\frac{d\mu}{d\mu\circ\sigma}d\mu$. If $H_\mu(\alpha)$ is finite, then $H(\alpha|\alpha_1^\infty)=h_\mu(\sigma)$. Therefore, Rokhlin's formula holds in whenever $H_\mu(\alpha)<\infty$.

A key step in the proof of (2) is to show that every ergodic equilibrium measure μ satisfies the Rokhlin formula, the difficulty being that in general $H_\mu(\alpha)$ may be infinite so it's not obvious apriori that $H(\alpha|\alpha_1^\infty)=h_\mu(\sigma)$.

Let μ be an ergodic probability measure for which $h_\mu(\sigma)+\int\phi d\mu$ is well defined and equal to zero. Note that $\int\phi d\mu$ must then be finite.

Fix some $[a]\in\alpha$ with positive measure and consider the induced transformation on $[a]$ given by $\overline{\sigma}(x)=\sigma^{\tau(x)}(x)$ where $\tau(x):=1_{[a]}(x)\inf\{n\geqslant 1:\sigma^n(x)\in[a]\}$. The induced transformation preserves the measure $\overline{\mu}(E):=\mu(E\cap[a])/\mu[a]$ and admits the Markov partition

$$\beta:=\{[a,\xi_1,\ldots,\xi_{n-1},a]:n\geqslant 1,\xi_i\neq a\}\setminus\{\varnothing\}.$$

Define a Bernoulli measure $\overline{\mu}_B$ on $[a]$ by $\overline{\mu}_B\big(\bigcap_{i=0}^{n-1}\overline{\sigma}^{-i}B_i\big)=\prod_{i=0}^{n-1}\overline{\mu}(B_i)$ whenever $B_i\in\beta$, and let μ_B be the measure on Σ given by

$$\mu_B(E)=\mu[a]\int_{[a]}\sum_{i=0}^{\tau-1}1_E\circ\sigma^i d\overline{\mu}_B.$$

This is a probability measure, since $\mu_B(\Sigma)=\mu[a]\int\tau d\overline{\mu}_B=\mu[a]\int\tau d\overline{\mu}=1$ (by Kac formula). μ_B is σ-ergodic, since $\overline{\mu}_B$ is $\overline{\sigma}$-ergodic and $[a]$ is a sweep-out set for μ_B (i.e. $\mu_B(\bigcup_{i\geqslant 1}\overline{\sigma}^{-i}[a])=1$).

We claim that $\int\phi d\mu_B>-\infty$. Set $C:=\sup_{n\geqslant 1}var_{n+1}\phi_n\leqslant\sum_{k\geqslant 2}var_k\phi$, and define $\overline{\phi}:=\sum_{i=0}^{\tau-1}\phi\circ\sigma^i$. Note that every $B\in\beta$ is a cylinder of length $n(B)+1$ where $n(B)$ is the unique value of τ on B. Now, $\mathbb{E}_{\overline{\mu}}(\overline{\phi}|\beta)=\sum_{B\in\beta}1_B\cdot\frac{1}{\overline{\mu}(B)}\int_B\overline{\phi}d\overline{\mu}$, so by the previous remark, $\|\overline{\phi}-\mathbb{E}_{\overline{\mu}}(\overline{\phi}|\beta)\|_\infty\leqslant C$. Therefore

$$\begin{aligned}\int\phi d\mu_B &= \mu[a]\int_{[a]}\overline{\phi}d\overline{\mu}_B\geqslant\mu[a]\int_{[a]}\mathbb{E}_{\overline{\mu}}(\overline{\phi}|\beta)d\overline{\mu}_B-C\\ &= \mu[a]\int_{[a]}\overline{\phi}d\overline{\mu}-C=\int_\Sigma\phi d\mu-C>-\infty.\end{aligned}$$

Consequently, $h_{\mu_B}(\sigma)+\int\phi d\mu_B$ is well-defined although it may be equal to $+\infty$. Therefore, by the variational principle, $h_{\mu_B}(\sigma)\leqslant-\int\phi d\mu_B<\infty$. Since μ_B is ergodic, Abramov's formula applies and so $h_{\mu_B}(\sigma)=\frac{1}{\mu[a]}h_{\overline{\mu}_B}(\overline{\sigma})=\frac{1}{\mu[a]}H_{\overline{\mu}_B}(\beta)=\frac{1}{\mu[a]}H_{\overline{\mu}}(\beta)$. Therefore,

$$H_{\overline{\mu}}(\beta)<\infty.$$

[3]Ledrappier considered the case of a finite alphabet but this part of his arguments apply to countable alphabet without modifications.

This implies that $\overline{\mu}$ satisfies the Rokhlin formula. Since μ is ergodic, the Kac and Abramov formulas both apply and so

$$\begin{aligned}\frac{1}{\mu[a]}h_\mu(\sigma) &= h_{\overline{\mu}}(\overline{\sigma}) = -\int_{[a]}\log\frac{d\overline{\mu}}{d\overline{\mu}\circ\overline{\sigma}}d\overline{\mu} = -\frac{1}{\mu[a]}\int_{[a]}\log\frac{d\mu}{d\mu\circ\sigma^\tau}d\mu\\ &= -\frac{1}{\mu[a]}\int_{[a]}\sum_{i=0}^{\tau-1}\log\frac{d\mu}{d\mu\circ\sigma}\circ\sigma^i d\mu = -\frac{1}{\mu[a]}\int_\Sigma\log\frac{d\mu}{d\mu\circ\sigma}d\mu.\end{aligned}$$

Therefore, μ satisfies the Rokhlin formula as well. This shows that every ergodic equilibrium measure satisfies the Rokhlin formula.

Our next step is to prove that if μ is an ergodic equilibrium measure, then $\frac{d\mu}{d\mu\circ\sigma} = g \mod \mu\circ\sigma$ where g is as in (1). Our argument is a variation on an argument of Ledrappier [15] (see also [26]).[4]

We begin by showing that $\log g \in L^1(\mu)$ and that $\int(\varphi-\varphi\circ\sigma)d\mu = 0$. Since $L_{\log g}1 \leqslant 1$, $\log g \leqslant 0$ and so $\log g$ is one-sided integrable. We show that it is absolutely integrable. Set $g^{(n)} := \prod_{i=0}^{n-1} g\circ\sigma^i$, $\phi_n := \sum_{i=1}^{n-1}\phi\circ\sigma^i$. By recurrence, $\liminf_{n\to\infty}\frac{1}{n}(\varphi-\varphi\circ\sigma^n) = 0$ a.e., and so,

$$\begin{aligned}\int\log g d\mu &= \lim_{n\to\infty}\frac{1}{n}\int\log g^{(n)}d\mu \geqslant \int\liminf_{n\to\infty}\frac{1}{n}\left(\log g^{(n)}+\varphi-\varphi\circ\sigma^n\right)d\mu\\ &= \int\liminf_{n\to\infty}\frac{1}{n}\phi_n d\mu = \int\phi d\mu > -\infty.\end{aligned}$$

Therefore, $\log g \in L^1(\mu)$. It follows that $\varphi-\varphi\circ\sigma = \phi-\log g$ is absolutely integrable, because ϕ must be absolutely integrable ($\int\phi^+ < \infty$ because $\sup\phi < \infty$ and $\int\phi^- < \infty$, because otherwise μ will not be an equilibrium measure). By the ergodic theorem,

$$\int(\varphi-\varphi\circ\sigma)d\mu = \lim_{n\to\infty}\frac{1}{n}(\varphi-\varphi\circ\sigma^n) = \liminf_{n\to\infty}\frac{1}{n}(\varphi-\varphi\circ\sigma^n) = 0$$

so $\int(\varphi-\varphi\circ\sigma)d\mu = 0$. Consequently, $h_\mu(\sigma)+\int\log g d\mu = 0$.

Set $g_\mu := \frac{d\mu}{d\mu\circ\sigma}$. One checks that the transfer operator of μ is given by $\widehat{\sigma}_\mu f = \sum_{\sigma y=x} g_\mu(y)f(y)$. The invariance of μ implies that $\sum_{\sigma y=x} g_\mu(y) = \widehat{\sigma}_\mu 1 = 1$ μ-almost everywhere, so g_μ is a g-function. Since μ satisfies Rokhlin's formula,

$$\begin{aligned}0 &= h_\mu(\sigma)+\int\log g d\mu = \int\log\frac{g}{g_\mu}d\mu\\ &= \int\left(\sum_{\sigma y=x} g_\mu(y)\log\frac{g(y)}{g_\mu(y)}\right)d\mu(x) \leqslant \int\log\left(\sum_{\sigma y=x, g_\mu(y)>0} g(y)\right)d\mu(x) \leqslant 0.\end{aligned}$$

All inequalities must be equalities, so

$$\sum_{\sigma y=x} g_\mu(y)\log\frac{g(y)}{g_\mu(y)} = \log\left(\sum_{\sigma y=x} g_\mu(y)\cdot\frac{g(y)}{g_\mu(y)}\right) \text{ for } \mu-\text{almost all } x.$$

By the lemma 2.4, for μ-almost all x there exists $c(x)$ such that

$$y\in\sigma^{-1}\{x\}, g_\mu(y) > 0 \text{ implies } g(y) = c(x)g_\mu(y).$$

[4]The arguments in [15],[26] only show that for almost every x, $\frac{d\mu}{d\mu\circ\sigma}(x) = g(x)$, but actually the following stronger statement is needed: for almost every x, $\forall y\in\sigma^{-1}x$ $\left(\frac{d\mu}{d\mu\circ\sigma}(y) = g(y)\right)$.

Summing over $y \in \sigma^{-1}\{x\} \cap [g_\mu > 0]$ gives $c(x) = \sum_{\sigma y = x, g_\mu(y)>0} g(y) \leqslant 1$. The last derivation shows that $\int \log c(x) d\mu(x) = 0$ so $c(x) = 1$ μ-almost everywhere. For every x such that $c(x) = 1$ and $\sum_{\sigma y = x} g_\mu(y) = 1$, $\sigma^{-1}\{x\} \subseteq [g_\mu > 0]$, else $c(x) < \sum_{\sigma y = x} g(y) \leqslant 1$ (we've used the positivity of g). Therefore, for μ-almost every x

$$y \in \sigma^{-1}\{x\} \text{ implies } g(y) = g_\mu(y).$$

This means that $L_{\log g_\mu} = L_{\log g}$ whence $L^*_{\log g}\mu = L^*_{\log g_\mu}\mu = \mu$. A calculation shows that $L^*_\phi(e^{-\varphi}\mu) = e^{-\varphi}\mu$ so (2) holds. By the discussion at the beginning of the proof, this is enough to prove the theorem. ∎

Proof of theorem 1.2: Assume without loss of generality that $P_G(\phi) = 0$. By lemma 2.3, $\phi = \log g + \varphi - \varphi \circ \sigma$ with φ continuous and g a sub g-function. The proof of theorem 1.1 shows that if μ is an equilibrium measure, then (2) holds. Consequently, if $\nu := e^{-\varphi}\mu$ and $h := e^{\varphi}$, then

$$L^*_\phi \nu = \nu \text{ and } \int h d\nu = 1.$$

For every $g \in L^\infty$,

$$\nu(gL_\phi h) \quad = \quad \nu(g \circ \sigma \cdot h) = \mu(g \circ \sigma) = \mu(g) = \nu(gh)$$

so $L_\phi h = h$ ν-almost everywhere. The identity $L^*_\phi \nu = \nu$ can be used to show that ν is finite and positive on cylinders, and so since both h and $L_\phi h$ are continuous, $L_\phi h = h$ everywhere. ∎

3. Equilibrium measures for piecewise expanding maps

We prove Theorem 1.3. Let (X, T, P) be a piecewise expanding map together with a piecewise Hölder-continuous potential $\phi : X \to \mathbb{R}$. We continue using the cylinder notation $[A_0, \dots, A_n] := \bigcap_{i=0}^n T^{-i}A_i$.

Fix some T-invariant probability measure m on X. We prove that $P_m(\phi, T) \leqslant P_{\text{top}}(\phi, T)$. We prove this under the extra assumption that m is ergodic (else use the ergodic decomposition).

The proof is based on a reduction to the symbolic dynamics of T, which we proceed to describe. Let $dom(T^n) \subseteq X$ denote the domain of definition of T^n. The *symbolic dynamics* of (X, T, P) is the left-shift σ on:

$$\Sigma(T) = Clos\big(\{A = (A_0, A_1, \dots) \in P^{\mathbb{N}\cup\{0\}} : \exists x \in X \text{ such that} \\ \forall n \geq 0 \; x \in dom(T^n) \text{ and } T^n x \in A_n\}\big)$$

where $Clos(\cdot)$ denotes the closure in the compact space $P^{\mathbb{N}\cup\{0\}}$ (it is endowed with the product topology of the discrete topologies on P). Define $\pi : \Sigma(T) \to X$ by $\{\pi(A)\} = \bigcap_{n\geqslant 0} \overline{[A_0 \dots A_n]}$. As T is piecewise expanding, π is well-defined. Moreover, if $\Delta := \pi^{-1}(\partial P)$, then

$$\pi : \Sigma(T) \setminus \bigcup_{k\geq 0} \sigma^{-k}\Delta \to X \setminus \bigcup_{k \geq 0} T^{-k}\partial P$$

is bi-measurable, injective, surjective and $\pi \circ \sigma = T \circ \pi$. Next, define $\Phi : \Sigma(T) \to \mathbb{R}$ by $\Phi(A) = \lim_{n\to\infty} \inf \phi([A_0, \dots, A_n])$ and note that

$$\Phi(A) = \phi(\pi(A)) \text{ whenever } \pi(A) \notin \partial P.$$

It is easy to check that Φ is Hölder-continuous, resp. continuous, if ϕ is piecewise Hölder-continuous, resp. piecewise uniformly continuous, (the distance on $\Sigma(T)$ is, as usual: $d(A,B) = 2^{-n}$ where n is the smallest integer such that $A_n \neq B_n$ or $n = \infty$).

Set $S := \bigcup_{k\geq 0} T^{-k}\partial P$. If $m(S) = 0$ then $\pi : (\Sigma(T), \mathcal{B}(\Sigma(T)), m\circ\pi, \sigma) \to (X, \mathcal{B}(X), m, T)$ is a measure-theoretic isomorphism, so $P_m(\phi, T) = P_{m\circ\pi^{-1}}(\Phi, \sigma)$. Also, $P_{m\circ\pi^{-1}}(\Phi,\sigma) \leqslant P_{\text{top}}(\Phi,\sigma)$, because of the variational principle for the topological pressure of a continuous function with respect to a homeomorphism of a compact metric space (see [25]). Therefore, $P_m(\phi,T) \leqslant P_{\text{top}}(\Phi,\sigma)$. It is routine to check that $P_{\text{top}}(\Phi,\sigma) = P_{\text{top}}(\phi,T)$, and it follows that $P_m(\phi,T) \leqslant P_{\text{top}}(\phi,T)$ whenever $m(S) = 0$.

Suppose now that $m(S) > 0$. We claim that $P_m(\phi,T) < P_{\text{top}}(\phi,T)$. This follows from our assumption that $P_{\text{top}}(\phi,\partial P,T) < P_{\text{top}}(\phi,T)$, and from the following observation (the proof of which is given in section 4):

Proposition 3.1. *Consider a piecewise expanding map τ together with a piecewise uniformly continuous potential f. Let m be an ergodic invariant probability measure. If $m(S) > 0$, then $P_m(f,\tau) \leqslant P_{\text{top}}(f,S,\tau)$.*

This proves that $\sup_{m\in\mathcal{P}_T(X)} P_m(\phi,T) \leqslant P_{\text{top}}(\phi,T)$.

We now show that $\exists m$ such that $P_m(\phi,T) = P_{\text{top}}(\Phi,\sigma)$. Note that $\sigma : \Sigma(T) \to \Sigma(T)$ is expansive, and that Φ is continuous (in fact it is Hölder-continuous). It follows that there is a σ-invariant probability measure μ on $\Sigma(T)$ for which $P_\mu(\Phi,\sigma) = P_{\text{top}}(\Phi,T)$ ([13], theorem 20.2.10). Set $m := \mu\circ\pi^{-1}$. If $m(S) > 0$ then $\mu(\pi^{-1}S) > 0$ and so

$$\begin{aligned} P_{\text{top}}(\Phi,\sigma) &\equiv P_\mu(\Phi,\sigma) \leqslant P_{\text{top}}(\Phi,\pi^{-1}\partial P,\sigma) \quad \text{(proposition 3.1)} \\ &\leqslant P_{\text{top}}(\phi,\partial P,T) \\ &< P_{\text{top}}(\phi,T) \quad \text{(by assumption)} \\ &= P_{\text{top}}(\Phi,\sigma) \end{aligned}$$

which is a contradiction. Therefore, $m(S) = 0$. It follows that π is an isomorphism between $(\Sigma(T), \mathcal{B}(\Sigma(T)), \mu, \sigma)$ and $(X, \mathcal{B}(X), m, T)$, and so $P_m(\phi,T) = P_\mu(\Phi,\sigma) = P_{\text{top}}(\Phi,s)$, and we already remarked that $P_{\text{top}}(\Phi,s) = P_{\text{top}}(\phi,T)$. This proves the first part (i) of theorem 1.3.

We now turn to the finite multiplicity of the equilibrium measure, point (ii) of the Theorem. We shall prove it by reduction to a Markov shift using the connected Markov diagram introduced in [5]. Let's recall its definition.

Definition 2. *The (connected)* Markov diagram *is the directed graph $\mathcal{D} = (V,E)$ with vertices $V = \{T^nC : n \geqslant 0, C$ is a connected component of $Z \in P_0^n\}$ and edges $E = \{A \to B : \exists Z \in P$ s.t. B is a connected component of $T(A)\cap Z\}$. This graph defines a one-sided Markov shift (Σ,σ).*

There is an natural projection onto the symbolic dynamics:

$$q : \Sigma \to \Sigma(T) \text{ defined by } q(\alpha) = A \text{ s.t. } A_n \in P \text{ contains } \alpha_n \text{ for all } n \geq 0.$$

Obviously $q\circ\sigma = \sigma\circ q$ and q is countable-to-one. We extend Φ to Σ by setting $\Phi := \Phi\circ q$ on Σ.

The proof of the second part of the theorem is based on constructing a certain type of isomorphism between the natural extensions of Σ and X. We begin with some generalities. A *weighted measurable dynamical system* (X,T,ϕ) is a measurable map $T:X\to X$ together with a measurable function $\phi:X\to\mathbb{R}$ called the *potential*. A set $S\subseteq X$ is *P-negligible* if there exists $p<\sup_\mu P_\mu(\phi,T)$ such that for every ergodic invariant probability measure μ, if $P_\mu(\phi,T)>p$ then $\mu(S)=0$.

DEFINITION 3. *Two weighted measurable dynamical systems (X,T,ϕ) and (Y,S,ψ) are said to be P-*isomorphic *if there exist invariant subsets $X'\subseteq X$, $Y'\subseteq Y$ and a bimeasurable bijection $h:X'\to Y'$, such that $X\setminus X'$, $Y\setminus Y'$ are P-negligible, $h\circ T=S\circ h$, and $\phi=\psi\circ h$.*

We will deduce part (ii) of the theorem from

THEOREM 3.2. *Let (X,P,T) be a piecewise invertible map with a piecewise uniformly continuous potential $\phi:X\to\mathbb{R}$. If $P_{\text{top}}(\phi,\partial P,T)<P_{\text{top}}(\phi,T)$, then*

(1) *the natural extensions*[5] *of (X,T) and of (Σ,σ) are P-isomorphic;*
(2) *(Σ,σ) contains only finitely many maximal irreducible subchains (defined just after this) with pressure close to $P_{\text{top}}(\phi,T)$.*

Recall that an *irreducible part* of an oriented graph is a subgraph with the property that any two vertices can be joined in both directions. A graph splits into its maximal irreducible parts (and a remaining part made of those vertices to which no path returns but this part plays no rôle dynamically speaking). Recall also that the support of any ergodic invariant measure is contained in exactly one *maximal irreducible subchain*, i.e., the subchain defined by one of these maximal irreducible parts of the graph.

Before proving this Theorem, let us see how it can be used to deduce part (ii) of theorem 1.3. Notice that $\Phi|_\Sigma$ is bounded from above and is Hölder continuous, and therefore has finite Gurevich pressure since

$$P_G(\Phi|_\Sigma)=\sup_{\mu\in\mathcal{P}_\sigma(\Sigma)}P_\mu(\Phi,\sigma)=\sup_{\mu\in\mathcal{P}_T(X)}P_\mu(\phi,T)=P_{\text{top}}(\phi,T)<\infty.$$

the second equality following from Theorem 3.2. Under these conditions, Theorem 1.1 implies that each irreducible sub-shift has at most one equilibrium measure. Point (2) of Theorem 3.2 says that Σ contains only finitely many maximal irreducible subchains with maximum pressure. Hence Σ has only finitely many ergodic equilibrium measures. By point (1) of Theorem 3.2 this must then also be true of T, concluding the proof of the second part (ii) of this theorem.

Proof of Theorem 3.2*:* The strategy of the proof is the same as in [5] where the same statements where proved in the case $\Phi\equiv 0$ (entropy replacing pressure).

Let $p_-:(X_-,T_-)\to(X,T)$ be the *natural extension*:

$$\begin{aligned}X_-&=\{x\in X^{\mathbb{Z}}:T(x_n)=x_{n+1}\ \forall n\in\mathbb{Z}\}\\T_-\big((x_n)_{n\in\mathbb{Z}}\big)&=\big(x_{n+1}\big)_{n\in\mathbb{Z}}\\p_-\big((x_n)_{n\in\mathbb{Z}}\big)&=x_0.\end{aligned}$$

We also extend ϕ to X_- by setting $\phi:=\phi\circ p_-$. Define $\tilde p_-:(\Sigma_-,\sigma_-)\to(\Sigma,\sigma)$ and Φ on Σ_- similarly. Let $\Pi_-:\Sigma_-\to X_-$ be the natural extension of $\Pi:=\pi\circ q:\Sigma\to X$ to a map from Σ_- to X_-.

It is well-known that the projection $m\mapsto m\circ p_-$ is a bijection between $\mathcal{P}_{T_-}(X_-)$ and $\mathcal{P}_T(X)$ which preserves ergodicity, entropy and metric pressure.

[5] see the definition at the beginning of the proof.

The following notion plays a key rôle in the construction.

Definition 4. *An invariant probability measure μ of T_- is said to be* shadowed by the boundary *if, for some integer N and for μ-a.e. $x \in X_-$, there exist infinitely many positive integers n such that $P_0^{n-1}(x_{-n}) \cap \Delta P \neq \varnothing$ where $\Delta P := \bigcup_{k=0}^{N} T^k \left(\bigcup_{A \in P} \partial TA\right)$ and $P_0^{n-1}(x)$ is the element of P_0^{n-1} containing x if it exists or $\varnothing$.*

Construct, as in [5] two invariant sets $X'_- \subseteq X_-$, $\Sigma'_- \subseteq \Sigma_-$ with the following properties:

(1) the restriction of the above Π_-, $\Pi_- : \Sigma'_- \to X'_-$ is a bi-measurable bijection and $\Pi_- \circ \sigma_- = T_- \circ \Pi_-$.
(2) Let μ be an ergodic T_--invariant probability on X'_-. If $\mu(X_- \setminus X'_-) = 1$, then μ is shadowed by the boundary.
(3) Let μ be an ergodic σ_--invariant measure on Σ_-. If $\mu(\Sigma_- \setminus \Sigma'_-) = 1$, then either $\mu \circ \Pi_-^{-1}$ is shadowed by the boundary, or $\mu(\tilde{p}_-^{-1}\Pi^{-1}\partial P) > 0$.

For the construction of X'_- and Σ'_- and (1) see [5], proposition 2.2. (2) and (3) are propositions 2.6 and 2.7 there (Observe that in contrast to Proposition 2.7 we work with the Markov shift defined by the Markov diagram and not the geometric Markov extension of [5]. To relate the two, we have to code, hence a possible problem if $\mu(\tilde{p}_-^{-1}\Pi^{-1}\partial P) > 0$).

To prove that the above restriction of Π_- is a P-isomorphism, we need to show that $X_- \setminus X'_-$ and $\Sigma_- \setminus \Sigma'_-$ are P-negligible.

We begin with the P-negligibility of $X_- \setminus X'_-$. Property (1) says that every ergodic invariant probability measure μ carried by $X_- \setminus X'$ must be shadowed by the boundary. The P-negligibility of $X_- \setminus X'_-$ follows from the following proposition, and our assumption that $P_{\mathrm{top}}(\phi, \partial P, T) < P_{\mathrm{top}}(\phi, T)$:

Proposition 3.3. *Let μ be an invariant measure of (X_-, T_-). If μ is shadowed by the boundary then $P_\mu(\Phi, T_-) \leqslant P_{\mathrm{top}}(\phi, \partial P, T)$.*

The proof of proposition 3.3 is given in section 4.

Next, we prove the P-negligibility of $\Sigma_- \setminus \Sigma'_-$. Consider some ergodic invariant probability μ measure carried by $\Sigma_- \setminus \Sigma'$. It is enough to prove that $P_\mu(\Phi, \sigma_-) \leqslant P_{\mathrm{top}}(\phi, \partial P, T)$, because $P_{\mathrm{top}}(\phi, \partial P, T) < P_{\mathrm{top}}(\phi, T) = \sup_{m \in \mathcal{P}_T(X)} P_m(\phi, T) \leqslant \sup_{m \in \mathcal{P}_{\sigma_-}(\Sigma_-)} P_m(\Phi, \sigma_-)$.

Property (3) above says that either $\mu(\tilde{p}_-^{-1}\Pi^{-1}\partial P) > 0$ or $\mu(\tilde{p}_-^{-1}\Pi^{-1}\partial P) = 0$ and $\mu \circ \Pi_-^{-1}$ is shadowed by the boundary. In the first case, $\nu := \mu \circ \tilde{p}_-^{-1} \circ q^{-1}$ is an ergodic σ invariant measure on $\Sigma(T)$ which satisfies $\nu(\pi^{-1}\partial P) > 0$. By proposition 3.1 $P_\nu(\Phi, \sigma|_{\Sigma(T)}) \leqslant P_{\mathrm{top}}(\Phi, \pi^{-1}\partial P, \sigma|_{\Sigma(T)}) \leqslant P_{\mathrm{top}}(\phi, \partial P, T)$. Note that $m \mapsto m \circ (\tilde{p}_-)^{-1}$ and $m \mapsto m \circ q^{-1}$ both preserve entropy, because the first is a natural extension, and the second is countable-to-one (see, e.g., proposition 2.8 in [5]). Therefore, $P_\mu(\Phi, \sigma_-) = P_\nu(\Phi, \sigma|_{\Sigma(T)}) \leqslant P_{\mathrm{top}}(\phi, \partial P, T)$.

In the second case, $\mu \circ \Pi_-^{-1}$ is shadowed by the boundary, and Proposition 3.3 implies that $P_{\mu \circ \Pi_-^{-1}}(\Phi, T_-) \leqslant P_{\mathrm{top}}(\Phi, \partial P, T)$. Since Π_- preserves entropy, being a natural extension, $P_\mu(\Phi, \sigma_-) = P_{\mu \circ \Pi_-^{-1}}(\Phi, T_-)$ and so again $P_\mu(\Phi, \sigma_-) \leqslant P_{\mathrm{top}}(\phi, \partial P, T)$.

This proves that $\Sigma_- \setminus \Sigma$ is P-negligible and concludes the proof of part (1) of Theorem 3.2.

We turn to part (2), which says that there are only finitely many subchains with large pressure. This is a strengthening of [5, Theorem C] even in the case of the zero potential, but under the additional assumption that T is expanding. We shall obtain it by adapting Proposition 1.1 of [9].

Let $B(x,r) \subseteq X$ denote the ball of radius r and center x, and define for $\alpha \in \Sigma$,

$$\epsilon(\alpha) := \sup\{r \geq 0 : \exists n \geqslant 0 \text{ s.t. } B(\Pi(\sigma^n\alpha), r) \subseteq \alpha_n\}.$$

LEMMA 3.4. *Let $\mathcal{V}$ be the set of vertices of the Markov diagram $\mathcal{D}$. For any $\delta_0 > 0$, there exists a finite subset $\mathcal{V}_0 \subseteq \mathcal{V}$ such that for all $\alpha \in \Sigma \setminus \bigcup_{n\geq 0} \sigma^{-n}\Pi^{-1}\partial P$:*

$$\text{if } \epsilon(\alpha) > \delta_0, \text{ then } \exists n \geq 0 \text{ s.t. } \alpha_n \in \mathcal{V}_0.$$

Proof: Let k_0 be a integer sufficiently large so that $\operatorname{diam} P_0^{k_0} < \frac{1}{2}\delta_0$. Define $\mathcal{V}_0$ to be the collection of the connected components C of the subsets of the form:

$$T^{k_0}A \text{ for some } A \in P_0^{k_0}$$

which satisfy: C contains a ball with radius $\delta_0/2$. The finiteness of $\mathcal{V}_0$ follows from the fact that T^{k_0} is piecewise uniformly continuous so that there is only a finite number of disjoint subsets $S \subseteq X$ such that $T^{k_0}S$ contains a ball with radius δ_0.

Indeed, set $\mathcal{V}_0(A) := \mathcal{V}_0 \cap \{C : C \text{ is a c.c. of } T^kA\}$. By definition

$$\mathcal{V}_0 = \bigcup_{\substack{A\in P_0^k \\ 0\leqslant k\leqslant k_0}} \mathcal{V}_0(A).$$

This is a finite union, because P is finite. Therefore, it is enough to prove that $\mathcal{V}_0(A)$ is finite for every A. If this is not true, there are distinct $C_1, C_2, C_3, \ldots \in \mathcal{V}_0(A)$ with $C_i \supseteq B_{\frac{\delta_0}{2}}(x_i)$. These balls are pairwise disjoint, because C_i are pairwise disjoint (being connected components of the same set). Thus $\{x_i\}$ is a $\frac{\delta_0}{2}$-separated sequence, which contradicts the compactness of X. This proves that $\mathcal{V}_0$ is finite.

Recall formula (2.1) of [5]: For $\alpha = (\alpha_0, \alpha_1, \ldots) \in \Sigma \setminus \bigcup_{n\geq 0} \sigma^{-n}\Pi^{-1}\partial P$

(3) $\quad \alpha_{n+k}$ is the connected component of

$$T^k(\alpha_n \cap [A_n \ldots A_{n+k}]) \text{ which contains } \Pi(\sigma^{n+k}\alpha).$$

Let $\alpha \in \Sigma$ be as in the statement of the Lemma with $\epsilon(\alpha) > \delta_0$. By definition, there is some integer $n \geqslant 0$ such that $\alpha_n \supseteq B(\Pi(\sigma^n\alpha), \frac{\delta_0}{2})$, so α_n contains a ball of radius $\frac{\delta_0}{2}$. By the choice of δ_0, $B(\Pi(\sigma^n\alpha), \frac{\delta_0}{2}) \supseteq [A_n \ldots A_{n+k_0}]$ and so $\alpha_n \supseteq [A_n \ldots A_{n+k_0}]$. Therefore, by (3), α_{n+k_0} is also a connected component of $T^{k_0}[A_n \ldots A_{n+k_0}]$. This shows that $\alpha_{n+k_0} \in \mathcal{V}_0$. ■

LEMMA 3.5. *For all $\epsilon > 0$, $\exists \delta > 0$ such that if μ is an invariant probability measure on (Σ, σ) and $\epsilon(\alpha) \leqslant \delta$ for μ-a.e. $\alpha \in \Sigma$, then $P_\mu(\Phi, \sigma) \leqslant P_{\text{top}}(\phi, \partial P, T)$.*

We defer the proof of this Lemma to the section on pressure estimates.

Proof of point (ii) of Theorem 1.3*:* Fix $\epsilon \in (0, P_{\text{top}}(\phi, T) - P_{\text{top}}(\phi, \partial P, T))$. Let $\delta > 0$ be given by Lemma 3.5. Let $\mathcal{V}_0$ be the finite part defined by Lemma 3.4.

Claim. Any invariant probability measure μ of Σ such that $\mu(\{\alpha \in \Sigma : \alpha_0 \in \mathcal{V}_0\}) = 0$ satisfies: $P_\mu(\Phi, \sigma) \leqslant P_{\text{top}}(\phi, \partial P, T) + \epsilon$.

To prove this claim observe that by Lemma 3.4 μ must satisfy: $\epsilon(\alpha) \leqslant \delta_0$ for μ-a.e. α. But then we can apply Lemma 3.5 and get the claim.

It follows from the claim that any ergodic measure with pressure closer to $P_{\rm top}(\phi,T)$ than this right hand side must live on an irreducible subchain meeting $\mathcal{V}_0$, hence it must live on a finite number of maximal irreducible subchains as $\mathcal{V}_0$ is finite.

∎

4. Proof of the Pressure estimates

Proof of Proposition 3.1*:* μ is an invariant and ergodic probability measure of τ with $\mu(S)>0$. Fix $\epsilon>0$ arbitrarily small.

By the ergodic theorem, there exist $n_1<\infty$ and a subset X_1 of measure $>1-\frac{1}{2}\mu(S)$ such that for all $x\in X_1$, all $n\geqslant n_1$,

$$\phi_n(x) := \frac{1}{n}\sum_{k=0}^{n-1}\phi(\tau^k x) \geqslant \int \phi\, d\mu - \epsilon. \tag{4}$$

Remark that $\mu(S\cap X_1)>\mu(S)/2>0$.

Observe that

$$|\phi_n(x)-\phi_n(y)|<\epsilon n \tag{5}$$

for all x,y in the same n-cylinder, for n large enough. This is because of the piecewise uniform continuity of ϕ and because of $\lim_{n\to\infty}\operatorname{diam}P^n=0$.

Consider C_n the collection of n-cylinders meeting S. By definition,

$$\sum_{A\in C_n}\sup_{x\in A} e^{\phi_n(x)} \leqslant C\cdot e^{n(P_{\rm top}(\phi,S,T)+\epsilon)}$$

for some constant $C<\infty$ and all n. Remove from C_n any cylinder that does not meet X_1. The resulting C_n' is a cover of $X_1\cap S$. By eq. (5) and then eq. (4),

$$\sum_{A\in C_n}\sup_{x\in A} e^{\phi_n(x)} \geqslant \sum_{A\in C_n'}\sup_{x\in A\cap X_1} e^{\phi_n(x)-\epsilon n} \geqslant \#C_n'\cdot e^{n(\int\phi\,d\mu-3\epsilon)}$$

We thus obtain:

$$\#C_n' \leqslant Ce^{(P_{\rm top}(\phi,S,\tau)-\int\phi\,d\mu+4\epsilon)n}$$

Rudolph's formula for the entropy [19] states that μ being an ergodic invariant probability measure and P being a finite generator

$$h_\mu(\tau)=\liminf_{n\to\infty}\frac{1}{n}\log R_n$$

if R_n the minimum cardinality of a collection of P-cylinders with union of measure at least some fixed constant $0<\lambda<1$. Hence,

$$h_\mu(\tau)\leqslant\liminf_{n\to\infty}\frac{1}{n}\log\#C_n'.$$

Hence, we get, after taking the limits $n\to\infty$ and $\epsilon\to 0$:

$$h_\mu(\tau)\leqslant P_{\rm top}(\phi,S,\tau)-\int\phi\,d\mu$$

from which the proposition follows immediately. ∎

Proof of proposition 3.3*:* Let μ_- be an invariant and ergodic probability measure of (X_-,T_-). Let μ be the corresponding measure on (X,T). As is well-known, $P_{\mu_-}(\Phi,T_-)=P_\mu(\Phi,T)$ so that it is enough to bound the latter quantity.

Pick $\epsilon>0$. Fix n_0 so large that for all $n\geqslant n_0$, the collection S_n of n-cylinders the closure of which meets ∂P satisfies:

$$\sum_{A\in S_n}\sup_{x\in A}e^{\phi_n(x)}\leqslant e^{(P_{\mathrm{top}}(\phi,\partial P,T)+\epsilon)n}.$$

Fix n_1 so large that for all $n\geqslant n_1$, $|\phi_n(x)-\phi_n(y)|<\epsilon n$ for all x,y in the same n-cylinder.

Let $n_*=\max(n_0,\epsilon^{-1}\log(\#P\cdot\|\phi\|_\infty)\cdot n_1)$. Increase n_* if necessary so that the binomial coefficient $C_n^{[2n/n_*]+1}$ is at most $e^{\epsilon n}/(n/n_*+1)$.

Indeed, by Sterling's formula $n!=C^{\pm1}(n/e)^n n^{1/2}$ where $C^{\pm1}$ represents a function of n with value in (C^{-1},C) where C is independent of n. Hence, writing $\alpha:=\frac{2}{n_*}$,

$$(n/n_*+1)C_n^{[2n/n_*]+1}\leqslant C^3\alpha n^{-1/2}\exp\left[(-\alpha\log\alpha-(1-\alpha)\log(1-\alpha))n\right]\leqslant e^{\epsilon n}$$

for n large enough if n_* has been chosen large enough.

Define the measurable integer-valued function n on X_- by:

$$n(x)=\min\{n\geqslant n_*:P^n(x_{-n})\cap\Delta P\neq\varnothing\}.$$

As μ_- is shadowed by the boundary, this is well-defined a.e.

Let N_- be so large that $\mu_-(\{x\in X_-:n(x)\geqslant N_-\})<\epsilon/(\#P\cdot\|\phi\|_\infty)$.

By the ergodic theorem, there exists $n_3<\infty$ and a measurable subset $X_1\subseteq X_-$ such that for all $x\in X_1$ and $n\geqslant n_3$,

$$\frac{1}{n}\#\{0\leqslant k<n:n(T_-^kx)\geqslant N_-\}<\epsilon/(\#P\cdot\|\phi\|_\infty). \tag{6}$$

By the same reasoning as in the proof of Proposition 3.1, it is enough to prove that for all large n the collection C_n of n-cylinders that meet X_1 satisfies:

$$\sum_{A\in C_n}\sup_{x\in A}e^{\phi_n(x)}\leqslant Ce^{n(P_{\mathrm{top}}(\phi,\partial P,T)+\epsilon)}$$

Let $x\in X_1$ and $n\geqslant n_*$. We define intervals $[a_i,b_i)$ by induction. We set $a_0=n$. Provided a_i is defined, we let $b_{i+1}=\max\{b\leqslant a_i:n(T_-^bx)\leqslant N_-\}$ and $a_{i+1}=b_{i+1}-n(T_-^{b_{i+1}}x)$.

Let i_0 be the largest i with $a_i\geqslant 0$. We claim:

(1) $\Delta P\cap P_0^{b_i-a_i}(p_-T_-^{a_i}x)\neq\varnothing$;
(2) $b_i-a_i\geqslant n_*$;
(3) $\#\left([0,n)\setminus\bigcup_{i=1}^{i_0}[a_i,b_i)\right)\leqslant(\epsilon/\log(\#P\cdot\|\phi\|_\infty))n$;

Therefore the itinerary $A_0A_1\dots A_{n-1}\in P^n$ (or more precisely in $(p_-^{-1}P)^n$) determining the n-cylinder containing x can be described completely by specifying:

(1) an integer $0\leqslant r\leqslant n/n_*$ giving the number of intervals $[a_i,b_i)$;
(2) the starting and ending positions of each one of these intervals;
(3) the symbols $A_{a_i}A_{a_i+1}\dots A_{b_i-1}$ picked among the itineraries of the same length of points in ΔP;
(4) the symbols at the remaining places

Hence, writing $n' = \sum_{i=1}^{i_0}(b_i - a_i)$,

$$\sum_{A\in C_n} \sup_{x\in A} e^{\phi_n(x)} \leqslant \sum_{r=0}^{[n/n_*]} \sum_{0\leqslant a_r<b_r<\cdots<a_1<b_1<n} \prod_{i=1}^{r} \sum_{B\in P_0^{b_i-a_i-1}|\bar{B}\cap\Delta P\neq\varnothing} \sup_{y\in B} e^{\phi_{b_i-a_i}(y)}$$
$$\times (\#P\cdot e^{\|\phi\|_\infty})^{n-n'}$$
$$\leqslant (n/n_*+1)C_n^{2[n/n_*]} e^{n(P_{\mathrm{top}}(\phi,\partial P,T)+2\epsilon)}e^{\epsilon n}$$
$$\leqslant e^{n(P_{\mathrm{top}}(\phi,\partial P,T)+4\epsilon)}$$

proving the claim. ∎

Proof of Lemma 3.5*:* Let μ be an invariant probability measure on (Σ,σ) such that for all a.e. α, $\epsilon(\alpha)<\delta$. Fix D_0 such that $\mu(\{\alpha:\alpha_0=D_0\})>0$. The proof of Proposition 1.1 in [9] shows that if $\delta>0$ is small enough then for every $n\geq 0$ there is a collection C_n of n-cylinders containing all the points $\alpha\in[D]$ that satisfy $\epsilon(\alpha)<\delta$ such that:

$$\sum_{A\in C_n}\sup_A e^{\Phi_n(x)} \leqslant C\cdot e^{(P_{\mathrm{top}}(\phi,\partial P,T)+\epsilon)n}$$

The union of the cylinders in C_n covers D_0 modulo μ, thus its μ-measure is positively lower-bounded. But we have seen (see the proof of Prop. 3.1) that this implies that $P_\mu(\Phi,\sigma)\leqslant P_{\mathrm{top}}(\phi,\partial P,T)+\epsilon$. ∎

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