ZETA FUNCTIONS FOR THE RENEWAL SHIFT

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Abstract. We exhibit a topological Markov shift on a countable alphabet with the property that for every sequence of complex numbers c_n such that $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < \infty$ there exists a weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates such that the corresponding weighted dynamical zeta function satisfies $\frac{1}{\zeta_A(z)} = 1 + \sum_{i\geq 1} c_i z^i$.

1. Introduction

Let S be a countable set and $\mathbf{A} = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. S is called the set of *states.* $\mathbf{A}$ is called a *topological transition matrix* if $\forall a \in S\ \exists i, j\ \ (t_{ai} = t_{ja} = 1)$. If this is the case then one defines the (one sided) *countable Markov shift* generated by $\mathbf{A}$ to be

$$X = \Sigma_{\mathbf{A}}^{+} = \left\{x \in S^{\mathbb{N}\cup\{0\}} : \forall i\ t_{x_i x_{i+1}} = 1\right\}.$$

We endow this set with the metric $d(x, y) := (\frac{1}{2})^{\min\{n: x_n \neq y_n\}}$, and equip it with the action of the *left shift* map:

$$T : \Sigma_{\mathbf{A}}^{+} \to \Sigma_{\mathbf{A}}^{+},\ \ (Tx)_i = x_{i+1}.$$

Let $Fix(T^n) := \{x \in \Sigma_{\mathbf{A}}^{+} : T^n x = x\}$.

Let $A : X \to \mathbb{C}$ be some function, called a *weight function.* The *generalized dynamical zeta function*, for the weight function A is

$$\zeta_A(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in Fix T^n} \prod_{k=0}^{n-1} A(T^k x).$$

These functions were introduced (in a more general context) by Ruelle [8],[9], as a generalization of certain generating functions which were considered by Artin and Mazur [1].

If $|S| < \infty$ and A is regular enough (e.g., when $\log A$ is Hölder continuous), then ζ_A is holomorphic in a neighborhood of zero and its first pole is in e^{-P}, where P is the topological pressure of $\log A$ (see [9]). A series of studies have focused on meromorphic extensions of ζ_A to larger domains (see for example [8], [5], [7], [4]).

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We show here that if $|S| = \infty$ then no such results are possible, even if one restricts one attention to locally constant potentials. We do this by exhibiting a specific topological Markov shift with the following property: Every function f such that $f(0) = 1$, which is holomorphic in a neighborhood of zero, can be represented a dynamical zeta function for a suitable weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates.

This topological Markov shift is the shift with set of states $\mathbb{N}$ and transition matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

We call this shift the *renewal shift* because of its obvious connection to renewal theory (see [2]). We prove:

Theorem 1. *Let X be the renewal shift and $\{c_n\}_{n=1}^{\infty}$ a sequence of complex numbers such that $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < \infty$. There exists a function $A : X \to \mathbb{C}$ which depends only on the first two coordinates, for which in the neighborhood of zero*

$$\frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i.$$

In particular, any type of singular behavior can occur away from zero. This should be contrasted with the case $|S| < \infty$, for which every zeta function with a weight function of the form $A(x) = A(x_0, x_1)$ is rational [6]. We remark that the dynamical zeta functions without meromorphic extensions have been constructed before [3].

2. Proof of Theorem 1

Set

$$c_i^* = \begin{cases} c_i & c_i \neq 0 \\ 1 & c_i = 0 \end{cases}$$

and

$$\alpha_1 = c_1^* \;\; ; \;\; \alpha_i = c_i^*/c_{i-1}^*$$

$$\beta_1 = -c_1 \;\; ; \;\; \beta_i = -c_i/c_{i-1}^*.$$

Let $\mathbf{A} = (a_{ij})_{\mathbb{N}\times\mathbb{N}}$ be the matrix given by

$$\mathbf{A} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ \alpha_1 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_3 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

Let $\mathbf{A}_n$ be the upper left $n \times n$ block. Set $r = \left(\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}\right)^{-1}$. This number is positive or infinite, by the assumptions of the theorem.

Lemma 1. *The following limit holds and is uniform on compacts in* $D_r := \{z : |z| < r\}$*:*

$$\lim_{n\to\infty} \det(1 - z\mathbf{A}_n) = 1 + \sum_{i=1}^{\infty} c_i z^i \tag{2}$$

Proof.

$$\begin{aligned}
\det(1 - z\mathbf{A}_n) &= \begin{vmatrix} 1-\beta_1 z & -\beta_2 z & \cdots & -\beta_{n-1} z & -\beta_n z \\ -\alpha_1 z & 1 & 0 & \cdots & 0 \\ 0 & -\alpha_2 z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\alpha_{n-1} z & 1 \end{vmatrix} \\
&= (1-\beta_1 z) \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\alpha_2 z & 1 & \cdots & 0 & 0 \\ 0 & -\alpha_3 z & \ddots & \vdots & 0 \\ \vdots & \vdots & & 1 & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1} z & 1 \end{vmatrix} \\
&\quad -(-\beta_2 z) \begin{vmatrix} -\alpha_1 z & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & -\alpha_2 z & \ddots & \vdots & 0 \\ & & & 1 & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1} z & 1 \end{vmatrix} + \dots \\
&\quad +(-1)^{n+1}(-\beta_n z) \begin{vmatrix} -\alpha_1 z & 1 & 0 & \cdots & 0 \\ 0 & -\alpha_2 z & 1 & & 0 \\ & 0 & -\alpha_3 z & & \vdots \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} z \end{vmatrix}
\end{aligned}$$

$$\begin{aligned} &= 1-\beta_1 z-\beta_2\alpha_1 z^2-\ldots-\beta_n\alpha_1\cdot\ldots\cdot\alpha_{n-1}z^n \\ &= 1+c_1z+\frac{c_2}{c_1^*}\cdot c_1^*\cdot z^2+\ldots+\frac{c_n}{c_{n-1}^*}\cdot c_1^*\cdot\frac{c_2^*}{c_1^*}\cdot\ldots\cdot\frac{c_{n-1}^*}{c_{n-2}^*}\cdot z^n \\ &= 1+c_1z+\ldots+c_nz^n \xrightarrow[n\to\infty]{} 1+\sum_{i=1}^{\infty}c_iz^i. \end{aligned}$$

This convergence is uniform on compacts in D_r, because r is the radius of convergence of this power series. □

Lemma 2. $E:=\{\lambda\in\mathbb{C}:\exists n\ \det(\lambda 1-\mathbf{A}_n)=0\}$ *is a bounded subset of* $\mathbb{C}$.

Proof. Else, $\exists\ n_k\nearrow\infty$ and $|\lambda_{n_k}|\to\infty$, such that $\det(\lambda_{n_k}1-\mathbf{A}_{n_k})=0$. Without loss of generality, assume that $\forall k\ \ |\lambda_{n_k}|\geq\frac{2}{r}$ (if $r=\infty$ assume that $|\lambda_{n_k}|\geq 1$).

According to the previous lemma, the following limit exists and is uniform on compacts in $D_r=\{z:|z|<r\}$:

$$f(z)=\lim_{n\to\infty}\det(1-z\mathbf{A}_n) \tag{3}$$

Note that $f(0)=1$, and that f is continuous in 0. In particular, since $\lambda_{n_k}^{-1}\to 0$ and $\lambda_{n_k}\in D_r$

$$\left|f(0)-f(\lambda_{n_k}^{-1})\right|\xrightarrow[k\to\infty]{}0.$$

By the uniform convegence of (3) in $\overline{D}_{r/2}$ (or in $\overline{D}_1$ if $r=\infty$) we have that

$$\left|f(\lambda_{n_k}^{-1})-\det(1-\lambda_{n_k}^{-1}\mathbf{A}_{n_k})\right|\xrightarrow[k\to\infty]{}0$$

Hence, since $\forall k\ \ \det(1-\lambda_{n_k}^{-1}\mathbf{A}_{n_k})=0$,

$$|f(0)-0|\leq\left|f(0)-f(\lambda_{n_k}^{-1})\right|+\left|f(\lambda_{n_k}^{-1})-\det(1-\lambda_{n_k}^{-1}\mathbf{A}_{n_k})\right|\xrightarrow[k\to\infty]{}0$$

which implies that $1=f(0)=0$, a contradiction. □

We are now ready to prove the theorem. Let $A:X\to\mathbb{C}$ be given by

$$A(x_0,x_1,\ldots)=\mathbf{A}_{x_0x_1}$$

where $\mathbf{A}$ is given by (1).

Set

$$Z_n=\sum_{x\in Fix(T^n)}\prod_{k=0}^{n-1}A(T^kx)$$

Then

$$\log\zeta_A=\sum_{n=1}^{\infty}\frac{z^n}{n}\cdot Z_n.$$

By the definition of A,

$$Z_n = \sum_{x\in Fix(T^n)} \mathbf{A}_{x_0x_1}\mathbf{A}_{x_1x_2}\cdot\ldots\cdot\mathbf{A}_{x_{n-1}x_0}.$$

$\forall x_0,\ldots,x_{n-1}\in\mathbb{N}$ if $\mathbf{A}_{x_0x_1}\mathbf{A}_{x_1x_2}\cdot\ldots\cdot\mathbf{A}_{x_{n-1}x_0}>0$ then

$$(x_0,x_1,\ldots,x_{n-1};x_0,x_1,\cdots,x_{n-1};\ldots)$$

belongs to $\Sigma^+_{\mathbf{A}}$ and constitutes a periodic point of order n. Thus

$$Z_n = \sum_{x\in Fix(T^n)}\prod_{i=0}^{n-1}A(T^ix) = \sum_{x_1\cdots x_n}\mathbf{A}_{x_0\,x_1}\cdot\ldots\cdot\mathbf{A}_{x_{n-1}\,x_0}.$$

By the definition of the renewal shift, if $(x_0,x_1,\ldots,x_{n-1},x_0)$ is admissible then $\forall i\ x_i\le n$ (if m appears, so must $m-1,m-2,\ldots,1$. Since there are at the most n different symbols x_i, m must be smaller than n). Thus,

$$\forall n\le N: Z_n = \sum_{x_0\ldots x_{n-1}=1}^{n}\mathbf{A}_{x_0\,x_1}\cdot\ldots\mathbf{A}_{x_{n-1}\,x_0} =$$

$$= \sum_{x_0\ldots x_{n-1}=1}^{N}\mathbf{A}_{x_0\,x_1}\cdot\ldots\mathbf{A}_{x_{n-1}\,x_0} = tr(\mathbf{A}_N^n).$$

This shows that

$$\left|\sum_{n=1}^{\infty}\frac{z^n}{n}\cdot Z_n - \sum_{k=1}^{\infty}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)\right| \le \left|\sum_{n>N}\frac{z^n}{n}\cdot Z_n\right| + \left|\sum_{n>N}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)\right|.$$

We estimate these tails. According to the previous lemma, $E=\{\lambda\in\mathbb{C}:\exists n\ \det(\lambda 1-\ \mathbf{A}_n)=0\}$ is bounded. Let $\lambda=\sup\{|z|:z\in E\}$. Let $\lambda_1(k),\ldots,\lambda_k(k)$ the eigenvalues of $\mathbf{A}_k$, written with multiplicities. Then $|\lambda_i(k)|\le\lambda$. Using the fact that every matrix can be triangulated, it is easy to verify that

$$|tr\,(\mathbf{A}_k^n)| = |\lambda_1\,(k)^n+\ldots+\lambda_k\,(k)^n| \le k\lambda^n$$

Thus, for every $|z|<\lambda^{-1}$,

$$\left|\sum_{n>N}\frac{z^n}{n}\cdot Z_n\right| = \left|\sum_{n>N}\frac{z^n}{n}\cdot tr(\mathbf{A}_n^n)\right|$$

$$\le \sum_{n>N}\frac{|z^n|}{n}\cdot n\lambda^n = \sum_{n>N}|z\cdot\lambda|^n \xrightarrow[N\to\infty]{} 0$$

and
$$\left|\sum_{n>N}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)\right|\le\sum_{n>N}|z\cdot\lambda|^n \xrightarrow[N\to\infty]{} 0.$$
Thus, $\forall\,|z|<\lambda^{-1}$
$$\left|\sum_{n=1}^{\infty}\frac{z^n}{n}\cdot Z_n-\sum_{k=1}^{\infty}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)\right|$$
$$\le\left|\sum_{n>N}\frac{z^n}{n}\cdot Z_n\right|+\left|\sum_{n>N}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)\right| \xrightarrow[N\to\infty]{} 0.$$

Using the Taylor expansion of $z\mapsto\log(1-z)$ and the identities
$$tr(\mathbf{A}_N^n)=\lambda_1(N)^n+\ldots+\lambda_N(N)^n$$
and
$$\det(1-z\mathbf{A}_N)=(1-z\lambda_1(N))\cdot\ldots\cdot(1-z\lambda_N(N))$$
it is not difficult to show that if $|z|<\lambda^{-1}$ then
$$-\sum_{n=1}^{\infty}\frac{z^n}{n}\cdot tr(\mathbf{A}_N^n)=\ln\det(1-z\mathbf{A}_N)$$
Thus, the following limit holds in $D_{\lambda^{-1}}$
$$\ln\det(1-z\mathbf{A}_N)\xrightarrow[N\to\infty]{} -\log\zeta_A(z).$$
But by (2) if $|z|<r$ then
$$\det(1-z\mathbf{A}_N)\xrightarrow[N\to\infty]{} 1+\sum_{i=1}^{\infty}c_iz^i$$
Hence, for $|z|<\min\{r,\lambda^{-1}\}$ we have
$$\frac{1}{\zeta_A(z)}=1+\sum_{i=1}^{\infty}c_iz^i$$
as required. □

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