Problem set 3: Spectacular explosions

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1. ***Hydro gymnastics.***

   (a) ***Lagrangian hydro eqs.*** Consider a one dimensional flow (i.e. with planar, cylindrical or spherical symmetry) of an ideal fluid. In this case the flow is described by functions of two variables, the time t and position r. Define $r(r_0, t)$ as the position r at which the fluid element, which was located at $r = r_0$ at time $t = t_0$, is located at time t. Derive the fluid flow equations in terms of r_0 and t (instead of r and t).

   (b) ***Simple simple waves.*** An ideal gas (adiabatic index γ) of uniform density ρ_0 and uniform speed of sound c_0 occupies the half space $x > 0$, bounded by a planar piston at $x = 0$. At $t = 0$ the piston begins moving with constant acceleration a in the x direction. Determine the resulting flow for both $a > 0$ and $a < 0$. For the $a > 0$ case, determine the time t_0 at which the solution (e.g. the function $v(x, t = t_0)$) becomes non single valued. What happens at this stage?

2. ***SNR emission.*** A spherical shell of mass M and kinetic energy E, ejected by a supernova explosion, expands into a uniform density ISM of density $n_{\rm ISM}$. Estimate the luminosity and spectrum of the (thermal) Bremsstrahlung emission of the shocked ISM at the onset of shell deceleration. Give numerical values normalized to $M = 1M_\odot$, $E = 10^{51}$ erg, $n_{\rm ISM} = 1{\rm cm}^{-3}$. Determine the temporal evolution of the luminosity and spectrum before and after the onset of deceleration.

3. ***Core collapse.***

   (a) ***Photodisintegration.*** Write the Saha equilibrium equation (relating the densities $n_{\rm Fe}$, n_α and n_n) for the Fe disintegration reaction, ${}^{56}_{26}{\rm Fe}+\gamma \leftrightarrow 13\alpha + 4n$, assuming that the nuclei are not degenerate (i.e. follow a Maxwell-Boltzmann distribution). You may assume that the statistical weights of the nuclear partition functions are $g_i = 1$, i.e. that the nuclei are at their ground states (for what temperature range, roughly, will this be a valid assumption?). Recall that the chemical potentials satisfy $\mu_{\rm Fe} = 13\mu_\alpha + 4\mu_n$. Estimate the temperature at which half the Fe is dissociated at the characteristic core density of 10^9g/cm^3 (the dissociation energy is 124 MeV).

(b) Repeat 3(a) for He disintegration, ${}^4_2$He$+\gamma \leftrightarrow 2p + 2n$ (the dissociation energy is 28 MeV).

(c) ***Neutronization.*** Consider a cold ($T = 0$) ideal gas of protons, electrons and neutrons. Assume that the neutrinos produced in the $e^- + p \to n + \nu$ interaction freely escape the gas, and that $n \to p + e^- + \bar{\nu}$ is not allowed due to electron degeneracy. Using the equilibrium requirement $\mu_n = \mu_p + \mu_e$, determine (i) The minimum density at which neutrons appear (you may assume that the protons are not-relativistic at this density, and then verify the validity of this assumption); (ii) The proton to neutron density ratio obtained in the limit of infinite density; (iii) The (approximate) density at which the proton to neutron density ratio is minimal (and the value of this minimum).

4. ***Accretion disks.*** We have mentioned that in case the gas in-falling onto a compact object has angular momentum, we expect the in-falling gas to form a thin planar accretion disk. Assume that a compact object of mass M is surrounded by a planar thin gas disk of inner radius R, in which the gas is rotating at circular Keplerian orbits. Since the angular velocity decreases outwards, viscosity between adjacent gas rings tend to slow down the inner ring (transferring angular momentum outwards). The loss of angular momentum leads to in-fall inward motion, i.e. to accretion. Assume that the inward velocity v_r is very small compared to the Keplerian velocity v_ϕ. For a steady disk, the mass accretion rate $\dot{M}$ is uniform (independent of r).

   (a) Assume that all the energy released by the in-fall, which is converted to heat by the viscosity, is not stored in the gas but rather radiated away instantaneously. Assume further that the energy and angular momentum of the gas falling inward of R are simply absorbed by the compact object. What is the luminosity per unit radius of the disk, dL/dr? What is the total luminosity L?

(b) Assume next that the compact object absorbs only a fraction β of the angular momentum flowing through R (at a rate of $\dot{M}\sqrt{GMR}$). This may happen if the compact object is rotating and exerting a torque on the gas. How are dL/dr and L modified? Hint: Show that the torque applied by the compact object leads to energy extraction from the compact object at a rate $(1-\beta)\dot{M}GM/R$.

(c) Assuming that the emission is thermal, what is the disk temperature $T(r)$?

(d) The disk has a finite thickness h. Neglecting the gravity of the disk (i.e. taking into account only the gravity of the compact object), assuming $h/r \ll 1$, and requiring the gas to be in hydrostatic equilibrium (supported by pressure against gravity) in the vertical direction (perpendicular the plane of the disk), show that $h/r \sim c_s/v_\phi$. What is then required for the disk to be thin?

(e) Consider accretion on a NS producing $L \sim 10^{37}$erg/s, assuming $R \sim R_{\rm NS}$ and $\beta \approx 1$. Derive the numerical values of rdL/dr, h, T and ρ at R. What is the optical depth (in the vertical direction) as function of r, assuming the opacity is dominated by Thomson scattering and that the gas is fully ionized?