# 4 Cosmology

Recall: Hubble's law, equivalent observers, $H_0 = 65 \pm 8$km/s Mpc.
The cosmological principle: homogeneous and isotropic.

## 4.1 Newtonian cosmology [2hr]

Consider first the evolution of a cold, pressureless, plasma.
***Birkhoff & Newtonian cosmology.***

$$\ddot{r} = -\frac{GM}{r^2} = -\frac{4\pi}{3} G\rho r \, , r(r_0, t) = a(t) r_0 \Rightarrow \; \ddot{a} = -\frac{4\pi}{3} G\rho a. \tag{48}$$

$\dot{a}_0 = H_0$.

$$\rho = \rho_0 a^{-3} \; \Rightarrow \; \dot{a}^2 = \frac{8\pi}{3} G\rho_0 a^{-1} + \left( H_0^2 - \frac{8\pi}{3} G\rho_0 \right). \tag{49}$$

Critical density $\rho_c = 3H_0^2/8\pi G = 10^{-29} h_{75}^2 \text{g/cm}^3$, $a_0 = 1$

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ \frac{\rho_0}{\rho_c} a^{-3} + \left( 1 - \frac{\rho_0}{\rho_c} \right) a^{-2} \right]. \tag{50}$$

Age, fate. $\rho_{b0}/\rho_c \sim 10^{-2}$. Recall DM, so that $\rho_0$ may be $\gg \rho_{b0}$.
***Modification:* $\lambda$.** $\lambda$ as additional length scale in GR- we'll be discussed later. Static:

$$\ddot{a} = -\frac{4\pi}{3} G\rho a + \frac{1}{3}\lambda c^2 a, \tag{51}$$

with $\lambda = 4\pi G\rho_0/c^2 = (2/3)(\rho_0/\rho_c)(H_0/c)^2$. Modified Friedmann

$$\dot{a}^2 = \frac{8\pi}{3} G\rho_0 a^{-1} + \frac{1}{3}\lambda c^2 a^2 + \left( H_0^2 - \frac{8\pi}{3} G\rho_0 - \frac{1}{3}\lambda c^2 \right), \tag{52}$$

or

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ \Omega a^{-3} + (1 - \Omega - \Lambda) a^{-2} + \Lambda \right], \tag{53}$$

with

$$\Omega = \frac{\rho_0}{\rho_c}, \quad \Lambda = \frac{\rho_\Lambda}{\rho_c}, \quad \rho_c = \frac{3H_0^2}{8\pi G} = 1.05 \times 10^{-29} h_{75}^2 \text{g/cm}^3, \quad \rho_\Lambda = \frac{\lambda c^2}{8\pi G}. \tag{54}$$

Revised discussion of age, fate.
For $a \ll 1$ we have $\dot{a} = (8\pi G\rho_0/3)^{1/2} a^{-1/2}$, giving $a \propto t^{2/3}$ and $tH = 2/3$ as for an $\{\Omega = 1, \Lambda = 0\}$ universe.

### 4.2 Radiation evolution

Recall ***CMB:*** Planck spec., $T = 2.73°$K, Dipole, $\delta T/T \sim 10^{-5}$. Radiation density:

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{T}{\hbar c}\right)^3 = 0.244\left(\frac{T}{\hbar c}\right)^3 = 420\text{cm}^{-3}, \quad u_\gamma = \frac{\pi^2}{15}\left(\frac{T}{\hbar c}\right)^3 T = 0.26\text{eV/cm}^3. \tag{55}$$

Denoting the baryon density by $n_b$,

$$\eta \equiv \frac{n_\gamma}{n_b} = 0.7 \times 10^8 \Omega_b^{-1} h_{75}^{-2}. \tag{56}$$

Assuming photons decoupled from matter. Photon redshift:

$$\begin{aligned} \lambda(t+dt) = \quad & (1+\beta)\lambda(t) = (1 + Hcdt/c)\lambda(t) \\ & \Rightarrow \tfrac{\dot\lambda}{\lambda} = H = \tfrac{\dot a}{a} \Rightarrow \lambda \propto a, \end{aligned} \tag{57}$$

and

$$1 + z \equiv \frac{\lambda_{\text{obs.}}}{\lambda_{\text{emt.}}} = \frac{1}{a}. \tag{58}$$

Conservation of Planck spec., $T \propto a^{-1}$. Comment on adiabatic expansion ($\hat\gamma = 4/3$).

$u_\gamma \propto a^{-4}$, $\rho \propto a^{-3}$, Matter-Radiation equality:

$$z_{\text{eq.}} = \frac{\rho_0 c^2}{u_\gamma} = \frac{\Omega\rho_c c^2}{u_\gamma} = 2.3 \times 10^3(\Omega h_{75}^2/0.1). \tag{59}$$

For $z > z_{\text{eq.}}$ energy density dominated by radiation, GR must be used to derive dynamics. $z > z_{\text{eq.}}$: Radiation domination, $z < z_{\text{eq.}}$: Matter domination.

Assume that at high $T$, for which the plasma is fully ionized, radiation and matter are strongly coupled and in thermal equilibrium. For $n_\gamma \gg n_b$, the internal energy and the pressure are dominated by radiation. The eos is then $e/n_b = m_p c^2 + 3p/n_b$, and $Td(S/N) = d(e/n_b) + pd(1/n_b)$ (assuming conserved number of baryons) gives $p \propto n_b^{4/3}$ for adiabatic processes, implying $T \propto n_b^{1/3}$ and $n_\gamma \propto n_b$. Thus, for thermal equilibrium we obtain $\eta = Const.$, i.e. $\eta \gg 1$ and given by its value today, and $T \propto 1/a$ (since $a^3 n_b = Const.$) as for free expansion.

The ionized fraction is approximately given by eq. (17), which may be written as

$$\begin{aligned}\frac{n_I}{n_0} &= \frac{g_I}{g_0}e^{-I/T} \cong \frac{(2mT/\hbar^2)^{3/2}}{n_e}e^{-I/T} \\ &= \frac{1}{0.244}\eta\left(\frac{2m_ec^2}{T}\right)^{3/2}e^{-I/T} \approx 10^{19}T_{\rm eV}^{-3/2}e^{-I/T}. \end{aligned} \tag{60}$$

The plasma becomes neutral ("recombination" occurs) therefore at $T_{\rm rec.} \approx 13.6{\rm eV}/\ln(10^{19}) \approx 0.3$ eV, $z_{\rm rec.} = T_{\rm rec.}/T_0 \approx 10^3$. Exact calculation gives

$$z_{\rm dec.} = z_{\rm rec.} = 1.1 \times 10^3. \tag{61}$$

Note that

$$\frac{z_{\rm eq.}}{z_{\rm dec.}} = 2.1(\Omega h_{75}^2/0.1). \tag{62}$$

$z_{\rm rec.}$ is also the redshift at which radiation decouples from matter. Comparing the photon scattering rate, $n_e\sigma c$, to the expansion rate, $\dot{a}/a = H = H_0\Omega^{1/2}a^{-3/2}$ (for $a_{\rm eq.} \ll a \ll 1$), we have

$$\frac{n_e\sigma c}{H} = \frac{n_{e0}\sigma c}{\Omega^{1/2}H_0a^{3/2}} \approx 2\times 10^3\frac{\Omega_b h_{75}}{\Omega^{1/2}}\frac{\sigma}{\sigma_T}\left(\frac{a}{10^3}\right)^{-3/2}. \tag{63}$$

For $z > z_{\rm rec.}$ the electrons are free, $\sigma = \sigma_T$ radiation and matter are coupled. For $z > z_{\rm rec.}$ the atoms are neutral, $\sigma \ll \sigma_T$, and the radiation is "decoupled". While $T_{\rm rad.} \propto 1/a$ also for $z > z_{\rm rec.}$, $T_{\rm mat.} \propto 1/a^2$.

## 4.3 Relativistic evolution, $z > z_{\rm eq.}$

At early time, $a_{\rm eq.} \ll a \ll 1$, the $a^{-3}$ term dominates and eq. (53) may be approximated as

$$H^2 = \frac{8\pi G}{3c^2}e, \tag{64}$$

where $e$ is the energy density, dominated by rest mass. We will show later, in the brief discussion of GR principles, that this equation is exact also for the relativistic phase. When radiation dominates we therefore have

$$H = \left[\frac{8\pi^3 G}{45c^2}\left(\frac{T}{\hbar c}\right)^3 T\right]^{1/2} = 0.9\times 10^{-5}T_{\rm eV}^2{\rm yr}^{-1}. \tag{65}$$

Since $T \propto 1/a$ we have $\dot{a} \propto 1/a$, i.e. $a \propto t^{1/2}$ and $tH = 1/2$, i.e.

$$t = t_H/2 = 1/2H = 0.5\times 10^5 T_{\rm eV}^{-2}{\rm yr} = 2T_{\rm MeV}^{-2}{\rm s}. \tag{66}$$

Mention entropy "injection" to photons at particle annihilation ($e^{\pm}$).

If eq. (64) is valid at all temperatures, $\dot{a}/a \propto a^{-2}$ and $\dot{a} \propto t^{1/2}$. The age is proportional therefore to $t_H$, $t = \int^a da/\dot{a} = t_H/2$, and the the size of causally connected regions is proportional to the Hubble distance, $ct = ct_H/2 = d_H/2$. The current size of regions which were causally connected at the time the temperature was $T$ is

$$a^{-1}d_H(T)/2 = 70T_{\rm eV}^{-1}{\rm Mpc}, \tag{67}$$

corresponding to angular size

$$\theta \approx \frac{a^{-1}d_H(T)/2}{c/H_0} \approx 1^\circ h_{75}T_{\rm eV}^{-1}. \tag{68}$$

Discuss causality problem, new physics at $T > 1$ TeV?

## 4.4 GR basics: I. Equivalence principle, metrics, Newtonian limit [2hr]

### 4.4.1 The equivalence principle and the concept of a metric

***Inertial and gravitational mass equivalence*** is naturally obtained by the assumption that the effects of $\vec{g}$ is equivalent to measurements in an "accelerating elevator," $x'^i = x^i + \frac{1}{2}g^i t^2$.

***The Equivalence principle:*** For any point $x^\mu = X^\mu$ a locally inertial coordinate system may be constructed, $\xi^\mu(x^\mu; X^\mu)$, in which the laws of physics in an infinitesimal region around $X^\mu$ are given by special relativity. Inertial and gravitational mass equivalence is automatically obtained.

***Remind special relativity ideas:*** 4-vectors, $\eta_{\mu\nu}$, $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, $ds^2 = -c^2dt^2 + \vec{v}^2dt^2 = -c^2(1-\beta^2)dt^2 = -c^2dt^2/\gamma^2 = -c^2d\tau^2$ for massive particles.

***Massive particles:***

$$-c^2d\tau^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}dx^\alpha dx^\beta, \tag{69}$$

i.e.

$$-c^2d\tau^2 = g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta \tag{70}$$

with

$$g_{\alpha\beta}(X^\mu) \equiv \left[\eta_{\mu\nu}\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha}\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta}\right]_{x^\mu=X^\mu}. \tag{71}$$

The metric is invariant under Lorentz transformations of the local inertial frames $\xi^\mu$.

The motion of a particle under gravity is described in the local inertial frame by $d^2\xi^\mu/d\tau^2 = 0$, which may be written as

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial \xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}. \tag{72}$$

multiplying by $\partial x^\alpha/\partial \xi^\mu$ we have

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\rho}(x^\mu)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}, \tag{73}$$

with the Christoffel symbol

$$\Gamma^\alpha_{\nu\rho}(X^\mu) \equiv \left[\frac{\partial^2 \xi^\mu(x^\mu; X^\mu)}{\partial x^\nu \partial x^\rho}\frac{\partial x^\alpha(\xi^\mu; X^\mu)}{\partial \xi^\mu}\right]_{x^\mu = X^\mu}. \tag{74}$$

For any given initial conditions, $x^\mu(\tau = 0)$ and $dx^\mu/d\tau(\tau = 0)$, the particle's trajectory is determined by $\Gamma^\alpha_{\nu\rho}(x^\mu)$. $\Gamma^\alpha_{\nu\rho}(x^\mu)$ determines therefore the effect of gravity. Similarly, for massless particles

$$0 = \frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\nu\rho}(x^\mu)\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma}. \tag{75}$$

*Without proof:* Given $\Gamma^\alpha_{\nu\rho}(x^\mu)$ and $g_{\alpha\beta}(x^\mu)$ at some point $x^\mu = X^\mu$, the local inertial frame at $x = X$, $\xi^\alpha(x; X)$ is determined in the vicinity of $x = X$ (up to a Lorentz transformation) up to (and including) $(x - X)^2$.

***The meaning of "infinitesimal region around $X$" in the equivalence principle*** is

$$\left[\frac{\partial g_{\alpha\beta}(x^\mu; X^\mu)}{\partial x^\lambda}\right]_{x^\mu = X^\mu} = 0, \tag{76}$$

where $g_{\alpha\beta}(x^\mu; X^\mu)$ is the metric in the coordinate system $\xi^\mu(x^\mu; X^\mu)$ at the point $\xi^\mu(x^\mu; X^\mu)$. That is, in the local inertial frame associated with the point $x = X$ the metric is $\eta_{\alpha\beta}$ at $x = X$ and its first order derivatives at $x = X$ vanish. *Without proof:* This implies

$$\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = g_{\beta\rho}\Gamma^\rho_{\mu\alpha} + g_{\alpha\rho}\Gamma^\rho_{\mu\beta}. \tag{77}$$

Adding the permutations with respect to $\alpha\beta\mu$ (and multiplying by the inverse $g^{\alpha\beta} \equiv \eta^{\mu\nu}(\partial x^\alpha/\partial \xi_\mu)(\partial x^\beta/\partial \xi_\nu)$ of $g_{\alpha\beta}$), we find

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right). \tag{78}$$

This relation implies that the effects of gravity, $\Gamma^{\alpha}_{\nu\rho}(x^\mu)$, are determined by the metric $g_{\alpha\beta}(x^\mu)$.

The condition $c^2 = -g_{\alpha\beta}(x^\mu)(dx^\alpha/d\tau)(dx^\beta/d\tau)$, with $g_{\alpha\beta}(x^\mu) \equiv g_{\alpha\beta}(x^\mu; X^\mu = x^\mu)$, can be chosen to hold at $\tau = 0$. Eq. (77) ensures that if it holds at $\tau = 0$ it holds also at $\tau > 0$.

### 4.4.2 Newtonian limit- slow motion in a stationary weak field

Slow motion implies that $dx^i/cd\tau \sim v/c \ll dx^0/cd\tau = dt/d\tau \sim 1$. The equation of motion is therefore approximately given by

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^{\alpha}_{00}(x^\mu)\left(\frac{dx^0}{d\tau}\right)^2. \tag{79}$$

Weak field implies $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h \ll 1$. Since the field is stationary we have

$$\Gamma^0_{00} = \frac{1}{2}g^{0\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho}\right) = -\frac{1}{2}g^{0i}\frac{\partial g_{00}}{\partial x^i} = O(h^2), \tag{80}$$

$$\Gamma^i_{00} = \frac{1}{2}g^{i\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho}\right) = -\frac{1}{2}g^{ij}\frac{\partial g_{00}}{\partial x^j} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} + O(h^2). \tag{81}$$

Eqs. (80) and (79) imply $dt/d\tau = const.$, which allows replacing $\tau$ with $t$ in eq. (79) leading to, using eq. (81),

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}. \tag{82}$$

In order to obtain the Newtonian limit we must have $h_{00} = -2\Phi/c^2 + const.$. Defining $\Phi$ to vanish at infinity (far away from sources) the constant must vanish and we have

$$g_{00} = -\left(1 + \frac{2}{c^2}\Phi\right). \tag{83}$$

- Discuss meaning of weak field: Earth, Sun.
- Discuss time: For a stationary clock $d\tau = \sqrt{-g_{00}}dt = (1 + \Phi/c^2)dt$. Connect to redshift, Doppler in "elevator".

### 4.5 The Friedmann-Robertson-Walker metric [2hr]

The cosmological principle: The universe is homogeneous and isotropic. Choosing time as $t(S)$ where is $S$ some scalar field, e.g. $T_{CMB}$, the subspaces $t = const.$ are homogeneous & isotropic. The metric is in general

$$-c^2 d\tau^2 = g_{00} c^2 dt^2 + 2 g_{i0} c dt dx^i + g_{ij} dx^i dx^j. \tag{84}$$

We must be able to choose a coordinate system where $g_{i0} = 0$, since otherwise there is a preferred direction, $g_{i0}(x)$. Homogeneity implies $g_{00}$ is a function of $t$ alone, so $t$ may be scaled to give

$$-c^2 d\tau^2 = -c^2 dt^2 + g_{ij} dx^i dx^j. \tag{85}$$

*Without proof:* The most general homogeneous and isotropic metric is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \right], \tag{86}$$

with $k = -1, 0$ or $+1$ (see below: Homogeneous and isotropic is equivalent to maximally symmetric; All maximally symmetric with the same curvature are equivalent). The spatial metric is invariant under rotations, $r'^i = R^i_j r^j$, and quasi translations,

$$\mathbf{r}' = \mathbf{r} + \mathbf{r}_0 \left[ \sqrt{1 - kr^2} - \left( 1 - \sqrt{1 - kr_0^2} \right) \frac{\mathbf{r}.\mathbf{r}_0}{r_0^2} \right], \tag{87}$$

which translates the origin to $\mathbf{r}_0$.

$$g_{tt} = -1, \quad g_{ij} = R^2(t) \tilde{g}_{ij}, \quad \tilde{g}_{rr} = \frac{1}{1 - kr^2}, \quad \tilde{g}_{\theta\theta} = r^2, \quad \tilde{g}_{\varphi\varphi} = r^2 \sin^2\theta. \tag{88}$$

At any $t = t_0$ we may define $\mathbf{x} = R(t_0)\mathbf{r}$ for which the metric is $\tilde{g}_{ij}$. The dimension of $g_{ij}$ is $L^2$, since $r$ is defined dimensionless. $g^{ij} = R^{-2} \tilde{g}^{ij}$.

Defining $r^{(1)} = r \sin\theta \cos\phi$, $r^{(2)} = r \sin\theta \sin\phi$ and $r^{(3)} = r \cos\theta$, the metric may be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left[ d\mathbf{r}^2 + \frac{k(\mathbf{r}.d\mathbf{r})^2}{1 - kr^2} \right], \tag{89}$$

where $d\mathbf{r}^2 = \delta_{ij} dr^i dr^j$ and $\mathbf{r}.d\mathbf{r} = \delta_{ik} r^k dr^i$, so that

$$g_{tt} = -1, \quad g_{ij} = R^2(t) \tilde{g}_{ij}, \quad \tilde{g}_{ij} = \delta_{ij} + \frac{k}{1 - kr^2} \delta_{ik} \delta_{jl} r^k r^l. \tag{90}$$

***The sphere example.*** Consider the metric on the surface of a sphere of radius $R$ in 3D Eucleadean space. The 3D metric is $ds^2 = dx^2+dy^2+dz^2$, and the sphere's surface is defined by $z^2 = R^2-x^2-y^2$. Defining $x = r\cos\theta$ and $y = r\sin\theta$ we have $dx^2+dy^2 = dr^2+r^2d\theta^2$ and $z = \sqrt{R^2-r^2}$, $dz^2 = r^2dr^2/(R^2-r^2)$, so that the metric on the sphere can be written as

$$ds^2 = \frac{R^2}{R^2-r^2}dr^2 + r^2d\theta^2 = R^2\left(\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2d\theta^2\right) \tag{91}$$

where $\tilde{r} = r/R$. Give example of circle circumference,

$$\begin{aligned} D(r) &= 2\int_0^r \frac{dr'}{\sqrt{1-r'^2/R^2}} = 2R\sin^{-1}\left(\frac{r}{R}\right) = 2r\left[1+\frac{1}{6}\left(\frac{r}{R}\right)^2 + O\left(\frac{r^4}{R^4}\right)\right], \\ \int_0^{2\pi} d\theta r &= 2\pi r < \pi D(r) = 2\pi r[1+(r/R)^2/6]. \end{aligned} \tag{92}$$

The spatial curvature is $1/R^2$.

***Comoving observers.*** For the FRW metric

$$\Gamma^\mu_{tt} = \frac{1}{2}g^{\mu\alpha}\left(2\frac{\partial g_{t\alpha}}{c\partial t} - \frac{\partial g_{tt}}{\partial x^\alpha}\right) = 0, \tag{93}$$

which implies that $\mathbf{r} = const.$ is a solution of the equations of motion of a particle. This in turn implies that $t$ is the time measured by a clock at rest in this coordinate system, i.e. at fixed $\mathbf{r}$. We define a "comoving observer" as an observer sitting at fixed $\mathbf{r}$. All such observers are equivalent, the universe appears the same to all of them. $\mathbf{r}$ is termed the "comoving coordinate." A comoving observer moves along $\mathbf{x} = R(t)\mathbf{r} = [R(t)/R(t_0)]R(t_0)\mathbf{r} = a(t)\mathbf{x}_0$.

***The proper distance*** is defined as

$$d_{prop}(r,t) \equiv \int_0^r dr'\sqrt{g_{rr}(r',t)} = R\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = R(t)rf(r), \tag{94}$$

with

$$f(r) = \begin{cases} \sin^{-1}(r)/r, & \text{k=+1;} \\ \sinh^{-1}(r)/r, & \text{k=-1;} \\ 1, & \text{k=0.} \end{cases} \tag{95}$$

***Redshift.*** Light travels along $cd\tau = 0$. For a photon emitted in the $\hat{r}$ direction, $c^2d\tau^2 = c^2dt^2 - g_{rr}dr^2$, so that $cdt = \sqrt{g_{rr}}dr = Rdr/\sqrt{1-kr^2}$. A photon emitted from a source at $r$ and received at $r = 0$ at time $t$ was emitted at time $t_i(r,t)$ given by

$$\int_{t_i(r,t)}^t \frac{cdt'}{R(t')} = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = rf(r) \tag{96}$$

Consider two signals emitted at $t_i$ and $t_i+\delta t_i$ at $r$, and received (at $r=0$) at $t$ and at $t+\delta t$. The relation between $\delta t$ and $\delta t_i$ is obtained by differentiating with respect to $t$, which gives $R^{-1}(t) - R^{-1}(t_i)\partial t_i/\partial t = 0$, i.e.

$$\frac{\delta t}{\delta t_i} = \frac{R(t)}{R[t_i(r,t)]} \equiv 1 + z(r,t). \tag{97}$$

A photon emitted at $r$ with wavelength $\lambda$ is measured as $(1+z)\lambda$.

***Angular diameter distance.*** What is the angular size of a sphere of diameter $D$ lying at $r$? Consider two photons emitted from the sphere's edges reaching us. They both move on fixed $\hat{\Omega}$, so that the size of the object is given by $D = rR[t_i(r,t)]d\theta$ (assuming of course $D \ll Rr$). The angular diameter distance is therefore

$$d_A(r) = R[t_i(r,t)]r = (1+z)^{-1}R(t)r. \tag{98}$$

***Luminosity distance.*** Consider a source emitting luminosity $L$ sitting at $r$. Our detector of diameter $D$ occupies a solid angle $\pi d\theta^2/4 = \pi[D/2rR(t)]^2$ as seen by the source. Since the arrival time between photons is larger than the emission time by $1+z$ and since the energy of each photon is decreased by $1+z$, the flux observed is $f = (\pi d\theta^2/4\pi)L/\pi D^2(1+z)^2 = L/4(1+z)^2\pi r^2 R(t)^2$. Defining $d_L$ by $f = L/4\pi d_L^2$,

$$d_L = (1+z)R(t)r. \tag{99}$$

***$r(z)$ for small $z$.*** Expand $\dot{R}$ as

$$\dot{R}(R) = \dot{R}_0 + \frac{\ddot{R}_0}{\dot{R}_0}\Delta R + O(\Delta R^2) = R_0 H_0 \left(1 + \frac{R_0\ddot{R}_0}{\dot{R}_0^2}\frac{\Delta R}{R_0}\right) + O(\Delta R^2) = R_0 H_0(1 - q_0\Delta a) + O(\Delta a^2), \tag{100}$$

where $a = R/R_0$, $q_0 = -R_0\ddot{R}_0/\dot{R}_0^2$, $\Delta a = a - 1$ and we have defined $H_0 \equiv \dot{R}_0/R_0$. We then have

$$\begin{aligned}\int_{t_i(r,t_0)}^{t_0} \frac{cdt'}{R(t')} &= R_0^{-1}\int_{a(r)}^{1}\frac{cda}{a\dot{a}} = \frac{c/R_0}{H_0}\int_{\Delta a(r)}^{0}\frac{dx}{(1+x)[1-q_0 x + O(x^2)]} \\ &= -\frac{c/R_0}{H_0}\left[\Delta a(r) + \frac{1}{2}(q_0 - 1)\Delta a^2(r) + O(\Delta a^3)\right]. \end{aligned} \tag{101}$$

Comparing with the rhs of the eq.,

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = r + \frac{1}{6}kr^3 + O(r^5), \tag{102}$$

we have

$$r = \frac{c/R_0}{H_0}\left[-\Delta a - \frac{1}{2}(q_0-1)\Delta a^2 + O(\Delta a^3)\right], \tag{103}$$

and

$$d_L = \frac{c}{aH_0}\left[-\Delta a - \frac{1}{2}(q_0-1)\Delta a^2 + O(\Delta a^3)\right], \tag{104}$$

or using $\Delta a/a = -z(r, t_0)$

$$d_L = \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + O(z^3)\right]. \tag{105}$$

Comoving observers satisfy Hubble's law for lowest order in $z$, deviations at higher $z$ provide information on $\ddot{a}$. Identifying $\dot{R}_0/R_0$ with the measured value of $H_0$ assumes the observed galaxies are "comoving observers".

## 4.6 GR basics: II. Physics laws, gravity, dynamics 1.5×[2hr]

### 4.6.1 Tensors and the laws of physics with gravity

The laws of physics are written in special relativity as equalities between tensors and their derivatives. EM, e.g., is described by $\eta^{\mu\nu}\partial_\mu\partial_\nu A^\alpha = -(4\pi/c)j^\alpha$ for $\partial_\mu A^\mu = 0$. These laws are the same in all frames, since the derivatives of tensors are also tensors. This, however, is no longer true in GR. If $V^\mu$ is a vector then

$$\frac{\partial V'^\mu}{\partial x'^\nu} = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\mu}{\partial x^\alpha}V^\alpha\right) = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha}V^\alpha. \tag{106}$$

The second term of the RHS destroys the tensor behavior (in special relativity the transformations are linear, and this term vanishes). *Without proof:* We replace the derivatives with covariant derivatives,

$$\frac{DV^\delta}{Dx^\gamma} \equiv \frac{\partial V^\delta}{\partial x^\gamma} + \Gamma^\delta_{\gamma\alpha}V^\alpha, \tag{107}$$

which conserve the tensor behavior (and reduce to ordinary derivatives in the local flat frame). Similarly, for a vector defined along a path (e.g. particle momentum), $V^\mu(p)$ where $p$ is a parameter along the path, we replace the ordinary derivative with

$$\frac{DV^\delta}{Dp} \equiv \frac{dV^\delta}{dp} + \Gamma^\delta_{\gamma\alpha}\frac{dx^\gamma}{dp}V^\alpha. \tag{108}$$

The prescription for writing the eqs. of phys.:

1. Write special relativity eqs. in tensor form;
2. Replace $d$ with $D$ and $\eta_{\mu\nu}$ with $g_{\mu\nu}$.

### 4.6.2 Gravity

The result for weak field, $g_{00} = -(1 + 2\Phi/c^2)$, may be written, recalling $\nabla^2\Phi = 4\pi G\rho$, as

$$\nabla^2 g_{00} = -\frac{2}{c^2}\nabla^2\Phi = -\frac{8\pi G\rho}{c^2} = -\frac{8\pi G}{c^4}T_{00}. \tag{109}$$

***Recall*** $T_{\mu\nu} = c^{-2}(p+e)u_\mu u_\nu + p g_{\mu\nu}$**.** We therefore guess that the field eqs. are

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{110}$$

where $G_{\mu\nu}$ is a tensor composed of $g_{\mu\nu}$ and its 1st and 2nd derivatives. The requirements from $G_{\mu\nu}$ are: (i) Symmetric tensor; (ii) Each of its terms contains 2 derivatives in order for the dimensions to be $L^{-2}$ (unless there is a new constant with dimensions of length in the theory); (iii) $g^{\mu\alpha}D_\alpha G_{\mu\nu} = 0$ to conserve energy; (iv) $G_{00} = \nabla^2 g_{00}$ in the Newtonian limit.

*Without proof:* The only tensor that can be formed from the metric and its 1st and 2nd derivatives is the Riemann-Christoffel tensor,

$$R^\alpha_{\beta\gamma\delta} \equiv \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} + \Gamma^\alpha_{\delta\mu}\Gamma^\mu_{\beta\gamma} - \Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\beta\delta}. \tag{111}$$

*Without proof:* A necessary and sufficient condition for the existence of a flat coordinate system (where the metric is $\eta_{\mu\nu}$) is that $R^\alpha_{\beta\gamma\delta} = 0$ everywhere (and that there is a point where $g_{\mu\nu}$ has 1 negative and 3 positive eigenvalues). This tensor is therefore called the curvature tensor. ***The parallel displacement*** of a $S^\mu$ along a path is determined by $DS^\mu/Dp = 0$. It is straightforward to show that for a closed infinitesimal path around some point $X^\mu$,

$$\Delta S_\mu = \frac{1}{2}R^\alpha_{\mu\beta\gamma}(X^\mu)\oint dx^\beta x^\gamma S_\alpha. \tag{112}$$

The Ricci tensor is $R_{\beta\delta} \equiv g^{\alpha\gamma}R_{\alpha\beta\gamma\delta}$ and the curvature scalar is $R \equiv g^{\alpha\beta}R_{\alpha\beta}$. For the FRW metric, the curvature scalar of the spatial part is $k/R^2$.

*Without proof:* The requirements (i-iii) imply $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$. Thus, the field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{113}$$

Contraction with $g^{\mu\nu}$ gives

$$R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu}, \tag{114}$$

which may be used to write the field eqs. as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta} \right). \tag{115}$$

If we allow a constant with the dimensions of length, we may have $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R - \lambda g_{\mu\nu}$ (where the dimensions of $\lambda$ are $L^{-2}$). This would not give the correct Newtonian limit, so $\lambda$ must be small enough. In this case we have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}. \tag{116}$$

Comment on $\lambda$ as part of $T_{\mu\nu}$, $p_\lambda = -e_\lambda$. Contraction with $g^{\mu\nu}$ gives

$$R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} - 4\lambda, \tag{117}$$

which may be used to write the field eqs. as

$$R_{\mu\nu} + \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta} \right). \tag{118}$$

### 4.6.3 Dynamics

For the FRW metric, the affine connections that do not vanish are

$$\begin{aligned}
\Gamma^t_{ij} &= \frac{1}{2} g^{t\alpha} \left( \frac{\partial g_{i\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\alpha} \right) = -\frac{1}{2} g^{tt} \frac{\partial g_{ij}}{c\partial t} = \frac{R\dot{R}}{c} \tilde{g}_{ij}, \\
\Gamma^i_{tj} &= \frac{1}{2} g^{i\alpha} \left( \frac{\partial g_{t\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{c\partial t} - \frac{\partial g_{tj}}{\partial x^\alpha} \right) = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{c\partial t} = \frac{\dot{R}}{Rc} \delta^i_j, \\
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk},
\end{aligned} \tag{119}$$

and the Ricci tensor is

$$\begin{aligned}
R_{tt} &= 3\frac{\ddot{R}}{c^2 R}, \\
R_{it} &= 0, \\
R_{ij} &= -\left( \frac{1}{c^2} R\ddot{R} + \frac{2}{c^2} \dot{R}^2 + 2k \right) \tilde{g}_{ij}.
\end{aligned} \tag{120}$$

We assume that the universe is filled with a plasma that may be considered a perfect fluid, for which $T_{\mu\nu} = c^{-2}(p+e)u_\mu u_\nu + p g_{\mu\nu}$. Using $u^\mu u_\mu = -c^2$ and $g^\mu_\mu = 4$ we find

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda = (p+e)\frac{u_\mu u_\nu}{c^2} + \frac{1}{2}(e-p) g_{\mu\nu}. \tag{121}$$

If the fluid is not at rest in the local "freely falling frame" $\mathbf{r} = const.$, its velocity would define a special direction. We therefore have $u^0 = dr^0/d\tau = c$ and $u^i = 0$, which gives

$$\begin{aligned} S_{tt} &= \frac{1}{2}(e+3p), \\ S_{it} &= 0, \\ S_{ij} &= \frac{1}{2}(e-p) g_{ij}. \end{aligned} \tag{122}$$

Comparing eqs. (120) and (122), we have

$$\begin{aligned} 3\frac{\ddot{R}}{R} - \lambda c^2 &= -\frac{4\pi G}{c^2}(e+3p), \\ R\ddot{R} + 2\dot{R}^2 + 2kc^2 - \lambda c^2 R^2 &= \frac{4\pi G}{c^2}(e-p)R^2. \end{aligned} \tag{123}$$

Comment on the similarity/difference from the Newtonian limit ($3p$, $p \ll e$). $\lambda$ may be thought of an addition to the energy momentum tensor with

$$e_\Lambda = \frac{\lambda c^4}{8\pi G}, \quad p_\Lambda = -e_\Lambda. \tag{124}$$

Substituting $\ddot{R}$ from the first eq. into the 2nd, we have

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 \equiv \frac{\dot{R}^2}{R^2} &= \frac{8\pi G}{3c^2}\left(e + \frac{c^4}{8\pi G}\lambda\right) - \frac{kc^2}{R^2} \\ &= H_0^2 \left(\frac{e}{\rho_c c^2} + \frac{\rho_\Lambda}{\rho_c} - \frac{kc^2/H_0^2}{a^2 R_0^2}\right) \end{aligned} \tag{125}$$

with

$$\rho_c = \frac{3H_0^2}{8\pi G}, \quad \rho_\Lambda = \frac{\lambda c^2}{8\pi G}. \tag{126}$$

Comparing this eq. with the Newtonian one we have derived,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\frac{\rho}{\rho_c} + \frac{\rho_\Lambda}{\rho_c} + \left(1 - \frac{\rho_0}{\rho_c} - \frac{\rho_\Lambda}{\rho_c}\right) a^{-2}\right], \tag{127}$$

we identify $a$ with $R$ (up to scaling factor) and the current curvature is

$$\frac{k}{R_0^2} = -\left(1 - \frac{\rho_0}{\rho_c} - \frac{\rho_\Lambda}{\rho_c}\right)\frac{H_0^2}{c^2}. \tag{128}$$

Comment on connection between curvature and energy content. Curvature radius is of order $c/H_0$ or larger, unless $|\rho_0 + \rho_\Lambda| >> \rho_c$.

### 4.7 Perturbations 1.5×[2hr]

Explain "Jeans scale": Pressure support, speed of sound, $\lambda_J = c_s(a/\dot{a}) = (c_s/c)\lambda_H$ (perturbations grow as power of $a$, see below), perturbations grow for $\lambda > \lambda_J$ and oscillate for $\lambda < \lambda_J$, $\lambda_J = \lambda_H$ for radiation domination, $\lambda_J \ll \lambda_H$ after decoupling. Consider $\lambda \gg \lambda_J$ at matter domination.

Consider a spherical region where the energy density is homogeneously perturbed. Since we are discussing scales $\lambda \gg \lambda_J$, the perturbed region may interact with the rest of the universe only through gravity- the thermal motion of particles leading to energy and momentum transfer (pressure and diffusion effects) may be neglected. According to Birkhoff's theorem, we may ignore the gravitational effects of the universe outside our perturbed sphere, provided that the universe is homogeneous outside the sphere. Although we are discussing the evolution of a non-homogeneous universe, we are considering only small, linear, perturbations. For such perturbations, the gravitational effects of the inhomogeneities outside the sphere on the evolution of the sphere, i.e. the interaction between perturbations, is a second order effect, which may therefore be ignored.

The perturbed sphere would thus evolve as if it were a part of a homogeneous universe, where the energy density is everywhere different than that of the unperturbed homogeneous universe. The evolution of the perturbed region is therefore described by a solution $a(t)$ of the equation (125), with parameters that differ from that of the unperturbed universe. Since we are free to choose the normalization of the expansion factor $a$, we shall choose a normalization of $a$ for the perturbed solution so that both the perturbed and non perturbed universes have the same density for $a = 1$ (of course, the two solutions may reach $a = 1$ at different times $t$). With this choice of $a$ normalization, Eq. (125) depends on a single parameter, the curvature term $\alpha_1 = kc^2/R_0^2$. The solution for $a(t)$ depends on one additional parameter, the integration constant of the first order differential equation. It will be useful below to consider time, $t$ to be a function of expansion factor, $a$,

$t(a;\alpha_1,\alpha_2)$, where $\alpha_2$ is the integration constant,

$$t = \alpha_2 + \int^a \frac{da}{\dot{a}(\alpha_1)}. \tag{129}$$

The energy density has a power law dependence on $a$, $e \propto a^{-m}$ with $m = 3$ for matter domination and $m = 4$ for radiation domination. Since we have normalized $a$ so that the energy density of both (perturbed and unperturbed) solutions is the same for given $a$ , the fractional energy density perturbation related to a perturbation corresponding to modification of the parameters $\alpha_i$ is $\delta \equiv \delta e/e = -m\delta a/a = -ma^{-1}(\partial a/\partial\alpha_i)\delta\alpha_i$. Note, that the derivative of $a$ with respect to $\alpha_i$ is taken at constant $t$, since the energy density perturbation is defined as the difference between the density of the two (perturbed and unperturbed) solutions at some fixed time $t$. The variation of the solution $a(t;\alpha_1,\alpha_2)$ with respect to $\alpha_i$, $\partial a/\partial\alpha_i$, may be obtained by the following consideration. Taking the partial derivatives with respect to $\alpha_i$ of the *rhs* and *lhs* of Eq. (129) at constant $t$ we find,

$$0 = \frac{\partial t}{\partial\alpha_1} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_1} - \int^a \frac{da}{\dot{a}^2}\frac{\partial\dot{a}}{\partial\alpha_1} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_1} - \int^a \frac{da}{2\dot{a}^3}\,, \tag{130}$$

and

$$0 = \frac{\partial t}{\partial\alpha_2} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_2} + 1\,. \tag{131}$$

Thus,

$$\frac{\partial a}{\partial\alpha_1} = \dot{a}\int^a \frac{da}{2\dot{a}^3}\,, \quad \frac{\partial a}{\partial\alpha_2} = -\dot{a}, \tag{132}$$

and the growth of the two perturbations modes is given by

$$\delta_1 \propto \frac{\dot{a}}{a}\int^a \frac{da}{2\dot{a}^3}\,, \quad \delta_2 \propto \frac{\dot{a}}{a}. \tag{133}$$

Note, that the evolution of perturbations is independent of the perturbation wavelength, as long as $\lambda \gg \lambda_J$.

At early time, $a \ll 1$, we may approximate $\dot{a}^2 \propto a^{2-m}$, with $m = 3$ for matter domination and $m = 4$ for radiation domination, which implies

$$\delta_1 \propto a^{m-2}\,, \quad \delta_2 \propto a^{-m/2}. \tag{134}$$

The growing mode evolves as $a^2$ during radiation domination, and as $a$ during matter domination.

### 4.8 The parameters of the universe

- Approximation for $d_A(a)$:

$$d_A(a) \approx \frac{2c}{H_0} a \times \begin{cases} \Omega^{-1}, & \Lambda = 0; \\ \Omega^{-1/3}, & \Omega \ll 1, \Omega + \Lambda = 1. \end{cases} \tag{135}$$

The angular scale corresponding to the size of the horizon at decoupling, $\lambda_{\rm dec.} = ac/\dot{a}_{\rm dec.} = a(c/H_0)(a_{\rm dec.}/\Omega)^{1/2}$, is given by

$$\begin{aligned} \theta_{\rm dec.} &\approx \frac{1}{2} a_{\rm dec.}^{1/2} \times \begin{cases} \Omega^{1/2}, & \Lambda = 0; \\ \Omega^{-1/6}, & \Omega \ll 1, \Omega + \Lambda = 1. \end{cases} \\ &= 1^\circ \times \begin{cases} \Omega^{1/2}, & \Lambda = 0; \\ \Omega^{-1/6}, & \Omega \ll 1, \Omega + \Lambda = 1. \end{cases} \end{aligned} \tag{136}$$

Determination of $\theta_{\rm dec.}$ from CMB observations, roughly the peak in the angular power spectrum, provides a stringent constraint on the value of $\Omega$ and $\Lambda$ and hence on the geometry. MAXIMA/Boomerang/WMAP give

$$\Omega + \Lambda = 1.02 \pm 0.02, \quad \text{where} \quad \Omega \equiv \frac{\rho_m}{\rho_c}, \quad \Lambda \equiv \frac{\rho_\lambda}{\rho_c}. \tag{137}$$

This implies that the geometry is nearly flat,

$$\frac{k}{R_0^2} = (0.02 \pm 0.02) \frac{H_0^2}{c^2}. \tag{138}$$

- Taking the derivative of Friedmann's eq. during matter domination,

$$\dot{a}^2 = H_0^2 \left[ \Omega a^{-1} + \Lambda a^2 + (1 - \Omega - \Lambda) \right], \tag{139}$$

with respect to time and dividing by $2\dot{a}$ we have

$$\ddot{a} = H_0^2 (a\Lambda - \Omega/2a^2), \tag{140}$$

from which we obtain

$$q_0 \equiv -\frac{a_0 \ddot{a}_0}{\dot{a}_0^2} = \frac{1}{2}\Omega - \Lambda = \frac{3}{4}(\Omega - \Lambda) - \frac{1}{4}(\Omega + \Lambda). \tag{141}$$

$q_0$ is therefore sensitive to $\Omega - \Lambda$. Given $\Omega + \Lambda$ from CMB, measuring

$$d_L(z) = (c/H_0)[z + (1 + \Lambda - \Omega/2) z^2/2] \tag{142}$$

using SNIa gives $\Omega - \Lambda$. Combining CMB and SNIa, $\Lambda \approx 0.7$. Comment on systematics.

- The neutron to proton ratio is kept at the equilibrium value, $N_n/N_p \approx \exp(-\Delta Mc^2/T)$ with $\Delta Mc^2 = 1.3$ MeV, by the interactions $p + e \leftrightarrow n + \nu_e$ and $p + \bar{\nu}_e \leftrightarrow n + e^+$ down to $T = 0.8$ MeV, at which point the interaction rate drops below $H$. The "freeze out" of $N_n/N_p$ at $N_n/N_p \approx 0.2$ leads to the production of D, $^3$He, $^4$He (and some Li and Be). The freeze out $T$, and the resulting isotopic ratios, depend on the number density of nucleons. From measurements of D:H in distant ("primordial") gas clouds, $\Omega_b h^2 \approx 0.02$. This is consistent with the baryon density we see today and with the secondary peaks in the CMB (recall $\lambda < \lambda_J$), which give $\Omega_b h^2 = 0.022 + 0.001$ assuming $k = 0$ with $\Omega + \Lambda = 1$.

- The amplitude of the perturbations observed in the CMB, $\delta T/T \sim 10^{-5}$, reflect the amplitude of perturbations in the baryons at $a = a_{\rm dec}$. For an $\{\Omega = 1, \Lambda = 0\}$ universe, the perturbations are amplified since then by a factor $1/a_{\rm dec}$, see eq. (133), while for an $\{\Omega \ll 1, \Lambda = 0\}$ universe the growth is suppressed by a factor $\sim \Omega$. Such growth is not sufficient to produce the large scale density fluctuations we see today in the galaxy distribution, with $\delta \sim 1$ on 10 Mpc scale (comment: 10 Mpc$< \lambda_{\rm dec}$). The common solution: adding non-baryonic "dark-matter", that has weak electromagnetic coupling and decoupled from radiation at $a \ll a_{\rm dec}$. It increases the amplitude of today's fluctuations by increasing the growth rate (through increase in $\Omega$) and by producing $\delta_m/\delta_b \approx a_{\rm dec}/a_{\rm eq.} = 6(\Omega h_{75}^2/0.3)$ at $a_{\rm dec}$. The latter effect is due to the growth of fluctuations in dark-matter between $a_{\rm eq.}$ and $a_{\rm dec}$, a period during which growth of fluctuations in the baryon component is suppressed due to its coupling to the radiation (recall Jeans). The required value of $\Omega$, $\Omega h_{75}^2 \simeq 0.3$, is consistent with the dark-matter to baryon ratio inferred from galaxy clusters, $\Omega/\Omega_b \simeq 7$.

- New dark matter particles- most popular are Weakly Interacting Massive Particles (WIMPs), both direct (recoil) and indirect (annihilation to $\gamma$, $\nu$) searches, so far only upper limits. What $\Lambda$ is- not known.

- Why the universe contains as much baryons as it does- unknown. $1/\eta = n_b/n_\gamma \sim 10^{-9}$, known matter-anti matter asymmetry predicts much smaller value.