Lecture 4: Lyapunov Exponents

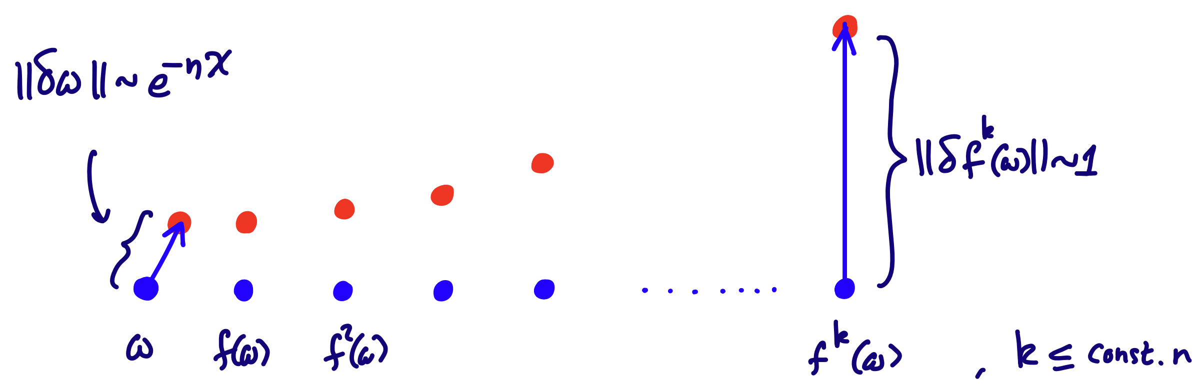
Overview & Spoilers

Last time, we found a necessary & sufficient condition for the ability of a prob. preserving map f to produce a purely random (= Bernoulli) time series: Positive entropy

Today we ask: What dynamical mechanisms produce positive entropy?

The Answer (volume preserving case) Exponential sensitivity to initial conditions:

"exponentially small perturbations in some directions grow to order one after linear number of iterations"



The precise statement is infinitesimal, so we begin with a review of some definitions from calculus on manifolds.

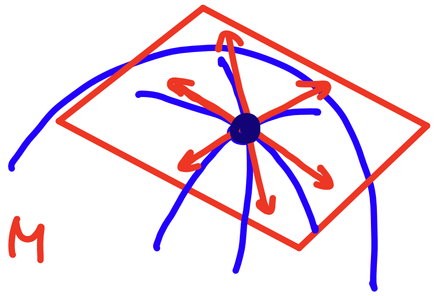
Review of Differential Calculus on Manifolds

Setup: M is a compact smooth Riemannian manifold of dimension d, and embedded in $\mathbb{R}^N$; $f: M \to M$ differentiable

Tangent Space at $p \in M$:

$$T_pM := \left\{ \dot\gamma(0) : \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \longrightarrow M \text{ is a smooth} \\ \text{curve s.t. } \gamma(0) = p \end{array} \right\}$$

Fact: T_pM is a linear vector space of dim d



Exponential Map: $\exp_p : T_pM \to M$

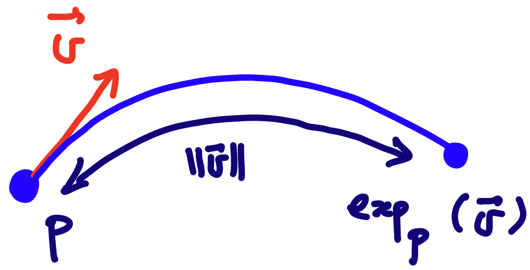
$\exp_p(\vec v) = g_{\vec v}(\|\vec v\|)$, where $g_{\vec v}(t) =$ geodesic from p in direction $\vec v / \|\vec v\|$.

Fact: There is $r > 0$ s.t.

$\exp_p : \{\vec v : \|\vec v\| < r\} \longrightarrow$ image

is smooth and invertible.

("radius of injectivity")



Corollary: Suppose $f : M \to M$ is smooth, then the following map is well-defined on a neigh of the origin

$$\underbrace{\exp^{-1}_{f(p)} \circ f \circ \exp_p}_{\text{non-linear}} : \underbrace{T_pM \longrightarrow T_{f(p)}M}_{\text{linear vector spaces}}$$

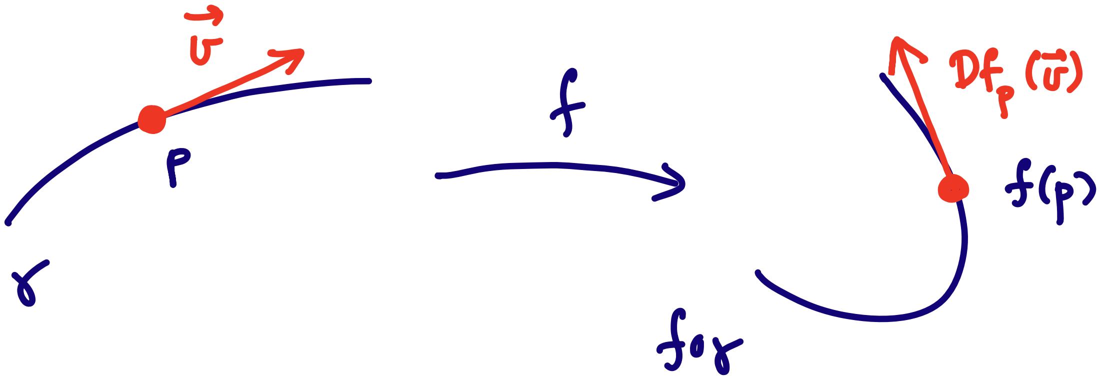
The Differential of f at p: $Df_p : T_pM \to T_{f(p)}M$

Def $\underline{=}$ 1: The linearization of $\exp^{-1}_{f(p)} \circ f \circ \exp_p : T_pM \to T_{f(p)}M$ at $\vec 0 \in T_pM$, i.e. the linear map s.t.

$$(\exp^{-1}_{f(p)} \circ f \circ \exp_p)(\vec 0 + t\vec v) = \underbrace{\cancel{(\exp^{-1}_{f(p)} \circ f \circ \exp_p)(\vec 0)}}_{\vec 0} + t \cdot Df_p(\vec v) + O(t^2)$$

Def$^{\underline{n}}$ 2 (<u>coordinate-free formula</u>): Every $\vec{v} \in T_pM$ is the velocity vector $\dot{\gamma}_{\vec{v}}(0)$ of some curve $\gamma_{\vec{v}}(t)$ on M s.t. $\gamma_{\vec{v}}(0) = p$.

$$\boxed{Df_p(\vec{v}) = (f\circ\gamma)^{\bullet}(0)}$$

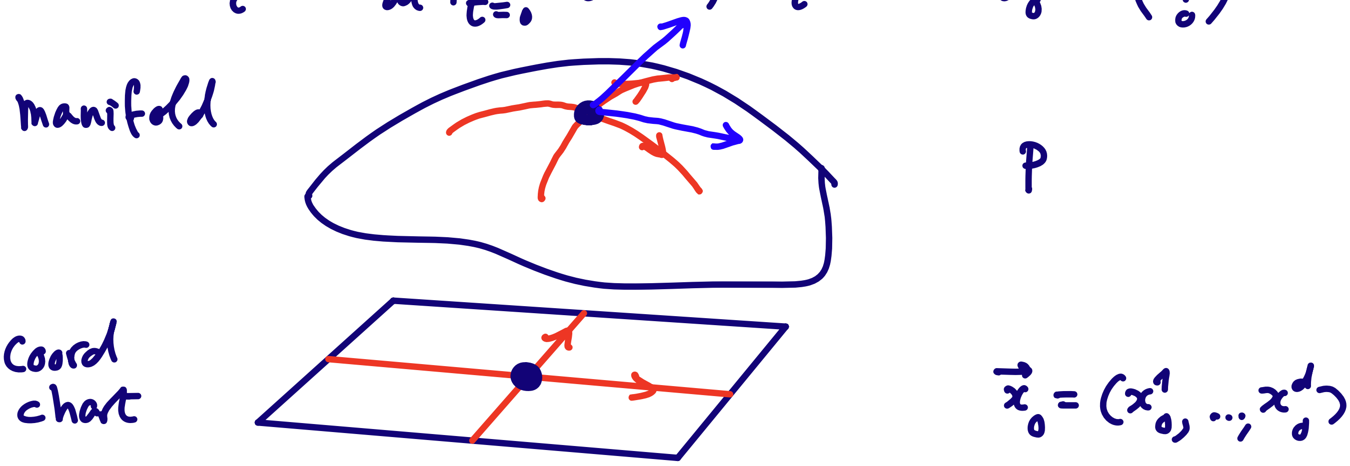


Def$^{\underline{n}}$ 3 (<u>formula in coordinates</u>): Fix $p \in M$ and two coordinate charts on neighborhoods of p, $f(p)$:

- $(x^1, \dots, x^d)$ near p, $p = (x_0^1, \dots, x_0^d)$
- $(y^1, \dots, y^d)$ near $f(p)$, $f(p) = (y_0^1, \dots, y_0^d)$

We have the following natural bases for T_pM, $T_{f(p)}M$:

- $\vec{e}_i^{\,p} := \frac{d}{dt}\Big|_{t=0} c_i^p(t)$, $c_i^p(t) = \vec{x}_0 + t\begin{pmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{pmatrix} \leftarrow i$ in coord.
- $\vec{e}_i^{\,f(p)} := \frac{d}{dt}\Big|_{t=0} c_i^{f(p)}(t)$, $c_i^{f(p)}(t) = \vec{y}_0 + t\begin{pmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{pmatrix} \leftarrow i$ in coord



Represent $f : \begin{pmatrix} \text{neigh} \\ \text{of } p \end{pmatrix} \longrightarrow \begin{pmatrix} \text{neigh} \\ \text{of } f(p) \end{pmatrix}$ in coordinates

$$f(x^1, \dots, x^d) = \begin{pmatrix} y^1(x^1, \dots, x^d) \\ \vdots \\ y^d(x^1, \dots, x^d) \end{pmatrix}$$

Then $(Df_p)\left(\sum_{i=1}^{d} \alpha_i \vec{e}_i^{\,p}\right) = \sum_{i=1}^{d} \beta_i \vec{e}_i^{\,f(p)}$, where

$$\boxed{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix} = \begin{pmatrix} \frac{\partial y_i}{\partial x_j}(x_0^1, \dots, x_0^d) \end{pmatrix}_{d \times d} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}}$$

Chain Rule: $\boxed{Df_p^n = Df_{f^{n-1}(p)} \circ \dots \circ Df_{f(p)} \circ Df_p}$

C^2-Diffeomorphism: An invertible map $f : M \to M$ s.t. f and f^{-1} are continuously differentiable twice, in coordinates.

Lyapunov Exponents

Oseledets Thm (part 1): Suppose $f: M \to M$ is a diffeomorphism on a compact manifold M of dim d, and μ is an invariant prob measure. Then for μ-a.e. $p \in M$

(1) The following limit exists for all non-zero $\vec{v} \in T_pM$:

$$\chi_p(\vec{v}) = \lim_{n \to \pm\infty} \frac{1}{n} \log \| Df_p^n \vec{v} \|$$

(2) There are finitely many possible values (for each p): $\chi_1(p) \gneq \cdots \gneq \chi_{s(p)}(p)$ s.t.

$$\forall \vec{v} \in T_pM \setminus \{\vec{0}\}, \quad \chi_p(\vec{v}) \in \{\chi_1(p), \ldots, \chi_{s(p)}(p)\}$$

(3) Invariance: $\chi_i(f(p)) = \chi_i(p)$, $s(f(p)) = s(p)$

If μ is <u>ergodic</u>, then $\chi_i(p)$, $s(p)$ are a.e. const.

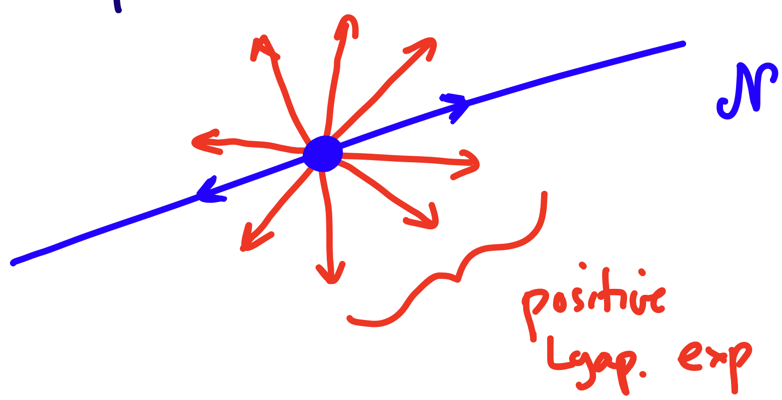
Terminology: $\chi_p(\vec{v})$ is called the <u>Lyapunov exponent</u> at p in direction $\vec{v}$.

Observe: If $\chi_p(\vec{v}) > 0$, then we have exponential sensitivity to initial condition at p:

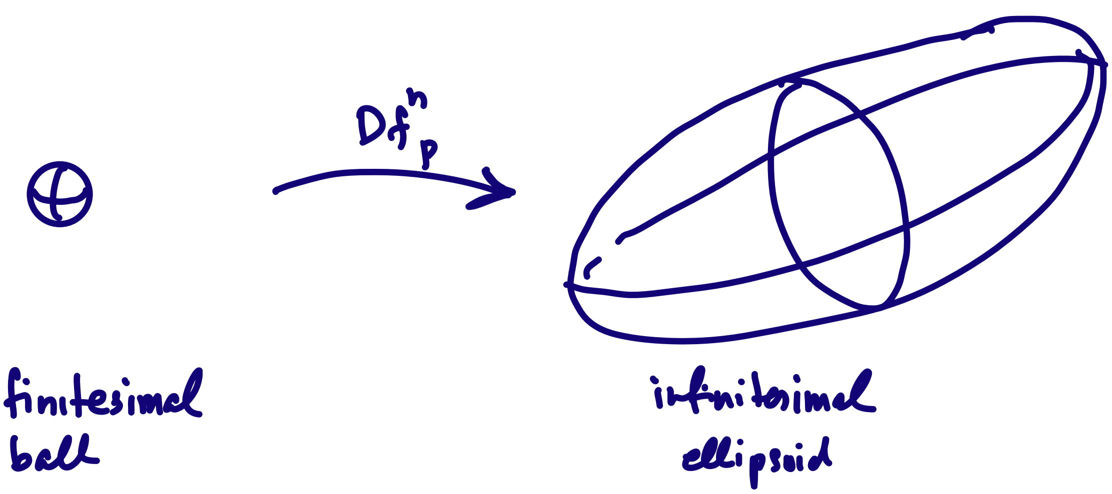
$$f^k(\underbrace{\exp_p(\vec{0} + e^{-n\chi/2}\vec{v})}_{\text{exp. small pert. of } p}) \approx \exp_{f(p)}(\underbrace{e^{-n\chi/2} Df_p^k(\vec{v})}_{\text{size } e^{(k-n/2)\chi} \text{ will be big for some } 1 \le k \le n.})$$

Remark: If $\chi_p(\vec{v}) > 0$ for some $\vec{v} \in T_pM$, then $\chi_p(\vec{v}) > 0$ for "most" $\vec{v} \in T_pM$.

Indeed, $\mathcal{N} = \{ \vec{u} \in T_pM \mid \chi_p(\vec{u}) \leq 0 \} \cup \{\vec{0}\}$ is a linear vector space (exercise), of dimension $\lneq \dim T_pM$ (because it's $\neq T_pM$).



Remark 2: In the volume preserving case, if there is a $\vec{v}$ with positive Lyap exponent, then there must also be a $\vec{v}$ with a negative Lyap. exp. Otherwise, f^n expands the volume exponentially, which is impossible, since M is compact.



Multiplicities of Lyapunov Exponents

Suppose the Lyapunov exponents at p are $\chi_1 \gneq \cdots \gneq \chi_s$. In general, $s \neq d$, so we feel that some χ_i must appear with a "multiplicity". How to define it?

Attempt 1 (fails): $m_i \overset{?}{=} \dim \{\vec{v} \in T_pM : \chi_p(\vec{v}) = \chi_i\} \cup \{\vec{0}\}$

This fails, because $\{\vec{v} : \chi_p(\vec{v}) = \chi_i\}$ is not a vector space. Choose $\vec{v}_1, \vec{v}_2$ s.t $\chi_p(v_1) < \chi_p(v_2)$.

Let $u_\pm = v_2 \pm v_1$

- $\chi_p(u_\pm) = \lim_{n\to\infty} \frac{1}{n} \log \|\underbrace{Df^n(v_2)}_{e^{n\chi_2}} \pm \underbrace{Df^n(v_1)}_{e^{n\chi_1}}\| = \chi_p(v_2)$ $\quad e^{n\chi_2} \gg e^{n\chi_1}$
- $\chi_p(u_+ - u_-) = \chi_p(2v_1) = \chi_p(v_1)$

So $u_+, u_- \in \{\vec{v} : \chi_p(\vec{v}) = \chi_i\}$, but $u_+ - u_- \notin \{\vec{v} : \chi_p(\vec{v}) = \chi_i\}$.

Attempt 2 (succeeds): Define

$$V_{\chi_i} = \{\vec{u} \in T_pM : \chi(\vec{u}) \le \chi_i\} \cup \{\vec{0}\}$$

These are linear spaces (check!), and

$$(*) \quad \{0\} \subsetneq V_{\chi_s} \subsetneq V_{\chi_{s-1}} \subsetneq \cdots \subsetneq V_{\chi_1} = T_pM$$

Def$^{\underline{n}}$. $(*)$ is called the Lyapunov filtration.

The multiplicity of χ_i is defined to be

$$m_i := \dim V_{\chi_i} - \dim V_{\chi_{i-1}}$$

Lyapunov Exponents and Entropy

Thm (Ruelle's Inequality): Suppose $f: M \to M$ is a diffeomorphism on a compact manifold, which preserves a finite invariant measure μ. Let $\chi_1(p) \gneq \cdots \gneq \chi_{s(p)}(p)$ be the Lyapunov exponents, and let $m_1(p), \dots, m_{s(p)}(p)$ be their multiplicities. Then

$$\boxed{h_\mu(f) \leq \int_M \sum_{i:\, \chi_i(p)>0} m_i(p)\, \chi_i(p)\, d\mu}$$

Corollary: If μ is ergodic, and $h_\mu(f) > 0$, then f has exponential sensitivity to initial conditions μ-a.e.

Thm (Pesin Entropy Formula): Under the above conditions, if μ is smooth (i.e. μ = density $\times$ volume measure), then

$$\boxed{h_\mu(f) = \int_M \sum_{i:\, \chi_i(p)>0} m_i(p)\, \chi_i(p)\, d\mu}$$

Corollary: In the volume preserving case, the entropy is positive if and only if there is exponential sensitivity to initial conditions on a set with positive volume.

The Oseledets Decomposition

Oseledets Thm (Part 2): Let f be a diffeomorphism on a compact smooth manifold M, and suppose μ is an invariant prob. measure. Let $\chi_1(p) \gneq \cdots \gneq \chi_{s(p)}(p)$ denote the Lyapunov exponents, and let $m_1(p), \ldots, m_{s(p)}(p)$ be their multiplicities. For μ-a.e. $p \in M$ we can decompose

$$T_p M = H_{\chi_1}(p) \oplus \cdots \oplus H_{\chi_{s(p)}}(p) \quad \text{s.t.}$$

(1) $H_{\chi_i}(p)$ are linear vector spaces of dim $m_i(p)$

(2) $\chi_p(\vec{v}) = \chi_i(p)$ on $H_{\chi_i}(p) \setminus \{\vec{0}\}$

(3) Invariance: $Df_p[H_{\chi_i}(p)] = H_{\chi_i}(f(p))$

(4) $\lim\limits_{n\to\infty} \frac{1}{n} \log \left| \measuredangle\left(H_{\chi_i}(f^n(p)), H_{\chi_j}(f^n(p))\right)\right| = 0 \quad (i \neq j)$

Application 1: For each $\vec{v}_i \in H_{\chi_i} \setminus \{0\}$, $\chi_p(\vec{v}_i) = \chi_i$

$$\Rightarrow \quad \|Df_p^n(\vec{v}_i)\| = e^{n\chi_i + o(n)} \|\vec{v}\|.$$

Thus, the Oseledets Decomp$^{\underline{n}}$ $\vec{v} = \vec{v}_{\chi_1} + \cdots + \vec{v}_{\chi_s}$ splits $\vec{v}$ into components with different exp rates of growth

Application 2: "Diagonalization" of Df_p^n.

For simplicity, suppose $\dim M = 2$, and assume μ is ergodic, with positive entropy.

- By Ruelle's inequality, some Lyap. exp. is positive
- Also, some Lyap exp. is negative:

  $h_\mu(f^{-1}) \equiv h_\mu(f) > 0$, so by Ruelle's inequality

  check!

  for μ-a.e. p, $\exists \vec{v} \in T_pM$ s.t.

  $$0 < \chi_p(f^{-1}, \vec{v}) = \lim_{n \to \pm\infty} \frac{1}{n} \log \| Df_p^{-n} \vec{v} \|$$

  $$= \lim_{n \to \pm\infty} \frac{1}{-n} \log \| Df_p^{+n} \vec{v} \| = -\chi_p(f, \vec{v})$$

  $$\Rightarrow \chi_p(f, \vec{v}) < 0.$$

Thus we must have two different Lyapunov exponents, one χ^u positive, and one χ^s negative.

Necessarily, in the Oseledets decomp^n

$$T_pM = H_{\chi^u}(p) \oplus H_{\chi^s}(p)$$

$H_{\chi^u}(p)$ and $H_{\chi^s}(p)$ are one-dimensional.

Choose unit vectors $\underbrace{\vec{e}^{u}(p) \in H_{\chi^u}(p)}_{\text{“unstable” direction}}$, $\underbrace{\vec{e}^{s}(p) \in H_{\chi^s}(p)}_{\text{“stable” direction}}$.

By the invariance property $Df_p(H_{\chi_i}(p)) = H_{\chi_i}(f(p))$,

$$Df_p\left[\operatorname{Span}\{\vec{e}^{\,t}(p)\}\right] = \operatorname{Span}\{\vec{e}^{\,t}(f(p))\} \quad (t = u, s)$$

Therefore

$$(Df_p)(\vec{e}^{\,u}(p)) = \pm \|Df_p \vec{e}^{\,u}(p)\| \cdot \vec{e}^{\,u}(f(p))$$

$$(Df_p)(\vec{e}^{\,s}(p)) = \pm \|Df_p \vec{e}^{\,s}(p)\| \cdot \vec{e}^{\,s}(f(p))$$

Define $C_p : \mathbb{R}^2 \to T_pM$ by

$$C_p\begin{pmatrix}1\\0\end{pmatrix} = \alpha(p)\vec{e}^{\,s}(p), \quad C_p\begin{pmatrix}0\\1\end{pmatrix} = \beta(p)\,\vec{e}^{\,u}(p$$

($\alpha(\cdot)$, $\beta(\cdot)$ to be determined later). Then

$$\begin{pmatrix}1\\0\end{pmatrix} \xmapsto{C_p} \alpha(p)\vec{e}^{\,s}(p) \xrightarrow{Df_p} \pm\alpha(p)\,\|Df_p e^s(p)\|\; e^s(f(p))$$

$$\xmapsto{C_{f(p)}^{-1}} \pm \frac{\alpha(p)}{\alpha(f(p))}\,\|Df_p e^s(p)\| \begin{pmatrix}1\\0\end{pmatrix}$$

We obtained:

$$\left(C_{f(p)}^{-1} \circ Df_p \circ C_p\right)\begin{pmatrix}1\\0\end{pmatrix} = \pm \frac{\alpha(p)}{\alpha(f(p))} \left\| Df_p\, e^s(p)\right\| \cdot \begin{pmatrix}1\\0\end{pmatrix}$$

Similarly,

$$\left(C_{f(p)}^{-1} \circ Df_p \circ C_p\right)\begin{pmatrix}1\\0\end{pmatrix} = \pm \frac{\beta(p)}{\beta(f(p))} \left\| Df_p\, e^s(p)\right\| \cdot \begin{pmatrix}0\\1\end{pmatrix}$$

In summary

$$C_{f(p)}^{-1} \circ Df_p \circ C_p = \begin{pmatrix} \lambda^u(p) & 0 \\ 0 & \lambda^s(p) \end{pmatrix}$$

where
$$\lambda^u(p) = \pm \frac{\alpha(p)}{\alpha(f(p))} \left\| Df_p\, \vec{e}^{\,s}(p)\right\|$$
$$\lambda^s(p) = \pm \frac{\beta(p)}{\beta(f(p))} \left\| Df_p\, \vec{e}^{\,u}(p)\right\|$$

A "good choice" of $\alpha(p), \beta(p)$ gives:

<u>Oseledets – Pesin Reduction:</u> Can construct $C_p : \mathbb{R}^2 \to T_pM$ so that

$$C_{f(p)}^{-1} \circ Df_p \circ C_p = \begin{pmatrix} \lambda^u(p) & 0 \\ 0 & \lambda^s(p) \end{pmatrix}, \qquad \begin{aligned} |\lambda^u(p)| &> e^{\chi_u - \varepsilon} > 1 \\ |\lambda^s(p)| &< e^{\chi_s + \varepsilon} < 1 \end{aligned}$$

In <u>higher dim</u>, the Oseledets–Pesin Reduction says that for some family of C_p,

$$C_{f^n(p)}^{-1} Df_p^n C_p = \begin{pmatrix} B_{\chi_1}^{(n)} & & 0 \\ & \ddots & \\ 0 & & B_{\chi_s}^{(n)} \end{pmatrix}$$

where $B_{\chi_i}^{(n)}$ are $m_i \times m_i$ blocks (m_i = mult. of χ_i) s.t.

$$\| B_{\chi_i}^{(n)} \vec{v} \| = e^{n(\chi_i \pm \varepsilon)} \| \vec{v} \| \quad \text{on } \mathbb{R}^{m_i}.$$

<u>Summary</u>: The Oseledets Thm allows to chose bases $\left\{ C_p \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, C_p \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$ for T_pM in such a way that $C_{f^n(p)}^{-1} \circ Df_p^n \circ C_p$ have diagonal block form.

In the two-dimensional, positive entropy, case, the resulting form is a hyperbolic diagonal matri

$$C_{f^n(p)}^{-1} \circ Df_p^n \circ C_p = \begin{pmatrix} \lambda_u^{(n)} & 0 \\ 0 & \lambda_s^{(n)} \end{pmatrix}, \qquad \begin{array}{l} |\lambda_u^{(n)}| > e^{n(\chi_u - \varepsilon)} \\ \phantom{|\lambda_u^{(n)}| > e^{n}} \gg 1 \\ |\lambda_s^{(n)}| < e^{n(\chi_s + \varepsilon)} \\ \phantom{|\lambda_s^{(n)}| < e^{n}} \ll 1 \end{array}$$