New Results on Second-Order Phase Transitions and Conformal Field Theories

Zohar Komargodski

Weizmann Institute of Science, Israel

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We have $L^d$ spins with some nearest-neighbor interaction energy $J > 0$ if they are misaligned. So the spins want to be aligned at low temperatures. The magnetization $M$ is the order parameter.
Alternatively, think of the 2nd order water-vapor transition diagram.
At $T_c$ there is a phase transition. Long range correlations develop. Lattice structure becomes irrelevant. Quantities depend singularly on the temperature and external field. Ginzburg-Landau theory:

$$H = \int d^d x \left( r (\nabla M)^2 + c M^2 + \lambda M^4 + \ldots \right)$$

The partition function

$$Z = \int [dM] e^{-H}$$

encodes all the thermodynamics properties at the phase transition.
Some experimentally interesting quantities are the usual $\alpha, \beta, \gamma, \delta, \eta, \nu$ exponents:

\[
C \sim (T - T_c)^{-\alpha}, \quad M \sim (T_c - T)^{\beta}, \quad \chi \sim (T - T_c)^{-\gamma},
\]
\[
M \sim h^{1/\delta}, \quad \langle M(\vec{n})M(0) \rangle \sim \frac{1}{|\vec{n}|^{d-2+\eta}}, \quad \xi \sim (T - T_c)^{-\nu}.
\]
Amazingly, one discovers four relations between these 6 quantities:

\[ \alpha + 2\beta + \gamma = 2 \, , \]
\[ \gamma = \beta(\delta - 1) \, , \]
\[ \gamma = \nu(2 - \eta) \, , \]
\[ \nu d = 2 - \alpha \, . \]
The explanation of this miracle is that at $T_c$ the symmetry of the system is enhanced.

$$SO(d) \times \mathbb{R}^d \rightarrow SO(d) \times \mathbb{R}^d \times \Delta ,$$

with

$$\Delta : x \rightarrow \lambda x$$

and $\lambda \in \mathbb{R}^+$. 
The dilaton charge $\Delta$ can be diagonalized. If we have a local operator $\mathcal{O}$ in the theory, $\Delta(\mathcal{O})$ would uniquely determine its two-point correlator

$$\langle \mathcal{O}(n)\mathcal{O}(0) \rangle \sim \frac{1}{n^{2\Delta(\mathcal{O})}}.$$

Local operators could also have spin $s$, but we suppress it in the meantime.
In the Ising model, two of the infinitely many operators in the theory are $M(x)$ and $\epsilon(x)$, which are the magnetization and energy operators. The four miraculous relations among $\alpha, \beta, \gamma, \delta, \eta, \nu$ can be simply understood from scale invariance:

\[
\alpha = \frac{d - \Delta \epsilon}{d - \Delta \epsilon}, \\
\beta = \frac{\Delta M}{d - \Delta \epsilon}, \\
\gamma = \frac{d - 2\Delta \epsilon}{d - \Delta \epsilon}, \\
\delta = \frac{d - \Delta M}{\Delta M}, \\
\eta = 2 - d + 2\Delta M, \\
\nu = \frac{1}{d - \Delta \epsilon}.
\]
This was essentially understood more than 70 years ago.

There has been a lot of recent progress based on the observation that the symmetry is actually bigger!!

\[ SO(d) \times \mathbb{R}^d \rightarrow SO(d) \times \mathbb{R}^d \times \Delta \rightarrow SO(d + 1, 1) \]

Theories with this big symmetry are called Conformal Field Theories.
$SO(d + 1, 1)$ is the conformal group that acts on $\mathbb{R}^d$. This is the set of all transformations that preserve orthogonal lines. It consists of

- $d(d - 1)/2$ rotations
- $d$ translations
- $1$ dilation ($\Delta$)
- $d$ special conformal transformations

The last $d$ symmetry generators were not known to Landau (I think...).
Essentially all the examples that we know of second order phase transitions which have

$$SO(d) \times \mathbb{R}^d \times \Delta$$

have the full $SO(d + 1, 1)$. It has been verified “experimentally” in some examples, it has been proven to be correct theoretically in some general situations etc. So we are quite sure about this.
The $d$ bonus special conformal transformations act on space as

$$x^i \rightarrow \frac{x^i - b^i x^2}{1 - 2 b \cdot x + b^2 x^2},$$

where $b^i$ is any vector in $\mathbb{R}^d$.

It turns out that these are enough to fix three-point correlation functions as follows

$$\langle O_1(n')O_2(n)O_3(0) \rangle = \frac{C_{O_1O_2O_3}}{n^{\Delta_2+\Delta_3-\Delta_1}(n-n')^{\Delta_1+\Delta_2-\Delta_3}n'^{\Delta_1+\Delta_3-\Delta_2}}$$

Remember

$$\langle O_i(n)O_j(0) \rangle \sim \frac{\delta_{ij}}{n^{2\Delta_i(O)}}.$$
SO(d+1,1) symmetry fixes the two- and three-point functions in terms of a collection of numbers

\[ \Delta_i, \quad C_{ijk} \]

which are called the “CFT data.”

It turns out that ALL the correlation functions are fixed in terms of the CFT data.
How can we say anything useful about this collection of numbers?

\[ \{ \Delta_i \}, \quad \{ C_{ijk} \} \]

This collection is infinite because there are infinitely many operators in every Landau-Ginzburg theory. It is easiest to measure the relevant, low dimension, operators (such as \( M, \epsilon \)), but also the others exist.
Consider a four-point function with operators at $n_1, n_2, n_3, n_4$. We can form conformally invariant ratios:

$$u = \frac{n_{12}^2 n_{34}^2}{n_{14}^2 n_{23}^2}, \quad v = \frac{n_{13}^2 n_{24}^2}{n_{13}^2 n_{24}^2}$$

So a four-point function could contain an arbitrary function of $u, v$

$$\langle \mathcal{O}_1(n_1)\mathcal{O}_2(n_2)\mathcal{O}_3(n_3)\mathcal{O}_4(n_4) \rangle \sim F(u, v).$$
We represent the four point function as a sum over infinitely many three point functions:

$$\sum_{X} \mathcal{C}_{12X} \mathcal{C}_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

Therefore,

$$F(u, v) \sim \sum_{X} \mathcal{C}_{12X} \mathcal{C}_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

The functions $G$ are partial waves. Thus, once we know the CFT data, the four-point function can be in principle computed.
The dynamics is in saying that we can make the decomposition in two different ways. And we get (roughly speaking)

$$\sum_{X} C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v) = \sum_{X} C_{13X} C_{X24} G(\Delta_{1,3,2,4,X}, v, u)$$

This equation is supposed to determine/constrain the allowed $\Delta_i$ and $C_{ijk}$ that can furnish legal conformal theories.
This is extremely surprising:

Maybe we could classify all the possible conformal theories by just solving self-consistency algebraic equations.

For the mathematically oriented: these equations are similar to the equations of associative rings. It is also very similar in spirit to the classification of Lie Algebras.
Additional Applications of Conformal Field Theories:

- Quantum phase transitions

- Fixed points of the Renormalization Group Flow.
Therefore, if we had a better idea about what the equations imply, that would be useful in many branches of physics. More ambitiously, we could hope to classify all the solutions!
Recently, there has been dramatic progress on this problem both from the analytic and numeric points of view.
I will quote, without proofs, three very general analytic results. They are experimentally and numerically testable.

**Result I: Additivity of the Spectrum**

If we have $(\Delta_1, s_1)$ and $(\Delta_2, s_2)$ in the spectrum, then there are operators $(\Delta_i, s_i)$ which have $\Delta_i - s_i$ arbitrarily close to $\Delta_1 - s_1 + \Delta_2 - s_2$.

Typically, to find operators with $\Delta_i - s_i \sim \Delta_1 - s_1 + \Delta_2 - s_2$ we would need to take large $\Delta_i, s_i$. 
This leads to a rather peculiar spectrum

(In the figure, \( \tau \equiv \Delta - s \).)
Result II: Convexity

$\Delta_i - s_i$ approaches the limiting value $\Delta_1 - s_1 + \Delta_2 - s_2$ in a convex manner.
These already lead to nontrivial results for the 3d Ising model. Since the spin field has $\Delta(M) = 0.518...$, there needs to be a family of operators with $\Delta - s$ approaching $1.037...$ from below, in a convex fashion:

This is beautifully verified by the measurement of the spin-4 operator dimension. Also for spin 6. The other predictions await confirmation.
Unfortunately there are not that many general analytic constraints on Conformal Field Theories at the moment.

However, there has been rapid and impressive progress on numerical constraints. In fact, for some problems, the new methods outdo 4 decades of attempts to measure these critical exponents or compute them using the $\epsilon$-expansion!
(El-Showk et al.)

So for example we know $\Delta(\sigma) = 0.518154(15)\ldots$
Monte Carlo relies on some microscopic realization of the Conformal Field Theory (e.g. with spin degrees of freedom) and there are no rigorous bounds on the errors. The new method only uses abstract algebraic properties of the theory (no microscopic realization is invoked) and there are rigorous results on its exponential convergence rate.

The $\epsilon$-expansion is a computation done near $d = 4$ extrapolated to $d = 3$. One needs to make assumptions to bound the errors and a microscopic realization is needed. Most of the interesting theories cannot be approached by an $\epsilon$-expansion.
One starts from the algebraic equations

The equations are coupled quadratic equations for the $C$’s and $\Delta$’s. Way too hard even if the equations are truncated.
One declares that one is only interested in theories that, for example, have their lowest and next to lowest operator at some dimensions $\Delta_1, \Delta_2$. Taking derivatives of the equations w.r.t. $u, v$ and using properties of the partial waves $G$, one is sometimes led to a mathematical contradiction. Then one scans these equations over all $\Delta_1, \Delta_2$ until no contradiction arises. This leads to bounds on $\Delta_1, \Delta_2$.

The combinations of derivatives that one needs to take in order to arrive at a contradiction is a nontrivial problem. The computer just scans over all the possibilities.

It is actually not completely clear why this procedure had to work, but it clearly does.
Some philosophical remarks: In quantum gravity one does not have local degrees of freedom, so there are “fewer” measurable quantities. For example, in asymptotically flat space-time we have the $S$-matrix
Quantum gravity in $AdS_{d+1}$ is described by a boundary Conformal Field Theory. The S-matrix of the quantum theory of gravity is basically the set of correlation functions of a $d$-dimensional CFT. So the program of classifying CFTs or learning exact things about their properties can shed light on quantum gravity in AdS space.
Conclusions

- Conformal Field Theories are abundant in physics.
- They are determined by an intricate self-consistency condition. We still don’t know much about the general consequences of this self-consistency condition.
- There are some results though on monotonicity, convexity, and additivity of the spectrum of dimensions.
- Extremely powerful numerical techniques were recently introduced. Not unlikely that in the near future would be able to say very precise things about strongly coupled models that appear in Condensed Matter systems (e.g. Herbertsmithite) or even determine the conformal window of Quantum Chromodynamics.