

# Subgraphs in random networks

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Understanding the subgraph distribution in random networks is important for modelling complex systems. In classic Erdős networks, which exhibit a Poissonian degree distribution, the number of appearances of a subgraph  $G$  with  $n$  nodes and  $g$  edges scales with network size as  $\langle G \rangle \sim N^{n-g}$ . However, many natural networks have a non-Poissonian degree distribution. Here we present mean-field equations for the average number of subgraphs in an ensemble of random sparse directed networks, characterized by an arbitrary degree sequence. We find new scaling rules for the commonly occurring case of directed scale-free networks, in which the outgoing degree distribution scales as  $P(k) \sim k^{-\gamma}$ . Considering the power exponent of the degree distribution,  $\gamma$ , as a control parameter, we show that random networks exhibit phase transitions between three regimes, where in each regime the subgraph number of appearances follows a different scaling law,  $\langle G \rangle \sim N^\alpha$ , where  $\alpha = n - g + m - 1$  for  $\gamma < 2$ ,  $\alpha = n - g + m + 1 - \gamma$  for  $2 < \gamma < \gamma_c$ , and  $\alpha = n - g$  for  $\gamma > \gamma_c$ , where  $m$  is the maximal outdegree in the subgraph, and  $\gamma_c = m + 1$ . We find that certain subgraphs appear much more frequently than in Erdős networks. These results are in very good agreement with numerical simulations. This has implications for detecting network motifs, subgraphs that occur in natural networks significantly more than in their randomized counterparts.

PACS numbers: 05, 89.75

## I. INTRODUCTION

Many natural systems are described as networks of interacting components ([1]-[7]). Random networks have been studied as models of these complex systems. The classic model for a random network is the Erdős model, in which each of the possible edges in the network exists with probability  $p$ . There exists an analytical solution to many of the properties of Erdős networks, such as the diameter, clustering coefficient, component size distributions, and subgraph distributions ([8]-[11]). The average number of appearances  $G$  of a subgraph with  $n$  nodes and  $g$  edges in a directed network of  $N$  nodes is

$$\langle G \rangle = \lambda \binom{N}{n} p^g (1-p)^{n(n-1)/2-g} \sim \lambda N^n \left( \frac{\langle K \rangle}{N} \right)^g$$

$$\langle G \rangle \sim N^{n-g} \quad (1)$$

assuming a fixed mean connectivity  $\langle K \rangle = pN$ .  $\lambda$  is a term of order 1 which stems from the symmetry of each subgraph. Erdős networks have been extensively used as models for analyzing real networks. An excellent example is the work of Davis, Holland and Leinhardt on subgraphs in social networks ([12]-[15]).

Erdős networks exhibit a Poissonian degree distribution: the distribution of the number of edges per node is  $P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$ . Nodes with a number of edges much higher than the mean are exponentially rare. Many naturally occurring networks, on the other hand, obey a long-tailed degree sequence, often described as a power law,  $P(k) \sim k^{-\gamma}$ , with  $\gamma$  often between 2 and 3 ([16]-[25]). These networks, termed scale-free networks, are characterized by the existence of nodes with high de-

gree, termed hubs (Fig1). The existence of hubs dramatically influences the properties of these networks. Some of the global properties of random networks with arbitrary degree distribution, and specifically scale-free networks, have been calculated from the network degree sequence. These include sizes of connected components ([3],[26],[27]), percolation thresholds [29],[30] and clustering coefficients ([31],[32],[33]).

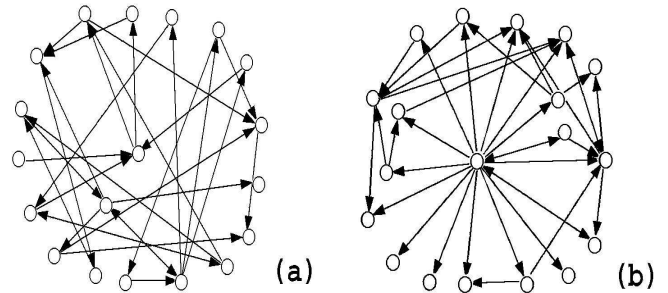


FIG. 1: Example of (a) Erdős network and (b) Scale-free network ( $\gamma = 2$ ). Mean connectivity is 1.85 in both. Notice the hub in the scale-free network.

There is much current interest in the local structure of networks ([5],[7],[31],[34]-[37]). Recently subgraph structure was analyzed in biochemical and technological networks ([38],[39]). It was found that these natural or designed networks contain network motifs, subgraphs that occur much more often than in an ensemble of randomized networks with the same degree sequence ([38],[39]). In these studies, random networks generation and the

enumeration of their subgraphs were performed numerically. To complement this numerical work, it would be important to theoretically characterize the subgraph distribution of random networks. Here we present approximate formulas for the average number of subgraphs in an ensemble of random networks with an arbitrary degree distribution. We apply these formulas to arrive at new scaling laws for networks with a scale-free degree distribution. We find that each subgraph has its own scaling exponent, influenced by its topology. Considering the power exponent of the degree distribution,  $\gamma$ , as a control parameter, we show that the random networks exhibit phase transitions between 3 regimes, where in each regime the subgraph number of appearances follows a different scaling law. We find that certain subgraphs appear much more frequently than in Erdős networks.

## II. NUMBER OF SUBGRAPHS, MEAN-FIELD SOLUTION

The following approximation assumes sparse networks ( $\langle K \rangle \ll N$ ). The network degree sequence is given by the outdegree  $\{K_i\}_{i=1}^N$  (the number of edges outgoing from each node), indegree  $\{R_i\}_{i=1}^N$  (the number of incoming edges at each node), and double degree  $\{D_i\}_{i=1}^N$  (the number of edges which are both incoming and outgoing, termed "double edges", at each node). We begin by computing the probability of obtaining a subgraph with  $g_s$  single edges,  $g_d$  double edges, subgraph outdegree sequence  $\{m_j\}_{j=1}^n$ , subgraph indegree sequence  $\{l_j\}_{j=1}^n$  and subgraph double degree sequence  $\{d_j\}_{j=1}^n$  in a given set of nodes. Consider the example of (Fig 2). The probability of obtaining a directed edge from node 1 to node 2 is approximately

$$P(\text{edge1}) = \frac{K_1 R_2}{N \langle K \rangle} \quad (2)$$

assuming  $K_1 R_2 \ll N \langle K \rangle$  (see Appendix A). This assumption is valid as long as there are few edges between very highly connected nodes. The probability of obtaining a second edge from node 1 to node 3 is :

$$P(\text{edge2}|\text{edge1}) = \frac{(K_1 - 1) R_3}{N \langle K \rangle} \quad (3)$$

This reasoning applies to all the subgraph edges. The mean number of appearances of a subgraph is found by taking the average of the resulting expression with respect to all choices of  $n$  distinct nodes  $\{\sigma_1 \dots \sigma_n\}$ , and multiplying by the number of possible choices of  $n$  nodes out of  $N$ :

$$\langle G \rangle = \frac{a N^{n-g_s-g_d}}{\langle K \rangle^{g_s} \langle D \rangle^{g_d}} \left\langle \prod_{j=1}^n \binom{K_{\sigma_j}}{m_j} \binom{R_{\sigma_j}}{l_j} \binom{D_{\sigma_j}}{d_j} \right\rangle_{\{\sigma\}} \quad (4)$$

Where  $\langle K \rangle$  is the average outdegree (equals the average

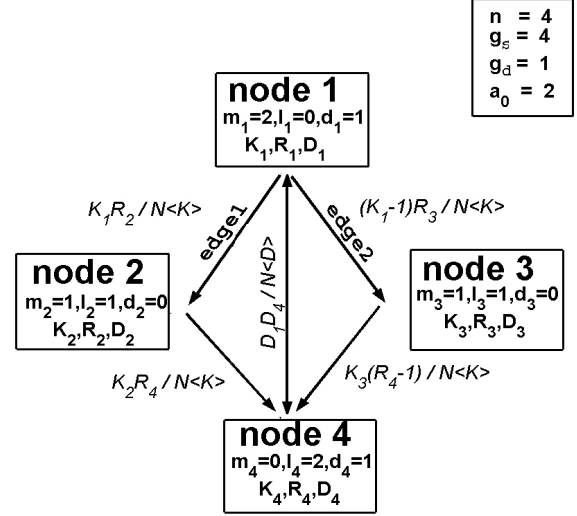


FIG. 2: A subgraph with one double edge and 4 single edges. The subgraph degree sequences  $\{m_i, l_i, d_i\}$  and node degrees  $\{K_i, R_i, D_i\}$  are displayed in bold. Edge probabilities are displayed in plain. Using Eq. (5), the mean subgraph number of appearances in an ensemble of random networks is  $\langle G \rangle = 2 \langle K(K-1)D \rangle \langle R(R-1)D \rangle \langle RK \rangle^2 / N \langle K \rangle^4 \langle D \rangle$

indegree  $\langle R \rangle$ ), and  $\langle D \rangle$  is the average double edge degree. The symmetry factor  $a$  is  $a_0^{-1} \prod_{j=1}^n m_j! l_j! d_j!$ , where  $a_0$  is the number of different permutations of the nodes that give an isomorphic subgraph.

The average (4) reduces to a product of moments of different orders of the indegree, outdegree and double degree distributions:

$$\langle G \rangle = \frac{a N^{n-g_s-g_d}}{\langle K \rangle^{g_s} \langle D \rangle^{g_d}} \prod_{j=1}^n \left\langle \binom{K_i}{m_j} \binom{R_i}{l_j} \binom{D_i}{d_j} \right\rangle_i \quad (5)$$

where the fact that each node should participate in the summation of only one term  $j$ , introduces higher order corrections, which we neglect.

For example, subgraph id102 (Table 1), has  $n=3$  nodes,  $g_s = 2$  single edges and  $g_d = 1$  double edge. The subgraph degree sequences are  $m_j = \{1, 1, 0\}$ ,  $l_j = \{0, 1, 1\}$ , and  $d_j = \{1, 0, 1\}$ . Using (5) we find :

$$\langle G \rangle = \langle id102 \rangle = \frac{\langle KD \rangle \langle RD \rangle \langle RK \rangle}{\langle K \rangle^2 \langle D \rangle} \quad (6)$$

The mean field approximation (Eq. 5) is exact in the case of Erdős networks. In Erdős networks, both indegree and outdegree are Poisson distributed and independent, and Eq. (5) reduces to Eq. (1).

We tested the mean field equations on random networks taken with the degree sequence of real world networks - transcription interactions in the yeast *S. cerevisiae* ([39]), synaptic connections between neurons in

| subgraph | id | equation   | transcription              | neurons                    | www                         |
|----------|----|--|----------------------------|----------------------------|-----------------------------|
| 6        |    | $N \langle K(K-1) \rangle / 2$   | $1.2 \times 10^4 (+0.2\%)$ | $4.6 \times 10^2 (+6\%)$   | $4.8 \times 10^7 (+0.2\%)$  |
| 12       |    | $N \langle KR \rangle$   | $3.5 \times 10^2 (+4\%)$   | $9 \times 10^2 (-10\%)$    | $2.8 \times 10^6 (+3\%)$    |
| 14       |    | $N \langle KD \rangle$   | $1.8 \times 10^1 (+1\%)$   | $8.9 \times 10^1 (-40\%)$  | $3.8 \times 10^6 (+0.5\%)$  |
| 36       |    | $N \langle R(R-1) \rangle / 2$   | $9.5 \times 10^2 (+2\%)$   | $6.1 \times 10^3 (-0.1\%)$ | $2.2 \times 10^8 (+0.04\%)$ |
| 38       |    | $\langle K(K-1) \rangle \langle RK \rangle \langle R(R-1) \rangle / \langle K \rangle^3$               | $1.4 \times 10^1 (-3\%)$   | $9.5 \times 10^1 (+10\%)$  | $8.7 \times 10^4 (+290\%)$  |
| 46       |    | $\langle KD \rangle^2 \langle R(R-1) \rangle / 2 \langle K \rangle^2 \langle D \rangle$                | $1 \times 10^{-1} (+50\%)$ | 4 (-11%)                   | $9.2 \times 10^3 (-8\%)$    |
| 74       |    | $N \langle RD \rangle$   | $2.9 \times 10^1 (+4\%)$   | $1.25 \times 10^2 (-29\%)$ | $4.8 \times 10^6 (+0.1\%)$  |
| 78       |    | $N \langle D(D-1) \rangle / 2$   | 0(0%)                      | 6.5 (+8%)                  | $2.5 \times 10^7 (+0.6\%)$  |
| 98       |    | $\langle KR \rangle^3 / 3 \langle K \rangle^3$   | 0(0%)                      | 2.6 (-25%)                 | $2.5 \times 10^1 (-10\%)$   |
| 102      |    | $\langle KD \rangle \langle RD \rangle \langle RK \rangle / \langle K \rangle^2 \langle D \rangle$     | 0(0%)                      | 1.7 (-58%)                 | $1.3 \times 10^2 (+4\%)$    |
| 108      |    | $\langle RD \rangle^2 \langle K(K-1) \rangle / 2 \langle K \rangle^2 \langle D \rangle$                | 0(0%)                      | 1.3 (-49%)                 | $1.6 \times 10^3 (+80\%)$   |
| 110      |    | $\langle KD \rangle \langle RD \rangle \langle D(D-1) \rangle / \langle K \rangle \langle D \rangle^2$ | 0(0%)                      | $4 \times 10^{-1} (-50\%)$ | $2.2 \times 10^3 (+4\%)$    |
| 238      |    | $\langle D(D-1) \rangle^3 / 6 \langle D \rangle^3$   | 0(0%)                      | 0(0%)                      | $4.8 \times 10^4 (+4\%)$    |

TABLE I: Mean numbers of the thirteen connected directed subgraphs in an ensemble of random networks with a given degree distribution. The degree distributions are those of transcription in the yeast *S. cerevisiae* ([39]), synaptic connections between neurons in *C. elegans* ([42]), and world-wide-web hyperlinks between web pages in a single domain ([16]). Shown are direct enumeration results - using the algorithms described in [39], where 1000 random networks with the same degree distributions as those of the real networks were generated and all subgraphs were counted. The value in parentheses is the percent deviation of the theoretical values (Eq. 5). Values below 0.1 were rounded to zero. Subgraph id is determined by concatenating the rows of the subgraph adjacency matrix and representing the resulting vector as a binary number. The id is the minimal number obtained from all the isomorphic versions of the subgraph.

*C. elegans* ([42]) and world-wide-web hyperlinks between web pages in a single domain ([16]). The mean field equations (5) are within a few percent of the numerical simulation results for most subgraphs (Table 1). There are some discrepancies (most notably a factor of almost 4 for subgraph id38 in the randomized world wide web network). These discrepancies are mainly due to the approximations made in Eq. (2,3), see Appendix A.

In addition, we find good agreement between our approximation and numerical enumeration of subgraphs in simulated random networks with scale-free outdegree and compact indegree (Fig. 3).

### III. SCALE-FREE NETWORKS

Scale-free networks have degree distribution that follows  $P(k) \sim k^{-\gamma}$  at large  $k$  ([16]-[25]). We consider directed networks in which the outgoing edge degree is scale-free, while the incoming edge degree distribution is Poissonian. Our results can be easily extended to scale-

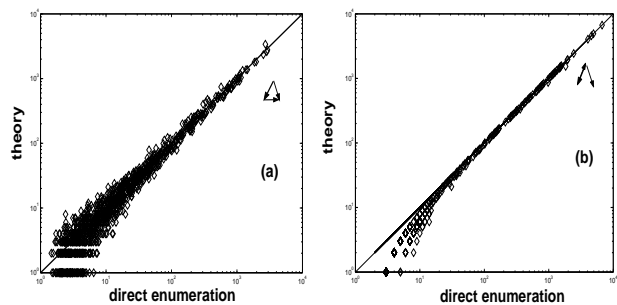


FIG. 3: Subgraph numbers in 1000 random networks with  $N=2000$  nodes, with scale-free outdegree and compact indegree. The outdegree of each node,  $K_i$  was picked from the distribution (7), with  $\gamma = 2$ . The networks were constructed using the algorithm of Newman, Strogatz and Watts [3] modified so that only a single edge in a given direction is allowed between any two nodes. Theoretical number of appearance were computed using the degree sequences of each network (equations in Table 1).

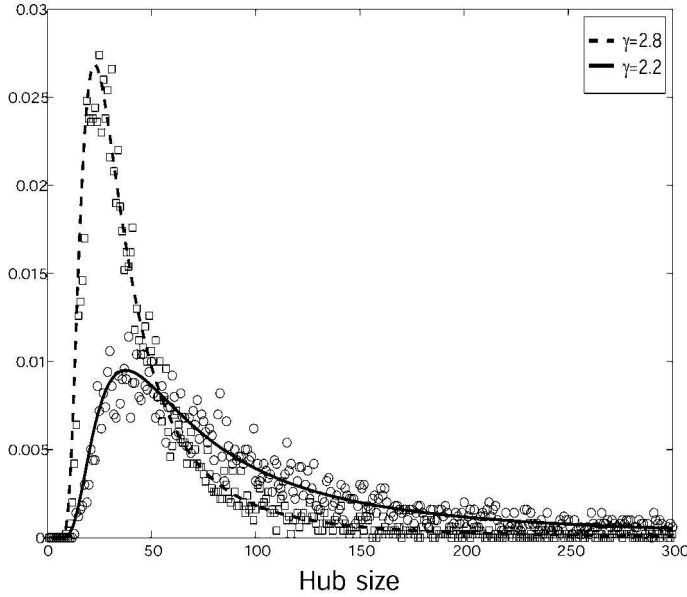


FIG. 4: Simulated and theoretical hub distribution for networks with  $N=3000$  nodes,  $\gamma = 2.2$  or  $\gamma = 2.8$ , and mean connectivity  $\langle K \rangle = 1.2$ . Lines : theoretical calculations, Eq. (8).

free indegree. For simplicity we choose the following form for the outgoing degree distribution for a network with  $N$  nodes:

$$P(k) = \frac{\gamma - 1}{k_0^{1-\gamma}} (k + k_0)^{-\gamma} \quad k < N \quad (7)$$

The mean connectivity  $\langle K \rangle$  is determined by  $k_0$ .

The hub is the node with the maximal number of outgoing edges,  $T$ . The hub size distribution (Fig. 4) is :

$$P(T) = NP(k = T)[P(k \leq T)]^{N-1} = \frac{N(\gamma - 1)}{k_0} (T/k_0)^{-\gamma} \left(1 - (T/k_0)^{-\gamma+1}\right)^{N-1} \quad (8)$$

assuming  $T \gg k_0$ . For  $2 < \gamma < 3$ , the mean hub scales as:

$$\langle T \rangle = \int_0^{N-1} TP(T)dT \sim N^{3-\gamma} \quad (9)$$

where the mean is over an ensemble of random networks with the same  $\gamma$  and mean connectivity. At  $\gamma \leq 2$ , there is a condensation effect [41], where a finite fraction of the nodes have outdegree  $\leq 1$ , and the mean hub size becomes proportional to  $N$ . Using (5), and assuming a compact distribution for the number of double edges, we find that the subgraph distribution is dominated by the hubs, and that the dominant term is that of the subgraph node with maximal outdegree,  $m$ . The number of

appearances of each subgraph can be shown to scale as :

$$\langle G \rangle \sim aN^{n-g-1} \langle K \rangle^{g-m} \sum_{i=1}^N \binom{K_i}{m} \sim N^\alpha \quad (10)$$

where  $g = g_s + 2g_d$  is the total number of edges in the subgraph [43]. We derive the scaling exponent  $\alpha$  in the following section.

#### IV. PHASE TRANSITIONS AT DIFFERENT $\gamma$

The subgraph numbers scale as

$$\langle G \rangle \sim N^\alpha \quad (11)$$

We find three different regimes, in each of which the scaling exponent  $\alpha$  behaves differently. Taking an ensemble average by integrating the largest term in Eq. (10) over the hub distribution (8) we get:

$$\langle G \rangle \sim N^{n-1-g} \int_1^{N-1} T^m P(T) dT \quad (12)$$

At  $\gamma \leq 2$  the network is in a condensed regime, where the hub  $T = O(N)$ . In this regime :

$$\langle G \rangle \sim N^{n-g+m-1} \quad (13)$$

For  $2 < \gamma < \gamma_c$  substituting Eq. 8 in Eq. 12 yields :

$$\langle G \rangle \sim N^{n-g+m-\gamma-1} \quad (14)$$

In this regime, the tail of  $P(T)$  is the dominant contribution to the integral. Finally at values above a critical  $\gamma$ , another transition occurs, where  $\alpha$  equals the scaling exponent in Erdős networks,  $\alpha = n - g = \alpha_{erdos}$ . The critical  $\gamma$  is  $\gamma_c$ :

$$\gamma_c = m + 1 \quad (15)$$

In this regime, the hubs no longer contribute significantly to the subgraph distribution. In summary,  $\langle G \rangle \sim N^\alpha$ , where  $\alpha$  is :

$$\alpha = \begin{cases} n - g + m - 1 & \gamma \leq 2 \\ n - g + m - \gamma + 1 & 2 < \gamma < m + 1 \\ n - g & \gamma \geq m + 1 \end{cases} \quad (16)$$

Table 2 shows the expected scaling exponent for the 13 connected directed 3-node subgraphs, as well as for several 4-node subgraphs. The scaling laws agree very well with numerical results (Fig. 5). The three regimes of scaling are clearly seen. Note that the topology of each subgraph effects its scaling, through the subgraph maximal outdegree,  $m$ . These results can be easily extended to the case of scale-free indegree.

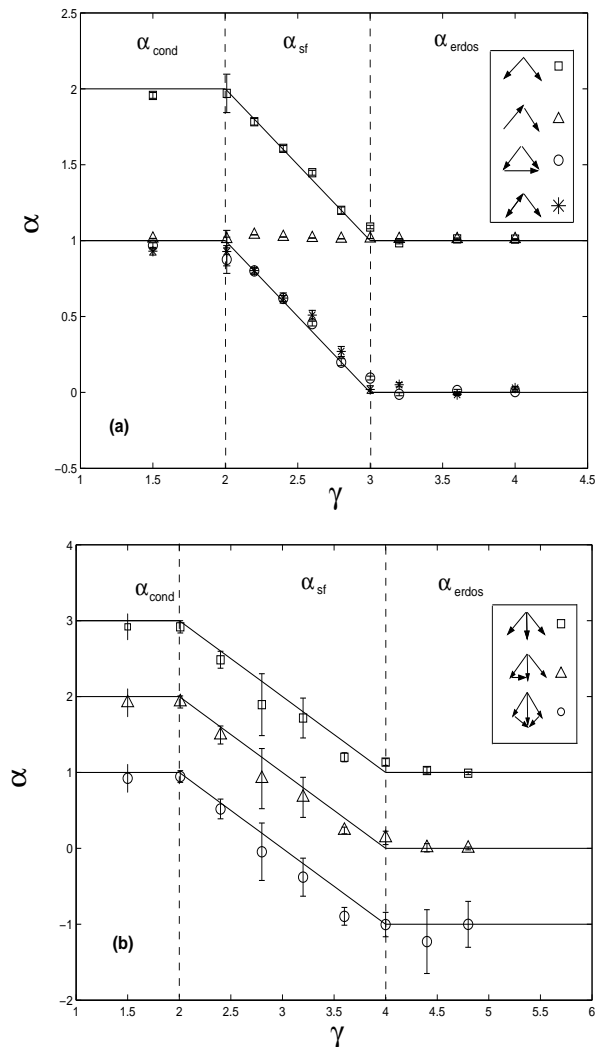


FIG. 5: Scaling exponent of 3-node subgraphs (a) and 4-node subgraphs (b) as a function of  $\gamma$ . The exponent  $\alpha$  was obtained from the slope of a log-log fit of the number of subgraphs vs. network size, for 9 different network size (30,100,300,500,1000,1500,2000,2500,3000) averaged over 5000 randomized networks for each size and outdegree powerlaw  $\gamma$ . All the networks had mean connectivity  $\langle K \rangle = 1.2$ . The exponent  $\alpha$  displays three regimes,  $\gamma < 2$  (the condensed regime),  $2 < \gamma < \gamma_c$  (the scale-free regime),  $\gamma > \gamma_c$  (Erdős regime).

## V. DISCUSSION

To summarize, we have presented a mean-field solution for the average number of directed connected subgraphs in an ensemble of random networks with arbitrary degree sequence. We have presented scaling formulas for the number of subgraphs in scale-free random networks, and showed that the subgraph numbers can be very different from those in Erdős random networks. Whereas in Erdős random networks the scaling exponent is strictly

| subgraph | id | n nodes | g edges | m | $\alpha_{erdos}$<br>$\gamma > \gamma_c$ | $\alpha_{sf}$<br>$2 < \gamma < \gamma_c$ | $\alpha_{cond}$<br>$\gamma \leq 2$ | $\gamma_c$ |
|----------|----|---------|---------|---|---|--|------------------------------------|------------|
| 6        |    | 3       | 2       | 2 | 1                                       | $4 - \gamma$                             | 2                                  | 3          |
| 12       |    | 3       | 2       | 1 | 1                                       | 1  | 1                                  | 2          |
| 14       |    | 3       | 3       | 2 | 0                                       | $3 - \gamma$                             | 1                                  | 3          |
| 36       |    | 3       | 2       | 1 | 1                                       | 1  | 1                                  | 2          |
| 38       |    | 3       | 3       | 2 | 0                                       | $3 - \gamma$                             | 1                                  | 3          |
| 46       |    | 3       | 4       | 2 | -1                                      | $2 - \gamma$                             | 0                                  | 3          |
| 74       |    | 3       | 3       | 1 | 0                                       | 0  | 0                                  | 2          |
| 78       |    | 3       | 4       | 2 | -1                                      | $2 - \gamma$                             | 0                                  | 3          |
| 98       |    | 3       | 3       | 1 | 0                                       | 0  | 0                                  | 2          |
| 102      |    | 3       | 4       | 2 | -1                                      | $2 - \gamma$                             | 0                                  | 3          |
| 108      |    | 3       | 4       | 2 | -1                                      | $2 - \gamma$                             | 0                                  | 3          |
| 110      |    | 3       | 5       | 2 | -2                                      | $1 - \gamma$                             | -1                                 | 3          |
| 238      |    | 3       | 6       | 2 | -3                                      | $-\gamma$                                | -2                                 | 3          |
| subgraph | id | n nodes | g edges | m | $\alpha_{erdos}$<br>$\gamma > \gamma_c$ | $\alpha_{sf}$<br>$2 < \gamma < \gamma_c$ | $\alpha_{cond}$<br>$\gamma \leq 2$ | $\gamma_c$ |
| 14       |    | 4       | 3       | 3 | 1                                       | $5 - \gamma$                             | 3                                  | 4          |
| 204      |    | 4       | 4       | 2 | 0                                       | $3 - \gamma$                             | 1                                  | 3          |
| 206      |    | 4       | 5       | 3 | -1                                      | $3 - \gamma$                             | 1                                  | 4          |
| 2190     |    | 4       | 5       | 3 | -1                                      | $3 - \gamma$                             | 1                                  | 4          |

TABLE II: the scaling exponent  $\alpha$  of subgraph numbers for random scale-free networks with outgoing degree exponent  $\gamma$ . The subgraph numbers scale as  $\langle G \rangle \sim N^\alpha$ . Shown are all thirteen 3-node connected directed subgraphs and 4 examples of 4 node subgraphs. n is the number of nodes in the subgraph, g, the number of edges and m, the maximal degree within the subgraph. The exponent  $\alpha$  has 3 regimes :  $\alpha_{erdos}$  in the "Erdős regime", when  $\gamma > \gamma_c$ ,  $\alpha_{sf}$  in the "scale-free regime", when  $2 < \gamma < \gamma_c$ , and  $\alpha_{cond}$  in the "condensed regime", when  $\gamma \leq 2$ .

determined by the number of nodes and edges of the subgraph, in scale-free random networks, the exact topology of the subgraph determines the scaling exponent. We showed that the scaling exponent  $\alpha$  exhibits three different scaling laws in three regimes, depending on the control parameter  $\gamma$  (the power of the degree distribution). In the common case of scale-free networks with  $\gamma$  between 2 and 3, there are many more subgraphs which

contain a node connected to more than one other node than in the corresponding Erdős networks with the same mean connectivity. For example, the feed-forward loop, (id38 in Table 1) is much more common for  $\gamma < 3$ . At  $\gamma = 2.5$ , the number of feed-forward-loops scales as  $N^{0.5}$ , as opposed to  $N^0$  in Erdős networks. On the other hand, subgraphs such as the 3-node cycle (id98 in Table 1) have the same scaling,  $N^0$ , as in Erdős networks.

This study adds to our understanding of the random network models to which real-world networks are compared. It highlights the importance of using random networks that preserve the single and double degree sequence of the real network. These results may be useful for enumerating subgraphs in very large random networks which are beyond the reach of current numerical algorithms.

## APPENDIX A: EDGE PROBABILITIES

Here we give a more detailed derivation for the edge probabilities used in Eq. (2,3). Without loss of generality we treat a network with single edges only. We denote by  $E = N \langle K \rangle$  the total number of edges. We begin by calculating the probability that no edge connects a source node with  $K$  outgoing edges and a target node with  $R$  incoming edges. This happens when all  $K$  edges connect to a set of nodes  $\{\sigma_i\}_{i=1}^k$  which does not contain the target node:

$$p(\text{no edge}|\{\sigma_i\}) = \prod_{k=0}^{K-1} \left( 1 - \frac{R}{E - R' - \sum_{i=1}^k R_{\sigma_i}} \right) \quad (\text{A1})$$

where  $R'$  is the indegree of the source node (we do not allow self edges). The probability of having no edge is obtained by summing over all possible sets  $\{\sigma_i\}_{i=1}^k$ :

$$p(\text{no edge}) = \frac{1}{K! \binom{N-2}{K}} \sum_{\{\sigma\}} \prod_{k=0}^{K-1} \left( 1 - \frac{R}{E - R' - \sum_{i=1}^k R_{\sigma_i}} \right) \quad (\text{A2})$$

Assuming  $\max \sum_{i=1}^k R_{\sigma_i} \ll E$ , and taking the complement as the probability of an edge existing, we obtain:

$$p(\text{edge}) = 1 - \left( 1 - \frac{R}{N \langle K \rangle} \right)^K = 1 - e^{-KR/N \langle K \rangle} \sim \frac{KR}{N \langle K \rangle} \quad (\text{A3})$$

where our last approximation assumes  $KR \ll N \langle K \rangle$ . Intuitively, this result can be understood as  $K$  attempts for the source node to connect to the target node with a probability of  $R/N \langle K \rangle$  at each attempt.  $R/N \langle K \rangle$  is the probability of an arbitrary edge connecting into the target node. Nodes in which  $KR$  is of the order of  $N \langle K \rangle$  will contribute multiple edges in the same direction in the approximation, which are not allowed in the simulations,

| subgraph | id | formula   |
|----------|----|---|
| 6        |    | $(\sum A' A \cdot \tilde{M} \cdot \tilde{M}' - \text{tr} A' A)/2$ |
| 12       |    | $\sum A^2 \cdot \tilde{M} \cdot \tilde{M}'$                       |
| 14       |    | $\sum S A \cdot \tilde{M} \cdot \tilde{M}'$                       |
| 36       |    | $(\sum A A' \cdot \tilde{M} \cdot \tilde{M}' - \text{tr} A A')/2$ |
| 38       |    | $\sum A^2 \cdot A$  |
| 46       |    | $(\sum A A' \cdot S)/2$   |
| 74       |    | $\sum S A' \cdot \tilde{M} \cdot \tilde{M}'$                      |
| 78       |    | $(\sum S^2 \cdot \tilde{M} \cdot \tilde{M}' - \text{tr} S^2)/2$   |
| 98       |    | $(\sum A'^2 \cdot A)/3$   |
| 102      |    | $\sum A^2 \cdot S$  |
| 108      |    | $(\sum A' A \cdot S)/2$   |
| 110      |    | $\sum S^2 \cdot A$  |
| 238      |    | $(\sum S^2 \cdot S)/6$  |

TABLE III: Matrix formulas for the numbers of all 3-node connected directed subgraphs.  $M$  is the adjacency matrix,  $S$  is its symmetric component, and  $A$  its asymmetric component.  $A'$  is the transposed matrix,  $\tilde{A}$  is the logical inverse of matrix  $A$ ,  $\text{tr} A$  is the matrix trace.

leading to over-estimation of subgraph numbers (Table I).

## APPENDIX B: SUBGRAPH ENUMERATION

In numerically enumerating the subgraphs we combine a dynamic programming method([39]), which is applied generally for  $n$ -node subgraphs with  $n \geq 4$ , and a more rapid calculation, based on adjacency matrix operations, used for 3-node subgraphs. Here we give formulas for the thirteen 3-node connected directed subgraphs based on the adjacency matrix. The network adjacency matrix is denoted by  $M$ , where  $M_{ij} = 1$  if a directed edge exists from node  $i$  to node  $j$ . We begin by dividing the network into a network containing only single arrows, whose adjacency matrix will be denoted by  $A$ , and a network containing only double arrows, whose symmetric adjacency matrix will be denoted as  $S$ .

$$M = A + S \quad (\text{B1})$$

We denote by  $AB$  the matrix multiplication of matrices  $A$  and  $B$ , and by  $A \cdot B$  the dot multiplication.  $\tilde{A}$  is the logical inverse of matrix  $A$ , where the 0 elements of  $A$  are the 1 of  $\tilde{A}$  and vice-versa.  $A'$  is the transpose matrix of  $A$ . A summation denotes summation of all the matrix indices. The matrix formulas for the 13 directed connected 3-node subgraphs are given in Table III. For

example id38 has two nodes which are connected by a path of 2 edges and a path of one edge.  $A^2_{ij}$  is the number of length 2 paths between node  $i$  and node  $j$ . Dot-multiplication with matrix  $A$  and summation of the terms of the resultant matrix gives the correct count. In some of the subgraphs a correction is made for the terms on the diagonal (id6,id36,id78).

## ACKNOWLEDGMENTS

We thank S. Maslov, A. Mayo and all members of our lab for valuable discussions. We acknowledge support from the Israel Science Foundation, the Human Frontier Science Program, and the Minerva Foundation.

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