Problem set 2: Cosmology

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1. ***Newtonian approximation.*** Consider the metric $d\tau^2 = dt^2 - a^2(t)d\mathbf{x}^2$ (FWR metric with $k = 0$). Define a new coordinate system by:

$$\xi^0 = t + \frac{1}{2}\dot{a}(t)a(t)\mathbf{x}^2, \quad \xi^i = a(t)x^i. \tag{1}$$

   Calculate the metric in the new coordinate system in the vicinity of $\xi^i = 0$ up to second order terms in the spatial coordinates ξ^i. Hint- it is easier to first calculate $g^{\mu\nu}$ up to second order terms in x^i. Show that this coordinate system is locally inertial at $\xi^i = 0$ (for any ξ^0). What are the velocities of the nearby (nearby = up to first order in ξ^i) comoving observers as a function of distance in this reference frame? Show that the accelerations of the nearby comoving observers are given by the Newtonian approximation $\ddot{\xi}^i = -\frac{1}{2}\nabla g_{00}$, are the conditions for the use of the Newtonian approximation satisfied? Show that the conditions for the validity of the expression of the gravitational mass of an ideal fluid, $\rho_G = \rho + 3p$, are satisfied at $\xi^i = 0$ and use it to write the (exact) Friedmann equation for $\ddot{a}$.

2. Comoving stars radiating at constant luminosity L and distributed homogenously throughout the universe (with a FRW metric) are lit at time t_0. What is the flux measured by a comoving observer at time t_1? Write the answer in terms of the function $a(t)$ and the density of stars at time t_1. Calculate in two ways: (i) Sum up the contribution of the fluxes of the stars from which light had enough time to arrive; (ii) Write an equation for the local energy density in the emitted photons and integrate it from t_0 to t. Does the result depend on the value of the spatial curvature k? For $a(t) \propto t^\alpha$, what is the condition α has to satisfy so that the result will not diverge for $t_0 \to 0$.

3. The evolution equations for the cosmic scale factor, $a(t)$, may be written as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \rho_i(1+3\omega_i), \quad (3)$$

where the present Hubble constant is positive, $H_0 > 0$, and the evolution of the energy density in component i is determined by

$$\frac{d\rho_i}{\rho_i} = -3(1+\omega_i)\frac{da}{a} \Rightarrow \rho_i \propto a^{-3(1+\omega_i)}. \quad (4)$$

(a) For all forms of normal matter, $\rho_i > 0$ and $\rho_i + 3p_i = \rho_i(1+3\omega_i) > 0$. Argue that a universe with $k \leq 0$ will expand forever, and that a universe with $k > 0$ will eventually recollapse.

(b) For a cosmological constant, ρ_Λ could be positive or negative and $\rho_\Lambda = -p_\Lambda$. Argue that a universe with $\rho_\Lambda < 0$ will eventually recollapse, independent of k, and that a universe with $k > 0$ and $\rho_\Lambda > 0$ may expand forever. Find a condition on $(\rho_\Lambda/\rho_m)_{t=t_0}$ for this to happen. Assume that $\Omega_0 \approx 1$. Find the (approximate) minimal ratio $(\rho_\Lambda/\rho_m)_{t=t_0}$ that will prevent forever recollapse.

(c) Consider a universe that is flat and contains both matter and a cosmological constant ($\Omega = \Omega_\Lambda + \Omega_m = 1$). Prove that the age of such a universe is given by

$$t_0 = \frac{2}{3}H_0^{-1}\Omega_\Lambda^{-1/2}\ln\left[\frac{1+\Omega_\Lambda^{1/2}}{(1-\Omega_\Lambda)^{1/2}}\right]. \quad (5)$$

Find the lower bound on Ω_Λ for the universe to be older than H_0^{-1}.

4. ***Reionization.*** We have mentioned that the Universe plasma has been "re-ionized" at some redshift $z_{\rm reion.}(\sim 10)$. CMB photons may be Thomson scattered on their way to us by the free electrons at $z < z_{\rm reion.}$. Assume, for simplicity, that all H atoms were instantaneously ionized at $z = z_{\rm reion.}$. Derive an approximate expression for the optical depth for scattering, $\tau(z = z_{\rm reion.})$, for $\{\Omega = 1, \Lambda = 0\}$ cosmology. For what $z_{\rm reion.}$ does $\tau(z = z_{\rm reion.}) = 1$? If $\tau(z = z_{\rm reion.}) > 1$, then CMB photons do not propagate freely, but rather diffuse, from $z = z_{\rm reion.}$ till today. What effect would this have on the observed CMB anisotropy $\delta T/T$? On what angular scales?