TAIL-IN Variant MEASURES FOR SOME SUSPENSION SEMIFLOWS

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Abstract. We consider suspension semiflows over abelian extensions of one-sided mixing subshifts of finite type. Although these are not uniquely ergodic, we identify (in the ergodic case) all tail-invariant, locally finite measures which are quasi-invariant for the semiflow.

1. Introduction.

1.1. The Tail Relations. We start with some background on equivalence relations, (see [F-M] for more detail). Let $(X,B)$ be a standard Borel space, and let $R \subset X \times X$ be an equivalence relation. Assume that $R \in B \otimes B$, and that each equivalence class $R(x) := \{ y : (x,y) \in R \}$ is countable. Then for any $A \in B$, the saturation $R(A) = \cup \{ R(x) : x \in A \}$ is again a Borel set. A $\sigma$-finite measure $\mu$ on $X$ is called non-singular for $R$ if $\mu(R(A)) = 0$ whenever $\mu(A) = 0$, and is, in addition, called ergodic if any saturated set $A = R(A)$ has either zero or full measure.

A Borel isomorphism $\phi$ defined on some $A \in B$ with image $B \in B$ is a holonomy if $(x,\phi(x)) \in R$ for any $x \in A$. A measure $\mu$ is invariant for $R$, if it is invariant under all the holonomies of $R$.

Let $S$ be a finite set, and let $\Sigma$ be a subshift of finite type over $S$:

$$\Sigma := \{ x \in S^\mathbb{N} : \forall k \geq 1, A_{x_k,x_{k+1}} = 1 \}$$

where $A = (t_{ij})_{S \times S}$ with $t_{ij} \in \{0,1\}$. We endow $\Sigma$ with the topology generated by cylinders $[a_1, \ldots, a_n] := \{ x \in \Sigma : x_1^a = a_1^a \}$, where $x_j^a := (x_1, \ldots, x_j)$. Note that the collection of cylinders of length $n$ is exactly $\alpha_n^{-1}$ where $\alpha := \{ [a] : a \in S \}$.

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Define the left shift $T: \Sigma \to \Sigma$ by $(Tx)_i = x_{i+1}$. Let $\mathcal{P}(\Sigma)$ denote the collection of Borel probability measures on $\Sigma$.

Henceforth we assume that $(\Sigma, T)$ is topologically mixing. It is well-known that this is equivalent to the existence of $N_0$ such that all the entries of $A^{N_0}$ are positive (see [Bo]).

Let $h: \Sigma \to \mathbb{R}_+$, $f: \Sigma \to \mathbb{Z}^d$ be Hölder continuous. Set

$$\Sigma^h := \{(x, s) : x \in \Sigma, 0 \leq s < h(x)\},$$

and define the semiflows $g_t: \Sigma^h \to \Sigma^h$ and $G_t: \Sigma^h \times \mathbb{Z}^d \to \Sigma^h \times \mathbb{Z}^d$ by

$$g_t(x, s) := (T^n x, s + t - h_n(x)),$$

$$G_t(x, s, \nu) := \left\{ T^n x, s + t - h_n(x), \nu + f_n(x) \right\} \quad \text{where} \; s + t \in [h_n(x), h_{n+1}(x)).$$

Define the tail equivalence relations $\mathcal{T}(g)$ on $\Sigma^h$, and $\mathcal{T}(G)$ on $\Sigma^h \times \mathbb{Z}^d$ as follows:

$$\mathcal{T}(g) := \{ (x, s), (x', s') \} \mid g_t(x, s) = g_t(x', s') \text{ for some } t > 0 \}$$

$$\mathcal{T}(G) := \{ (x, s, \nu), (x', s', \nu') \} \mid G_t(x, s, \nu) = G_t(x', s', \nu') \text{ for some } t > 0 \}.$$

It is not difficult to verify that

$$((x, s), (x', s')) \in \mathcal{T}(g) \iff \exists n, m > 0 \text{ s.t. } \begin{cases} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \end{cases}$$

and that

$$((x, s, \nu), (x', s', \nu')) \in \mathcal{T}(G) \iff \exists n, m > 0 \text{ s.t. } \begin{cases} T^n(x) = T^m(x') \\ s - h_n(x) = s' - h_m(x') \\ \nu + f_n(x) = \nu' + f_m(x') \end{cases}$$

As shown in [B-M], the relation $\mathcal{T}(g)$ is a symbolic model for the strong stable foliation of a topologically mixing basic set $\Omega_h$ of an Axiom A flow, in the sense that, given such a flow, there exists $\Sigma$, $h$ as above, and a one-to-one correspondence between invariant measures for the strong stable foliation of $\Omega_h$ and locally-finite invariant measures for $\mathcal{T}(g)$. The reader is referred to [B-M] for the definition of these geometric objects.

In the same sense, $\mathcal{T}(G)$ is a symbolic model for the strong stable foliation of a $\mathbb{Z}^d$-extension of an Axiom A flow, see [B-L],[Po],[C].

1.2. The Babillot–Ledrappier Measures. The relation $\mathcal{T}(g)$ is uniquely ergodic [B-M], but $\mathcal{T}(G)$ is not: [B-L] provides a $d$-parameter family of pairwise disjoint $\mathcal{T}(G)$-invariant measures, called here Babillot-Ledrappier (B-L) measures. These are given as follows. Fix $\alpha \in \mathbb{R}^d$. By [Bo], [Ru] there exists a unique $\tau_\alpha \in \mathbb{R}$ and a unique Borel probability measure $\mu_\alpha$ on $\Sigma$ which is $(e^{-\tau_\alpha h + \langle \alpha, f \rangle}, T)$-conformal in the sense that $\mu_\alpha \circ T \sim \mu_\alpha$ and

$$\frac{d\mu_\alpha \circ T}{d\mu_\alpha} = e^{-\tau_\alpha h + \langle \alpha, f \rangle}.$$ 

The B-L measure indexed by $\alpha \in \mathbb{R}^d$ is the measure on $X = \Sigma^h \times \mathbb{Z}^d$ given by

$$m_\alpha(A \times B \times \{\nu\}) := e^{-\langle \alpha, \nu \rangle} \mu_\alpha(A) \int_B e^{\tau_\alpha r} dr.$$

These are $\mathcal{T}(G)$-invariant measures. They are infinite, but locally finite: compact subsets of $\Sigma^h \times \mathbb{Z}^d$ have finite measure.
13. Main Results. It is known that ([C] and [Po])

Proposition 1.1. $m_b$ is $\mathfrak{T}(G)$-ergodic iff $T_{(\sigma, f)} : \Sigma \times \mathbb{R} \times \mathbb{Z}^d \to \Sigma \times \mathbb{R} \times \mathbb{Z}^d$ given by $T_{(\sigma, f)}(x, s, \nu) = (Tx, s - h(x), \nu + f(x))$ is ergodic with respect to $\mu_\sigma \times m_\mathfrak{T} \times \mathbb{Z}^d$, where $m_\mathfrak{T}$ denotes Haar measure.

The purpose of this note is

1. To characterize this situation of ergodicity in terms of a cocycle condition for $(\sigma, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ by showing that if one of the B-L measures is ergodic, then $(\sigma, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d$ is non-arithmetic (as defined below) and that this implies that all the B-L measures are ergodic (see [C], and theorem 2.1 and corollary 2.4 below, which imply proposition 1.1).

2. To identify the locally finite $\mathfrak{T}(G)$-invariant measures by showing that in the case when the B-L measures are ergodic, that every locally finite, $\mathfrak{T}(G)$-

invariant, ergodic measure which is $G$-quasi-invariant must be proportional to a B-L measure (Theorem 3.1 below). Theorem 2.2 in [A-N-S-S] can be viewed as a (more complete) discrete time version of this result.

As shown in [B-L], horocycle foliations of $\mathbb{Z}^d$-covers of compact manifolds of constant negative curvature are ergodic with respect to the B-L measures. This is implied (via theorem 2.1 below) by ergodicity with respect to Lebesgue measure which was established earlier in [L-S] (see also [K] and [Po]).

It follows from our results that a locally finite measure which is ergodic and invariant for the stable foliation of a basic set $\Omega_b$ of an Axiom A flow, and which is quasi-invariant under the flow must be proportional to a B-L measure. (In the case of a surface of constant negative curvature this can also be shown via a geometric argument, [Ba].)

2. Ergodicity and non-arithmeticity of $\mathbb{G}$-extensions. Let $\mathbb{G}$ be a locally compact, second countable, Abelian topological group, let $(X, B, m, T)$ be a probability preserving transformation and let $\phi : X \to \mathbb{G}$ be measurable. Consider the skew product $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ defined by $T_\phi(x, g) := (Tx, g + \phi(x))$ with respect to the (invariant) product measure $m \times m_\mathbb{G}$ where $m_\mathbb{G}$ denotes Haar measure.

Following [G], we say that $\phi$ is non-arithmetic if

$$\gamma(\phi) = T \cdot g \circ T$$

has no nontrivial solution in $\gamma \in \hat{\mathbb{G}}$ and $g : X \to S^1$ measurable; and that $\phi$ is aperiodic if

$$\gamma(\phi) = z \cdot g \circ T$$

has no nontrivial solution in $\gamma \in \hat{\mathbb{G}}$, $z \in S^1$ and $g : X \to S^1$ measurable. It is not hard to show that if $T_\phi$ is ergodic, and $T$ is weakly mixing, then $\phi$ is non-arithmetic, and in this case $T_\phi$ is weakly mixing iff $\phi$ is aperiodic (see e.g. [K-N]).

Since $\mathbb{G}$ is a locally compact Abelian polish group topological group, there are norms $\| \cdot \|$ generating the topology of $\mathbb{G}$ which are Lipschitz in the sense that each character $\gamma : \mathbb{G} \to S^1$ is $\| \cdot \|$-Lipschitz. Indeed, if $Y$ is a metric space, and $f : Y \to \mathbb{G}$ is such that $\gamma \circ f : Y \to S^1$ is Lipschitz $\forall$ characters $\gamma$, then $\exists$ a Lipschitz norm $\| \cdot \|$ such that $f : Y \to \mathbb{G}$ is $\| \cdot \|$-Lipschitz.

Livsic's theorem (see [L]) states that if $(\Sigma, B, m, T)$ is a mixing subshift of finite type equipped with a Gibbs measure, $\phi : X \to \mathbb{G}$ is H"older continuous (w.r.t some Lipscitz norm), and $\gamma \in \hat{\mathbb{G}}$ and $g : X \to S^1$ measurable with $\gamma(\phi) = T \cdot g \circ T$ a.e., then $g : X \to S^1$ is also H"older continuous (w.r.t the same Lipschitz norm). Thus
if a Hölder continuous \( \phi : X \to \mathbb{G} \) is non-arithmetic with respect to some Gibbs measure, then it is non-arithmetic with respect to all Gibbs measures.

Recall that a non-singular subshift of finite type \((\Sigma, B, m, T)\) has the Rényi property if there is a constant \( C > 0 \) such that for every cylinder of positive measure \( a = [a_1, \ldots, a_n] \)

\[
\frac{v_n'(x)}{v_n'(y)} \leq C \quad \text{for } m \times m \text{ a.e. } (x, y) \in a \times a,
\]

where \( v_n := (T^n|_a)^{-1} \) and \( v_n' := \frac{d\text{vol}}{dm} \). The following is a generalization of a theorem in [C].

**Theorem 2.1.** Suppose that \((\Sigma, B, m, T)\) is a mixing subshift of finite type with the Rényi property and that \( \phi \) is Hölder continuous and non-arithmetic; then \( T_\phi \) is ergodic.

**Lemma 2.2.** Assume \( u : \Sigma \to \mathbb{S}^1 \) is Hölder continuous. At least one of the following statements is true:

1. \( u = \mathcal{G} \cdot g \circ T \) for some Hölder continuous \( g : \Sigma \to \mathbb{S}^1 \).
2. Let \( \epsilon \in (0,1) \) and \( N \in \mathbb{N} \) be arbitrary constants. There exists \( n \geq N \) such that for every \( z \in \Sigma \) there are \( x \in \Sigma \) and \( k \leq n \) such that

\[
x_1^N \neq z_1^N, \quad T^k x = T^m z \quad \text{and} \quad |u_n(z) - u_k(x)| \geq \epsilon.
\]

**Proof.** Let \( \mu \) be the Parry measure (i.e. measure of maximal entropy on \( \Sigma \)), then \( d\mu = \psi d\nu \) where \( \nu \in \mathcal{P}(\Sigma) \) is \((1, T)\)-conformal and \( \psi > 0 \) is Hölder continuous. Let \( P : L^1(\nu) \to L^1(\nu) \) be the transfer operator, then

\[
P f(x) = \sum_{T^y = x} e^{-h_{\psi \circ \varphi} (T^y f(y))}
\]

and \( P^n f \to \psi \int_X f d\nu \) uniformly \( \forall f \in C(X) \). Define \( P_u : C(\Sigma) \to C(\Sigma) \) by \( P_u(f) := P(uf) \), then \( P^n_u f = P^n(u_n f) \) where \( u_n := \prod_{i=0}^{n-1} u \circ T^i \). By [G-H] either \( \exists \varphi : \Sigma \to \mathbb{S}^1 \) Hölder continuous such that \( P_u(\varphi) = \varphi \) (which implies (1) with \( g := \varphi/\psi \)), or \( \frac{1}{n} \sum_{i=0}^{n-1} P^k_u f \to 0 \forall f \in C(\Sigma) \). If (2) fails, then \( \exists \epsilon \in (0,1), \ N \geq 1 \) such that \( \forall n \geq N, \ \exists z = z^{(n)} \) satisfying

\[
k \leq n, \ x \in T^{-k} \{T^n z\}, \ x_1^N = z_1^N \Rightarrow |u_k(x) - u_n(z)| < \epsilon.
\]

There are only finitely many possibilities for the \( N\)-prefix of \( z^{(n)} \). We may therefore assume without loss of generality that \( \exists a = [a_1, \ldots, a_N] \) such that \( z^{(n)} \in a \).
for all $n$.

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} P_{u_k}^k x \right\|_\infty \geq \frac{1}{n} \sum_{k=0}^{n-1} P_{u_k}^k \left( T^n z \right) \left( n \right)
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} e^{-h_{top}(x)} \sum_{y \in T^k \left( T^n z \right)} u_k(y) \left( n \right)
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} e^{-h_{top}(x)} \sum_{y \in T^k \left( T^n z \right)} u_k(y) \left( n \right)
\]

\[
= \left( 1 - \epsilon \right) \frac{1}{n} \sum_{k=0}^{n-1} P_{u_k}^k \left( T^n z \right) \left( n \right)
\]

Now \( \sum_{k=0}^{n-1} P_{u_k}^k \rightarrow \nu(a) \psi \) uniformly, whence

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{u_k}^k \left( T^n z \right) \left( n \right) \geq \nu(a) \inf \psi > 0.
\]

Let $W_n$ denote the collection of admissible words of length $n$ in $\Sigma$, that is $W_n := \{ (e_1, \ldots, e_n) \in S^n : A_{e_j, e_{j+1}} = 1 \forall 1 \leq j \leq n - 1 \}$. We denote the concatenation of $a \in W_n$ and $b \in W_m$ with $A_{a_n, b_1} = 1$, by $a \cdot b$, and the concatenation of $a \in W_n$ and $x \in \Sigma$ with $A_{a_n, x_1} = 1$ by $(a, x)$.

**Lemma 2.3.** Suppose that $\phi$ is Hölder continuous, $\gamma \in \hat{G}$ is non-constant, $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$. If $\phi$ is non-arithmetic, then there exists $\ell \geq 1$ arbitrarily large and infinitely many $n \geq N$ with the following property:

- $a \in W_n$
- $c \in W_{\ell}$
- $a \cdot c \in W_{n+\ell}$
- $\exists k \in [N, n]$ and $\exists b \in W_{k}$
- $\forall x \in c, |\gamma \circ \phi_n(a, x) - \gamma \circ \phi_k(b, x)| \geq \epsilon$

**Proof.** Fix $\gamma \in \hat{G}$ non-constant, $\epsilon \in (0, 1)$, and $N \geq 1$. Choose $0 < \delta < \frac{1-\epsilon}{2}$ and $\ell \geq 1$ such that

\[
y_{\epsilon} := \sup\{|\gamma \circ \phi_n(x) - \gamma \circ \phi_n(y)| : n \geq 1, x, y \in \Sigma, x^{n+\ell} = y^{n+\ell}\} < \delta.
\]

By lemma 2.2, $\exists n \geq N$ such that $\forall z \in \Sigma, \exists k \leq n, x \in T^{-k} \{ T^n z \}, x_1 = z_1$ such that

\[
|\gamma \circ \phi_n(z) - \gamma \circ \phi_k(x)| \geq \epsilon + 2\delta.
\]

Now fix $a \in W_n$, $c \in W_{\ell}$ with $a \cdot c \in W_{n+\ell}$, choose some $u \in \Sigma$ such that $A_{c, u} = 1$, and set $z = (a, c, u)$. Let $k \leq n$, $x(z) \in T^{-k} \{ T^n z \}$, $x(z)_1 = z_1$ be such that $|\gamma \circ \phi_n(z) - \gamma \circ \phi_k(x(z))| \geq \epsilon + \delta$ and let $b = x(z)_1$. Since $T^k x(z) = T^n z$, $x(z) = (b, c, u)$. For any $v \in \Sigma$ with $A_{c, v_1} = 1$ we have that

\[
|\gamma \circ \phi_n(a, c, u) - \gamma \circ \phi_k(b, c, v)| < \delta, |\gamma \circ \phi_k(b, c, u) - \gamma \circ \phi_k(b, c, v)| < \delta
\]

whence $|\gamma \circ \phi_n(a, c, v) - \gamma \circ \phi_k(b, c, v)| \geq \epsilon$. Since this is true for all $v \in \Sigma$ with $A_{c, v_1} = 1$, the lemma is proved.
Proof of theorem 2.1 (c.f. §2 “Proof of theorem 1” in [AD]). For a nonsingular transformation \((Y, C, \mu, Q)\), define the Grand Tail Relation of \(Q\):

\[
\mathcal{Q}(Q) := \{(x, y) \in Y \times Y : \exists n, k > 0, Q^n x = Q^k y\}.
\]

This is an equivalence relation, and if \((Y, C, \mu)\) is standard, then \(\mathcal{Q}(Q) \subseteq C \otimes C\). If \(Q\) is locally invertible, then \(\mathcal{Q}(Q)\) has countable equivalence classes and is nonsingular. It is easy to check that every \(Q\)-invariant subset of \(Y\) is \(\mathcal{Q}(Q)\)-saturated. It follows that if \(\mathcal{Q}(Q)\) is ergodic, then \(Q\) is ergodic.

It is therefore enough to prove that \(\mathcal{Q}(T_\phi)\) is ergodic. Define

\[
\bar{T}_\phi : \mathcal{Q}(T) \setminus \{(x, y) \in X \times X : x \text{ and } y \text{ are pre-periodic}\} \to \mathbb{G}
\]

by \(\bar{T}_\phi(x, y) = \phi_n(x) - \phi_k(y)\) whenever \(T^n x = T^k y\). This is independent of the choice of \(n, k\) whenever \(x, y\) are not pre-periodic.

The grand tail relation of \(T_\phi\) is given by

\[
\mathcal{Q}(T_\phi) = \left\{ ((x, s), (y, t)) \in (X \times \mathbb{G})^2 : \exists n, k > 0 \text{ such that } T^n x = T^k y,
\quad \text{and } s = t = \phi_n(x) - \phi_k(y) \right\}
\]

\[
= \left\{ ((x, s), (y, t)) \in (X \times \mathbb{G})^2 : (x, y) \in \mathcal{Q}(T), \bar{T}_\phi(x, y) = s - t \right\}
\]

We prove that \(\mathcal{Q}(T_\phi)\) is ergodic by the method of Schmidt (explained in [S]), by considering the group of essential values which we now proceed to define. Set \(B_+ :=\{B \in B : m(B) > 0\}\). For every \(B \in B_+\), let \(\operatorname{Hol}(B) = \operatorname{Hol}(B, \mathcal{Q}(T))\) be the collection of non-singular \(\mathcal{Q}(T)\)-holonomies with domain \(B\):

\[
\operatorname{Hol}(B) := \{\tau : B \to X : \tau \text{ is a non-singular Borel isomorphism } B \to \tau(B)
\quad \text{such that } \forall x \in B, (x, \tau(x)) \in \mathcal{Q}(T)\}.
\]

Now define

\[
E(\mathcal{Q}(T_\phi)) := \left\{ t \in \mathbb{G} : \forall U \text{ open neighborhood of } t \text{ and } \forall A \in B_+, \exists B \in B_+ \text{ and } \exists \tau \in \operatorname{Hol}(B) \text{ such that } B, \tau(B) \subseteq A
\quad \text{and } m(B \cap \tau^{-1} B \cap \{x \in X : \bar{T}_\phi(x, \tau(x)) \in U\}) > 0 \right\}.
\]

It is shown in [S] that \(E(\mathcal{Q}(T_\phi))\) is a closed subgroup of \(\mathbb{G}\). To prove ergodicity, we show that \(E(\mathcal{Q}(T_\phi)) = \mathbb{G}\) (see [S]).

Suppose that \(E(\mathcal{Q}(T_\phi)) = H \subseteq \mathbb{G}\), then \(\exists \gamma \in \mathbb{G}, \gamma \neq 0\) with \(\gamma|_H \equiv 1\). Fix a precompact neighborhood of the identity \(V \subseteq \mathbb{G}\), and let \(N \in \mathbb{N}\) be so large that

\[
j \geq 1, n \geq N, \ x_1^{j+n} = y_1^{j+n} \Rightarrow \phi_j(x) - \phi_j(y) \in V.
\]

Fix \(\epsilon \in (0, 1)\) and let \(\ell \geq 1\) and \(n \geq N\) be as in lemma 2.3 with \(\ell\) so large that

\[
\eta_\ell := \sup \left\{|\gamma \circ \phi_j(x) - \gamma \circ \phi_j(y)| : j \geq 1, x, y \in \Sigma, x_1^{j+\ell} = y_1^{j+\ell}\right\} < \frac{\epsilon}{5}.
\]

It follows that \(\forall u \in W_n, \forall c \in W_\ell\) s.t. \(a \cdot c \in W_{n+\ell}\), \(k \leq n, b \in W_k\) with \(b_1^n = a_1^n, b_k = a_n\) such that \(\forall j \geq 1, \forall u \in W_j\) s.t. \(A_{a_j, a_j} = 1,\)

\[
|\gamma \circ \phi_{j+n}(u, a, c, x) - \gamma \circ \phi_{j+k}(u, b, c, x)| \geq \frac{4\epsilon}{5} \forall x \in Tc_\ell.
\]
Let
\[ K := \left\{ \phi_{j+n}(u,a,c,x) - \phi_{j+k}(u,b,c,x) : j \geq 1, u \in W_j, a \in W_n, A_{u_j,a} = 1, \\
\begin{array}{c}
c \in W_t, a \cdot c \in W_{n+t}, k \leq n, b \in W_k, b_k^N = a_k^N, b_k = a_n, \\
x \in Tc_t, |\gamma \circ \phi_{n+j}(u,a,c,x) - \gamma \circ \phi_{j+k}(u,b,c,x)| \geq \frac{4k}{3} \right\}. \]

By the choice of \( N \) and \( \gamma \), \( K \subset V \setminus E(\Theta(T_0)) \) and \( K \) is compact. The methods of [S] show that \( \exists A \in B_+ \) such that
\[ (A \times A) \cap \Theta(T) \cap \{ \emptyset \in K \} = \emptyset. \]

By the Rényi property, \( \exists M > 1 \) such that
\[ M^{-1}m(u)m(v) \leq m(u \cap T^{-k}v) \leq Mm(u)m(v) \ \forall \ u \in \alpha_0^{-k}, \ v \in \alpha_0^{-1}, \ [v_1] \subset T[u_k]. \]

Given \( j \geq 1, \ u = [u_1, \ldots, u_j] \subset \Sigma \) and \( a \in W_n, b \in W_k, c \in W_t \) as above, define \( \tau : [u \cdot a \cdot c] \to [u \cdot b \cdot c] \) by
\[ \tau(u,a,c,y) = (u,b,c,y). \]

It follows that \( \tau : [u,a,c] \to [u,b,c] \) is invertible, nonsingular and \( \frac{dm\sigma}{dm} = M^{+4}\frac{m(\emptyset)}{m(\emptyset)}. \)

Let \( \delta > 0 \) be so small that for all \( k \leq n, a \in W_n, b \in W_k, c \in W_t, k \leq n, \)
\[ \delta < \frac{m(b)}{M^{+4}m(a)} \left( \frac{m([u,a,c])}{M} - \delta \right). \]

\( \exists j \geq 1 \) and \( u = [u_1, \ldots, u_j] \subset \Sigma \) such that \( m(u \setminus A) < \delta m(u) \). Let \( a \in W_n \)
be such that \( [u,a] \neq \emptyset \) and let \( k \leq n, b \in W_k, c \in W_t \) be as above. Consider the corresponding \( \tau : [u,a,c] \to [u,b,c] \). Evidently \( T^{j+k} \circ \tau \equiv T^{j+n} \) so \( (x,\tau(x)) \in \Theta(T) \ \forall \ x \in [u,a,c], \) and \( \phi_{j+k} \circ \tau(x) - \phi_{j+n}(x) \in K \ \forall \ x \in [u,a,c]. \)

To complete the proof we claim that \( \exists B \in B_+, B \subset A \cap [u,a,c] \) such that \( \tau B \subset A. \) To see this we show that \( m(\tau([u,a,c] \cap A)) \geq m(u \setminus A) \), because this implies \( m(A \cap \tau([u,a,c] \cap A)) > 0 \) since \( \tau(u,a,c] \cap A) \subset u. \) Now
\[ m(\tau([u,a,c] \cap A)) \geq \frac{m(b)}{M^{+4}m(a)} m([u,a,c] \cap A) \]
\[ \geq \frac{m(b)}{M^{+4}m(a)} \left( m([u,a,c]) - m(u \setminus A) \right) \]
\[ > \frac{m(b)}{M^{+4}m(a)} \left( \frac{m([u,a,c])}{M} - \delta \right) m(u) \]
\[ > \delta m(u) > m(u \setminus A). \]

and this shows that \((A \times A) \cap \Theta(T_0) \cap \{ \emptyset \in K \} \neq \emptyset \) which is a contradiction.

The following amplifies proposition 1:

**Corollary 2.4.** Let \( m_\alpha \) be a \( B \)-\( L \) measure on \( \Sigma^h \times \mathbb{Z}^d \). The following are equivalent:
1. \( (\Sigma^h \times \mathbb{Z}^d, m_\alpha, \mathcal{F}(G)) \) is ergodic;
2. the cocycle \((-h, f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d\) is non-arithmetic;
3. \( T_{(-h,f)} \) is ergodic on \( \Sigma \times \mathbb{R} \times \mathbb{Z}^d \) with respect to \( \mu_\alpha \times m_\mathbb{R} \times \mathbb{Z}^d \) where \( m_\mathbb{R} \) denotes Haar measure and \( \mu_\alpha \) is as in \( \S 1.2. \)

**Proof.** Set \( X = \Sigma^h \times \mathbb{Z}^d \). As shown in [Po],
\[ \Theta(T_{(-h,f)}) \cap (X \times X) = \mathcal{F}(G) \] (2.1)
(1) ⇒ (2). Suppose (1) and that \( s \in \mathbb{R}, \gamma \in \mathbb{R}^d \) and \( g : \Sigma \to \mathbb{S}^1 \) satisfy \( e^{-is\theta+i(\gamma,f)} = \frac{g}{g(x)} \), and define \( F : X \to \mathbb{C} \) by 
\[
F(x,y,z) = g(x)e^{-is\theta+i(\gamma,f)}
\]
then
\[
F \circ T_{(-h,f)}(x,y,z) = F(Tx,y - h(x),z + f(x)) = g(Tx)e^{-is\theta+i(\gamma,f)} = g(Tx)\frac{g(x)}{g(Tx)}e^{-is\theta+i(\gamma,f)x}(y,x,z) = F(x,y,z).
\]
It follows that \( F \) is constant, since \( F \circ T_{(-h,f)} = F \) and so every set of the form \([F \leq t]\) is \( \Theta(T_{(-h,f)}) \)-saturated whence also \( \mathfrak{S}(G) \)-saturated.

Now consider \( F_0 : X \to \mathbb{C} \) the restriction of \( F \) to \( X \). It follows that for \( (x,y,z) \in X \), \( t \geq 0 \) (choosing \( n \geq 0 \) such that \( h_n(x) \leq t < h_{n+1}(x) \)):
\[
F_0 \circ G_t(x,y,z) = F_0(T^nx,y + t - h_n(x),z + f_n(x)) = F_0 \circ T_{(-h,f)}^t(x,y + t,z) = F(x,y + t,z) = e^{-is\theta}F_0(x,y,z)
\]
and \( F_0 \) is \( \mathfrak{S}(G) \)-invariant, whence constant. It follows that \( s = 0 \), \( \gamma = 0 \) and \( g \equiv 1 \), so \((-h,f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d \) is non-arithmetic. (2) ⇒ (3) by theorem 2.1. (3) ⇒ (1) follows from (2.1).

Thus:

**Corollary 2.5.** If \( \mathfrak{S}(G) \) is ergodic with respect to some B-L measure, then the cocycle \((-h,f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d \) is non-arithmetic and \( \mathfrak{S}(G) \) is ergodic with respect to all B-L measures.

3. Identification of ergodic, locally finite \( \mathfrak{S}(G) \)-invariant measures.

**Theorem 3.1.** Let \( X := \Sigma^h \times \mathbb{Z}^d \) and let \( G_t \) \((t \geq 0)\) be the suspension semi-flow. Assume that \((-h,f) : \Sigma \to \mathbb{R} \times \mathbb{Z}^d \) is non-arithmetic and Hölder continuous. Suppose that \( m \) is a locally finite, \( \mathfrak{S}(G) \)-invariant, ergodic measure on \( X \) and that \( m \circ G_t^{-1} \sim m \forall t > 0 \), then \( m \) is proportional to a B-L measure.

**Proof.** By assumption, \( f : \Sigma \to \mathbb{Z}^d \) is Hölder continuous, and every such function is of the form \( f(x) = f(x_1, \ldots, x_m) \) for some \( m \). Recoding \( \Sigma \) if necessary, we assume without loss of generality that \( f(x) = f(x_1, x_2) \).

For \( t > 0 \), define the measure \( m \circ G_t \) by \( m \circ G_t(A) := \sum_{a \in \alpha} m(G_t(A \cap a)) \) where \( \alpha \) is a countable partition of \( X \) such that \( G_t|_a \) is \( 1 \)-1 \( \forall a \in \alpha \). Evidently \( m \circ G_t \sim m \).

Let \( \mathfrak{M}(\Sigma \times \mathbb{Z}^d) \) denote the collection of all (possibly infinite) Borel measures on \( \Sigma \times \mathbb{Z}^d \).

Claim 1: \( \exists \tau \in \mathbb{R} \) such that \( \frac{d(m \circ G_t)}{dm} = e^{\tau t} \), and \( \exists \mu \in \mathfrak{M}(\Sigma \times \mathbb{Z}^d) \) locally finite, such that \( \frac{d(m \circ G_t)}{dm} = e^{\tau t} \) and
\[
m(A \times B) = \mu(A) \int_B e^{\tau r} dr \quad (A \in B(\Sigma \times \mathbb{Z}^d), \ B \in B(\mathbb{R}), \ A \times B \subset X). \quad (3.2)
\]
Moreover \( (\Sigma \times \mathbb{Z}^d, B(\Sigma \times \mathbb{Z}^d), T_f, \mu) \) is ergodic.

**Proof.** Fix \( t_0 > 0 \). We prove first that \( \frac{d(m \circ G_{t_0})}{dm} \) is \( \mathfrak{S}(G) \)-invariant and hence constant. Suppose that \( A \subset X \) is Borel, and that \( K : A \to KA \) is a \( \mathfrak{S}(G) \)-holonomy. Without loss of generality, \( G_{t_0}|A \), \( G_{t_0}|KA \) are 1-1. It follows that
\[
K_1 := G_{t_0} \circ K \circ G_{t_0}^{-1} : G_{t_0}A \to G_{t_0}KA
\]
is a well-defined \( \mathfrak{S}(G) \)-holonomy. By the \( \mathfrak{S}(G) \)-invariance of \( m \),
\[
m(G_{t_0}KA) = m(K_1G_{t_0}A) = m(G_{t_0}A).
\]
This shows that \( \frac{d\mu}{dm} G_{\alpha} \) is indeed \( \mathcal{T}(G) \)-invariant and hence constant. Disintegrating the measure \( m \) over \( \Sigma \times \mathbb{Z}^d \), we see that \( \exists \lambda \in M(\Sigma \times \mathbb{Z}^d) \) locally finite, and \( m_x \in \mathcal{M}(\mathbb{R}_+) \) such that \( x \mapsto m_x \) is measurable, and such that

\[
m(A \times B) = \int_A m_x(B) d\lambda(x).
\]

It follows that \( m_x(J + t) = e^{rt} m_x \) for open \( J \subset (0, h(x)) \) and \( t \in \mathbb{R} \) small, whence \( dm_x(y) = c(x)e^{ty} dy \) and (3.2) follows with \( d\mu(x) := c(x)d\lambda(x) \). The equation \( \frac{du}{d\mu} = e^{rh} \) now follows from \( \frac{d\mu}{dm} G_{\alpha} = e^{rt} \), and the ergodicity of \((\Sigma, T_f, \mu)\) is standard.

\[\Box\]

Claim 2: \( \exists \) a homomorphism \( \alpha : \mathbb{Z}^d \to \mathbb{R} \) and \( c > 0 \) such that \( \mu(A \times \{n\}) = ce^{-\alpha(n)}(A) \) where \( \nu \in \mathcal{P}(\Sigma) \) is \((e^{\alpha}f^{\tau}, T)\)-conformal.

\[\text{Proof.}\] We first claim it suffices to show that \( H := \{ n \in \mathbb{Z}^d : \mu \circ Q_n \sim \mu \} = \mathbb{Z}^d \) where \( Q_n(x, k) := (x, k + n) \). To see this, note that

\[
\frac{d\mu \circ Q_n \circ T_f}{d\mu} = \frac{d\mu \circ T_f}{d\mu} \circ Q_n = e^{rh} \quad \forall n \in \mathbb{Z}^d.
\]

The ergodicity of \((\Sigma, T_f, \mu)\) ensures that \( \forall n \in \mathbb{Z}^d \), either \( \mu \circ Q_n \perp \mu \) or \( \mu \circ Q_n = c_n \mu \) for some \( c_n > 0 \). The condition \( H = \mathbb{Z}^d \) ensures that \( \mu \circ Q_n = e^{-\alpha(n)} \mu \) where \( \alpha : \mathbb{Z}^d \to \mathbb{R} \) is a homomorphism. Thus, \( \mu(A \times \{n\}) = ce^{-\alpha(n)}(A) \) where \( c > 0 \) and \( \nu \in \mathcal{P}(\Sigma) \). The \((e^{\alpha}f^{\tau}, T)\)-conformality of \( \nu \) follows from the \((e^{rh}, T_f)\)-conformality of \( \mu \).

We now prove that \( H = \mathbb{Z}^d \). Suppose otherwise that \( H \neq \mathbb{Z}^d \), then \( \exists \gamma \in \hat{\mathbb{Z}}^d \) non-constant, such that \( \gamma|_H \equiv 1 \). Using non-arithmeticity and lemma 2.3, we fix \( n \geq 1 \) so that \( \forall a \in W_n \) and \( c \in S \) s.t. \( a \cdot c \in W_{n+1} \), \( \exists k = k(a) \leq n \) and \( b = b(a, c) \in W_k \) such that \( a_1 = b_1 \), \( a_n = b_k \) and \( c \circ f_n(a, c) \neq c \circ f_k(b, c) \). By choice of \( \gamma \), this means that \( f_n(a, c) - f_k(b, c) \notin H \).

Set \( J := \{ f_n(a, c) - f_k(b, c) \} : a \in W_n, c \in S, a \cdot c \in W_{n+1} \) then \( J \subset \mathbb{Z}^d \) and \( J \) is finite. Set \( P := \sum_{j \in J} \mu \circ Q_j \), then \( P \perp \mu \) and \( \exists K \subset \Sigma \) compact and \( g \in \mathbb{Z}^d \) such that \( \mu(K \times \{g\}) > 0, P(K \times \{g\}) = 0 \).

Set \( I := \sup \{ |h_j(x) - h_j(y)| : j \geq 1, x_j = y_j \}, L := 2 \max_{k \leq n} \sup |h_j| \) and \( M := \|W_{n+1}\|_{e^{\tau(I+\ell)}} \). Approximating \( K \) by larger open sets, we see that \( \exists U \subset \Sigma \) open, such that \( K \subset U \) and \( P(U \times \{g\}) < \frac{\mu(U \times \{g\})}{2M} \). It follows that \( \exists \) a cylinder set \( d = [d_1, \ldots, d_N] \) such that \( \mu(d \times \{g\}) > 0 \) and \( P(d \times \{g\}) < \frac{\mu(d \times \{g\})}{2M} \).

Since \( d \times \{g\} = \bigcup_{a \in W_n, c \in S} \{d, a, c \times \{g\}, \exists a \in W_n, c \in S \) with \( a \cdot c \in W_{n+1} \) such that \( \mu(d, a, c \times \{g\}) \geq \frac{\mu(d \times \{g\})}{M} \). Next, \( \exists b = (b_1, \ldots, b_k) \in W_k \) such that \( a_1 = b_1, a_n = b_k \) and \( f_n(a, c) - f_k(b, c) \in J \). Define \( \kappa : [d, a, c] \times \{g\} \to d \times \mathbb{Z}^d \) by \( \kappa((d, a, x), g) := ((d, b, x), g + f_k(b, c) - f_n(a, c)) \). Since \( \frac{d\mu \circ T_f}{d\mu} = e^{rh} \), we have that

\[
\frac{d\mu \circ \kappa(x, v)}{d\mu} = e^{rh} \in [e^{-\tau(I+\ell)}, e^{\tau(I+\ell)}],
\]

\footnote{We are using here the assumption \( f(x) = f(x_0, x_1) \) to note that lemma 2.3 can be used with \( \ell = 1 \) and that \( f_n \) (resp. \( f_k \)) is constant on \( [a, c] \in W_{n+1} \) (resp. \( [b, c] \in W_{k+1} \)) so that the notation \( f_n(a, c), f_k(b, c) \) makes sense.}
where the last estimate follows from
\[ |h_{N+k}(d, b, x) - h_{N+n}(d, a, x)| \leq |h_N(d, b, x) - h_N(d, a, x)| + |h_k(b, x)| + |h_n(a, x)| \leq I + L. \]
Thus
\[
(\mu \circ \kappa)([d, a, c] \times \{g\}) = \int_{[d, a, c] \times \{g\}} \frac{d\mu \circ \kappa}{d\mu} d\mu \\
\geq e^{-\tau(I+L)}\mu([d, a, c] \times \{g\}) \\
\geq e^{-\tau(I+L)}\frac{\mu(d \times \{g\})}{M} \\
= \frac{\mu(d \times \{g\})}{M}.
\]
On the other hand, \( \kappa([d, a, c] \times \{g\}) \subset Q_{f_k(b, a) - f_n(a, c)}(d \times \{g\}) \) whence
\[
\frac{\mu(d \times \{g\})}{M} \leq \mu(\kappa([d, a, c] \times \{g\})) \leq \mu(Q_{f_k(b, c) - f_n(a, c)}(d \times \{g\})) \leq \\
\frac{\mu(d \times \{g\})}{2M} < \frac{\mu(d \times \{g\})}{2M}
\]
and \( 1 < \frac{1}{2} \). This contradiction establishes claim 2. \( \square \)

Since \((e^{\alpha\phi} + \tau h, T)\)-conformal probability is unique, it follows from claim 2 that \( m \) is proportional to the corresponding B-L measure. \( \square \)

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