SUPERCONVERGENT DISPERSION RELATIONS AND ELECTROMAGNETIC MASS DIFFERENCES*

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Using dispersion relations and the convergence properties of t-channel isospin amplitudes, we show that (a) $\Delta I=2$ electromagnetic mass differences should be correctly obtained by summing the self-energy contributions of a few low-lying states; (b) $\Delta I=1$ mass differences cannot be obtained in this way, and a subtraction term is always necessary; (c) the subtraction term has the correct sign for explaining the proton-neutron mass difference.

The problem of computing the electromagnetic mass differences between particles in a given isomultiplet has always been one of the greatest puzzles of elementary-particle physics. It is well known that the simple, naive calculations which include only the contributions of a few low-lying states to the self-energy diagram lead in most cases to totally wrong results (including the notorious wrong signs for the proton-neutron and K^+-K^0 mass differences). On the other hand, the same simple approach gives the correct sign and magnitude in a few other cases (such as the π^+-π^0 difference). In this paper we propose simple, reasonable assumptions on the energy dependence of t-channel isospin amplitudes for forward Compton scattering and, using these assumptions, we reach the following conclusions:

(a) All $\Delta I=2$ mass differences should be correctly obtained when we approximate the self-energy diagram by the contributions of a few low-lying states.

(b) There is no reason to expect that the same simple approximation will give the correct order of magnitude or even the correct sign for the $\Delta I=1$ mass differences.

(c) A consistent calculation of the $\Delta I=1$ terms must include an additional "subtraction" term. We show that this term has the correct sign and, roughly, the correct order of magnitude required by the experimental masses.

We also demonstrate that the statements (a)-(c) are correct for all six electromagnetic hadron mass differences which are experimentally known, and we propose further experimental tests of our assumptions.

In perturbation theory, the electromagnetic self-energy of a hadron is given by[1]

$$\Delta M=-\frac{1}{8\pi^2}\int_{-\infty}^{+\infty}\frac{T_{\mu\nu}(q^2,\nu)g^{\mu\nu}}{q^2-i\epsilon}d^4q, \tag{1}$$

where $\epsilon^\mu\epsilon^\nu T_{\mu\nu}(q^2,\nu)$ is the forward amplitude for Compton scattering of a virtual photon with mass q^2, energy $q^0=\nu$, and polarization ϵ^μ from a hadron with momentum p and mass M ($M\nu=-pq$). We write $T_{\mu\nu}$ as

$$T_{\mu\nu}(q^2,\nu)=t_1(q^2,\nu)[q^2g_{\mu\nu}-q_\mu q_\nu]+t_2(q^2,\nu)[\nu^2g_{\mu\nu}+\frac{q^2}{M^2}p_\mu p_\nu+\frac{\nu}{M}(p_\mu q_\nu+p_\nu q_\mu)]. \tag{2}$$

Cottingham[1] has shown that by rotating the integration contour in (1) from the real to the imaginary axis in the complex ν plane, one can express ΔM in terms of scattering amplitudes for spacelike photons, allowing us, in principle, to use experimental electron scattering data in order to compute the integral. Substituting $\nu\to i\nu$ and integrating over the angular variables in (1), we find

$$\Delta M=-\frac{1}{4\pi}\int_0^\infty\frac{dq^2}{q^2}\int_{-q}^{+q}d\nu(q^2-\nu^2)^{1/2}[3q^2t_1(q^2,i\nu)-(q^2+2\nu^2)t_2(q^2,i\nu)]. \tag{3}$$

We can now write, for t_1 and t_2, fixed-q^2 dispersion relations in ν and compute the t_i's in terms of their absorptive parts. The main obstacle at this point is, of course, the question of possible subtractions in the dispersion relations, since only in the case of no subtractions can we hope that the contribution of the first few low-lying states will dominate the expressions for t_1, t_2 and hence for ΔM. The convergence properties of the dispersion integrals are determined by the asymptotic behavior of the absorptive parts of the amplitudes.

To lowest order in α, the electromagnetic mass differences transform according to $\Delta I=1$ or $\Delta I=2$. For any given isomultiplet we can separate $\Delta M^{(1)}$ and $\Delta M^{(2)}$, and using Eq. (3) we can express each one of them in terms of Compton scattering amplitudes with t-channel isospins $I=1$ and $I=2$, respectively. We now face the following question: What is the asymptotic energy dependence of the forward spin-nonflip $I=1$ and $I=2$ t-channel amplitudes for Compton scattering of photons with mass q^2? At this point we propose to use the most successful and least controversial prediction of Regge pole theory.[2] We assume that a forward spin-nonflip amplitude with a given set of t-channel quantum numbers will be proportional at high energies to $\nu^{\alpha(0)}$, where $\alpha(0)$ is the $t=0$ intercept of the leading Regge trajectory with the appropriate quantum numbers.[3] Following de Alfaro, Fubini, Rosetti, and Furlan,[4] we further assume that, in view of the absence of low-lying $I=2$ mesons, all $I=2$ trajectories have $\alpha_{I=2}(0)<0$.

An unsubtracted dispersion relation for $t_i^{(I)}(q^2,\nu)$ $(i=1,2;\ I=1,2)$ would have the form

$$t_i^{(I)}(q^2,\nu)=\frac{4Mq^2f_i^{(I)}(q^2)}{q^4-4M^2\nu^2}+\frac{2}{\pi}\int_{\nu_t}^{\infty}\frac{\operatorname{Im}t_i^{(I)}(q^2,\nu')\nu'd\nu'}{\nu'^2-\nu^2}, \tag{4}$$

where $t_i^{(I)}(q^2,\nu)=t_i^{(I)}(q^2,-\nu)$ and ν_t is the inelastic threshold;

$$f_1^{(I)}(q^2)=\frac{\alpha}{\pi}\frac{G_M^{\ 2}(q^2)_I-G_E^{\ 2}(q^2)_I}{q^2+4M^2}, \tag{5}$$

$$f_2^{(I)}(q^2)=\frac{\alpha}{\pi}\frac{q^2G_M^{\ 2}(q^2)_I+4M^2G_E^{\ 2}(q^2)_I}{q^2(q^2+4M^2)}. \tag{6}$$

$G_{E,M}^{\ 2}(q^2)$ is the appropriate linear combination of the squared form factors [e.g., for the nucleon, $G_M^{\ 2}(q^2)_{I=1}=G_M^{\ 2}(q^2)_p-G_M^{\ 2}(q^2)_n$].

Assuming that the total cross section is bounded by a constant, Eq. (2) gives

$$\left|\nu^2t_2^{(I)}(q^2,\nu)\right|_{\nu\to\infty}\le\text{const.}\times\nu. \tag{7}$$

$t_2^{(I)}$ will therefore satisfy Eq. (4) for $I=1,2$. The amplitude $t_1^{(2)}$ satisfies

$$|t_1^{(2)}(q^2,\nu)|_{\nu\to\infty}\propto\nu^{\alpha_{I=2}(0)}, \tag{8}$$

If $\alpha_{I=2}(0)<0$, then $t_1^{(2)}$ obeys the unsubtracted dispersion relation (4). It is then perfectly reasonable to assume that the integral over the absorptive part is dominated by a few low-lying states and that $\Delta M^{(2)}$ can be determined by this approximation.

For $I=1$, the energy dependence of t_1 is determined by the intercept of the leading trajectory with quantum numbers $I=1$, $C=1$, $G=-1$, $P=(-1)^J$. This is the trajectory of the A_2 meson which has[5] $\alpha_{A_2}(0)\sim0.4>0$. We therefore predict

$$|t_1^{(1)}(q^2,\nu)|_{\nu\to\infty}\propto\nu^{0.4}. \tag{9}$$

The integrand in (4) will fall off like $\nu^{-0.6}$ and the dispersion integral will not converge. We must introduce a subtraction and we find (we subtract at $\nu=0$)

$$t_1^{(1)}(q^2,\nu)=t_1^{(1)}(q^2,0)+\frac{16M^3\nu^2f_1^{(1)}(q^2)}{q^2(q^4-4M^2\nu^2)}+\frac{2\nu^2}{\pi}\int_{\nu_t}^{\infty}\frac{\operatorname{Im}t_1^{(1)}(q^2,\nu')d\nu'}{\nu'(\nu'^2-\nu^2)}. \tag{10}$$

We can now safely assume that the contribution of the low-mass states dominates the integral in (10). However, we have an additional, unknown term $t_1^{(1)}(q^2,0)$ which essentially results from the presence of high-energy contributions and which must be included in our final expression for $\Delta M^{(1)}$. We conclude that for all $\Delta I=1$ mass differences there is absolutely no reason for the simple approximation of "a few low-mass states" to give the correct magnitude or sign. What one really computes in this approximation is a combination of $\Delta M^{(1)}$ and an unknown term, and it is always possible that this expression will have the opposite sign to that of $\Delta M^{(1)}$. The correct expression for $\Delta M^{(1)}$ is

$$\Delta M^{(1)} = \Delta M_1^{\text{sub}} + \Delta M_1^{E} + \Delta M_2^{E} + \Delta M^{I}, \tag{11}$$

where

$$\Delta M_1^{\text{sub}} = -\tfrac{3}{8}\int_0^\infty q^2 t_1^{(1)}(q^2,0)dq^2, \tag{12}$$

$$\Delta M_1^{E} = \tfrac{3}{2}\int_0^\infty f_1^{(1)}(q^2)dq^2\left\{M - q\left[\left(1+\frac{q^2}{4M^2}\right)^{1/2} - \frac{q}{2M}\right]\right\}, \tag{13}$$

$$\Delta M_2^{E} = \tfrac{1}{2}\int_0^\infty q f_2^{(1)}(q^2)\left\{2\left(\frac{q}{2M}\right)^3 + \left[1-2\left(\frac{q}{2M}\right)^2\right]\left[\left(\frac{q}{2M}\right)^2+1\right]^{1/2}\right\}dq^2. \tag{14}$$

ΔM^I is the inelastic contribution obtained by substituting the integrals of Eq. (4) (for $t_2^{(1)}$) and Eq. (10) (for $t_1^{(1)}$) in (3).

If we now consider the present status of the calculations of ΔM for mesons and baryons we find the following picture:

(1) There are two experimentally known $\Delta I=2$ mass differences: $m(\pi^+)-m(\pi^0)=4.61$ MeV; $m(\Sigma^+)+m(\Sigma^-)-2m(\Sigma^0)=1.76\pm0.23$ MeV. It has been known for a long time[6,7] that the simplest calculation gives the correct π^+-π^0 mass difference, and recent calculations[7,8] for $\Delta M^{(2)}(\Sigma)$ indicate that, again, the correct sign and magnitude are obtained. The actual numbers depend on the details of the assumed q^2 dependence of the form factors and on the number of intermediate states included in the calculation. Typical numbers are[7] $\Delta M^{(2)}(\pi)\sim 5\pm1$ MeV; $\Delta M^{(2)}(\Sigma)\sim 1.5\pm0.5$ MeV.

(2) There are four known $\Delta I=1$ mass differences: $m(n)-M(p)=1.3$; $m(\Sigma^-)-m(\Sigma^+)=7.9\pm0.1$; $m(\Xi^-)-m(\Xi^0)=6.5\pm1.0$; and $m(K^0)-m(K^+)=3.90\pm0.25$ (in MeV). In all these cases we find indeed that the simple approximation fails completely,[7] and at least for N and K even the sign is wrong. This was recognized a few years ago by Coleman and Glashow[9] who proposed that an arbitrary $\Delta I=1$ "tadpole" term should be added to all these calculations. We interpret this "tadpole" contribution as our subtraction term [Eq. (12)] and we believe that our approach explains why "tadpoles" are always needed for $\Delta I=1$ and are not required for $\Delta I=2$.[10]

What can we say about the sign and magnitude of the subtraction term (12)? We multiply Eq. (10) by q^2 and take the limit $q^2\to0$. For $\nu\neq0$, $t_1^{(1)}(q^2,\nu)$ does not have a pole[11] at $q^2=0$. We therefore find (for the nucleon case[12])

$$\lim_{q^2\to0} q^2 t_1^{(1)}(q^2,0) = 4M \lim_{q^2\to0} f_1^{(1)}(q^2) = \frac{\alpha}{\pi M}(\mu_p^{\,2}-\mu_n^{\,2}-1). \tag{15}$$

Equation (12) can now be rewritten as

$$\Delta M_1^{\text{sub}} = -\frac{3\alpha M}{2\pi}(\mu_p^{\,2}-\mu_n^{\,2}-1)\int_0^\infty g(\tau)d\tau, \tag{16}$$

where $\tau=q^2/4M^2$ and $g(\tau)$ is a dimensionless unknown function of τ satisfying $g(0)=1$. Although we do not know the explicit form of $g(\tau)$, we can safely assume that the integrand in (16) converges very rapidly [possibly as fast as q^{-6}, since $t_1^{(1)}(q^2,0)$ is probably proportional to the product of two ordinary form factors, each presumably falling like q^{-4}]. The sign of ΔM_1^{sub} will then be determined by Eq. (15) and will be negative for $m(p)-m(n)$! Neglecting ΔM^I in (11) and using the experimental nucleon form factors and mass difference we find that $\int_0^\infty g(\tau)d\tau$ is required in this case to be of the order of $\frac{1}{4}$. In order to see whether this is reasonable we parametrize $g(\tau)=e^{-a\tau}$ and find $a\sim4$, leading to

$$q^2 t_1^{(1)}(q^2,0) \propto e^{-q^2/M^2}. \tag{17}$$

This has a reasonable slope[13] at $q^2=0$. A more detailed numerical analysis for the various terms in (11) will be given elsewhere.

There are two "families" of additional experimental tests for our general approach. Measuring some more $\Delta I=2$ differences such as $\rho^+-\rho^-$, $Y_1^{*+}+Y_1^{*-}-2Y_1^{*0}$, and $N^{*++}+N^{*0}-2N^{*+}$, is one possible set of tests. However, even the simplest calculation of these differences requires some information on the magnetic moments of these particles and, therefore, suffers from very large ambiguities. A second test which will be much more crucial and will be performed in the next few years at the Stanford Linear Accelerator Center is to measure the total γp and γn cross sections.[3] We predict that at high energies,

$$|\sigma_t(\gamma p)-\sigma_t(\gamma n)| \propto \nu^{-0.6}. \qquad (18)$$

In conclusion let us summarize our results: We have explained both the success (for $\Delta I=2$) and the failure (for $\Delta I=1$) of the simple approximation of including only a few low-lying states in the expression for the electromagnetic self-energies of hadrons. We have presented an explicit formula for $\Delta M(I=1)$ which includes a "natural" subtraction term, and we have demonstrated that this term has the correct sign and, roughly, the correct order of magnitude.

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[1]We will closely follow the work of W. N. Cottingham, Ann. Phys. (N. Y.) 25, 424 (1963), and M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters 2, 7 (1959).

[2]We assume nothing about the t dependence of the residue functions; our results will not be modified by cuts; we use only the sign of $\alpha(0)$, and not its numerical value.

[3]Throughout this paper, when we write "Compton amplitude" or "total γN cross section," we refer to the contribution to lowest order in α. For $q^2 \neq 0$ photons, this is practically equivalent to the observed amplitude or cross section. For real photons there is a large contribution of electron pair production in which a factor α^2 is balanced by $(M/m_e)^2$. The contribution of this to ΔM is, however, negligible because of the q^2 integration. In particular, our prediction (18) cannot be tested by measuring total photoabsorption by matter, since we must first separate the pair-production cross section. Equation (18) can be directly tested in bubble-chamber experiments.

[4]V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters 21, 576 (1966).

[5]R. J. N. Phillips and W. Rarita, Phys. Rev. 140, B200 (1966); V. Barger and M. Olsson, Phys. Rev. 146, 1080 (1966).

[6]R. P. Feynman and G. Speisman, Phys. Rev. 94, 500 (1954).

[7]R. H. Socolow, thesis, Harvard University, 1964 (unpublished); S. Coleman and H. J. Schnitzer, Phys. Rev. 136, B223 (1964).

[8]R. E. Norton and G. H. Thomas, to be published.

[9]S. Coleman and S. L. Glashow, Phys. Rev. 134, B671 (1964).

[10]We can extend our superconvergence assumptions to SU(3) and assume that since there are no known 10 and 27 meson multiplets, they all have $\alpha(0)<0$. This immediately gives the octet transformation properties for the tadpoles.

[11]For $\nu \neq 0$, $q^2\,\mathrm{Im}t_1(q^2,\nu)$ is a longitudinal inelastic cross section and it vanishes at $q^2=0$. This implies that

$$\lim_{q^2 \to 0} q^2\,\mathrm{Re}t_1(q^2,\nu)$$

is independent of ν (for $\nu \neq 0$), since the limit $q^2 \to 0$ of the Born term in Eq. (10) is ν independent. For large ν the ratio between $\mathrm{Re}t_1$ and $\mathrm{Im}t_1$ approaches a constant and since $\mathrm{Im}t_1$ has no $q^2=0$ pole $\mathrm{Re}t_1$ will also be finite. We conclude that the $q^2=0$ pole of the Born term must be compensated by a $q^2=0$ pole of $\mathrm{Im}t_1(q^2,0)$.

[12]We do not know the magnetic moments of Σ and Ξ. However, if we use their SU(3) values, we find that (15) has the right sign for these cases as well.

[13]The function (17) decreases at small q^2, more slowly than $q^2G^2(q^2)$ where $G(q^2)$ is any electromagnetic form factor of the nucleon. We have to know much more about the detailed q^2 dependence before we can draw any final conclusions.