GR Basics

April 28, 2021

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1 Equivalence principle, metrics, Newtonian limit

Much of this chapter can be found in the excellent book *Gravitation & Cosmology* of Weinberg.

1.1 The equivalence principle and the concept of a metric

Inertial and gravitational mass equivalence is naturally obtained by the assumption that the effects of $\vec{g}$ is equivalent to measurements in an "accelerating elevator," $x'^i = x^i + \frac{1}{2}g^i t^2$.

The Equivalence principle: For any point $x^\mu = X^\mu$ a locally inertial coordinate system may be constructed, $\xi^\mu(x^\mu; X^\mu)$, in which the laws of physics in an infinitesimal (space-time) region around X^μ are given by special relativity. Inertial and gravitational mass equivalence is automatically obtained.

The "photon in elevator" example. $\Delta h\nu = -\Delta\Phi(h\nu/c^2)$. Discuss E and gravitational mass, clock tick rate (via freq. shift at a stationary system).

If the local coordinate systems are known for every point $x = X$, the effects of gravity are completely determined. In what follows we will first derive the eqs. of physics under the assumption that $\xi^\mu(x^\mu; X^\mu)$ are given. We will then derive the equations that determine the local inertial coordinate systems given the distribution of gravitating mass.

Remind special relativity ideas: 4-vectors, $\eta_{\mu\nu}$, $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, $ds^2 = -c^2dt^2 + \vec{v}^2dt^2 = -c^2(1-\beta^2)dt^2 = -c^2dt^2/\gamma^2 = -c^2d\tau^2$ for massive particles. The motion of a free particle is described by $d^2x^\mu/d\tau^2 = 0$, where the proper time is determined by $c^2d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$.

Massive particles: Let us first determine $d\tau$ in the general coordinate system.

$$-c^2d\tau^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}dx^\alpha dx^\beta, \tag{1}$$

i.e.

$$-c^2d\tau^2 = g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta \tag{2}$$

with

$$g_{\alpha\beta}(X^\mu) \equiv \left[\eta_{\mu\nu}\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha}\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta}\right]_{x^\mu=X^\mu}. \tag{3}$$

The metric is invariant under Lorentz transformations of the local inertial frames ξ^μ. Discuss the general relation between metrics in different coordinate systems, $g'(x')$ vs. $g(x)$.

The motion of a particle under gravity is described in the local inertial frame by $d^2\xi^\mu/d\tau^2 = 0$, which may be written as

$$0 = \frac{d}{d\tau}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}. \tag{4}$$

multiplying by $\partial x^\alpha/\partial\xi^\mu$ we have

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\rho}(x^\mu)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}, \tag{5}$$

with the Christoffel symbol

$$\Gamma^\alpha_{\nu\rho}(X^\mu) \equiv \left[\frac{\partial^2\xi^\mu(x^\mu;X^\mu)}{\partial x^\nu\partial x^\rho}\frac{\partial x^\alpha(\xi^\mu;X^\mu)}{\partial\xi^\mu}\right]_{x^\mu=X^\mu}. \tag{6}$$

For any given initial conditions, $x^\mu(\tau=0)$ and $dx^\mu/d\tau(\tau=0)$, the particle's trajectory is determined by $\Gamma^\alpha_{\nu\rho}(x^\mu)$. $\Gamma^\alpha_{\nu\rho}(x^\mu)$ determines therefore the effect of gravity. Similarly, for massless particles

$$0 = \frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\nu\rho}(x^\mu)\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma}. \tag{7}$$

g,Γ and the local "freely falling" frame. Given $\Gamma^\alpha_{\nu\rho}(x^\mu)$ and $g_{\alpha\beta}(x^\mu)$ at some point $x^\mu = X^\mu$, the local inertial frame at $x = X$, $\xi^\alpha(x;X)$ is determined in the vicinity of $x = X$ (up to a Lorentz transformation) up to (and including) $(x-X)^2$. Define

$$\xi^\mu(x,X) = a^\mu_\alpha dx^\alpha + \frac{1}{2}b^\mu_{\alpha\beta}dx^\alpha dx^\beta, \tag{8}$$

where $dx = x - X$. Eq. (3) gives

$$\eta_{\alpha\beta}a^\alpha_\mu a^\beta_\nu = g_{\mu\nu}(x=X), \tag{9}$$

and eq. (6) gives

$$b^\nu_{\beta\gamma} = \left[\frac{\partial\xi^\nu}{\partial x^\alpha}\Gamma^\alpha_{\beta\gamma}\right]_{x=X} = a^\nu_\alpha\Gamma^\alpha_{\beta\gamma}(x=X). \tag{10}$$

It is straightforward to verify that if these conditions are satisfied for some ξ^μ, they are also satisfied for any other coordinate system obtained by a Lorentz transformation $\tilde{\xi}^\mu = \Lambda^\mu_\nu\xi^\nu$.

The meaning of "infinitesimal region around X" in the equivalence principle is

$$\left[\frac{\partial g_{\alpha\beta}(x^\mu;X^\mu)}{\partial x^\lambda}\right]_{x^\mu=X^\mu}=0, \tag{11}$$

where $g_{\alpha\beta}(x^\mu;X^\mu)$ is the metric in the coordinate system $\xi^\mu(x^\mu;X^\mu)$ at the point $\xi^\mu(x^\mu;X^\mu)$. That is, in the local inertial frame associated with the point $x=X$ the metric is $\eta_{\alpha\beta}$ at $x=X$ and its first order derivatives at $x=X$ vanish. Note that $\partial_\xi g(\xi;X)=(\partial x/\partial\xi)\partial_x g(\xi(x);X)=(\partial x/\partial\xi)\partial_x g(x;X)$.

We prove below that this implies

$$\frac{\partial g_{\alpha\beta}}{\partial x^\mu}=g_{\beta\rho}\Gamma^\rho_{\mu\alpha}+g_{\alpha\rho}\Gamma^\rho_{\mu\beta}. \tag{12}$$

Adding the permutations with respect to $\alpha\beta\mu$ (and multiplying by the inverse $g^{\alpha\beta}$ of $g_{\alpha\beta}$), we find

$$\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^\nu}+\frac{\partial g_{\nu\rho}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right). \tag{13}$$

This relation implies that the effects of gravity, $\Gamma^\alpha_{\nu\rho}(x^\mu)$, are determined by the metric $g_{\alpha\beta}(x^\mu)$.

Proof: Consider a coordinate system x^μ with metric $g_{\alpha\beta}(x^\mu)$. Let us define $g_{\alpha\beta}(x^\mu;X^\mu)$ as the metric in the coordinate system $\xi^\mu(x^\mu;X^\mu)$. We know that $g_{\alpha\beta}(x^\mu=X^\mu;X^\mu)=\eta_{\alpha\beta}$. We will now show that eq. (11) leads to eq. (12).

The metrics are related through

$$g_{\alpha\beta}(x^\mu)=g_{\mu\nu}(x^\mu;X^\mu)\left[\frac{\partial\xi^\mu(x'^\mu;X^\mu)}{\partial x'^\alpha}\frac{\partial\xi^\nu(x'^\mu;X^\mu)}{\partial x'^\beta}\right]_{x'^\mu=x^\mu}. \tag{14}$$

Taking the derivative with respect to x^μ and setting $x^\mu=X^\mu$ we obtain

$$\begin{aligned}
\left[\frac{\partial g_{\alpha\beta}(x^\mu)}{\partial x^\lambda}\right]_{x^\mu=X^\mu} &= \left[\frac{\partial g_{\mu\nu}(x^\mu;X^\mu)}{\partial x^\lambda}\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha}\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta}\right]_{x^\mu=X^\mu}\\
&+ \eta_{\mu\nu}\left[\frac{\partial^2\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha\partial x^\lambda}\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta}+\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha}\frac{\partial^2\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta\partial x^\lambda}\right]_{x^\mu=X^\mu}\\
&= \eta_{\mu\nu}[\Gamma^\rho_{\alpha\lambda}(X^\mu)\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\rho}\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\beta}\\
&\quad +\Gamma^\rho_{\beta\lambda}(X^\mu)\frac{\partial\xi^\nu(x^\mu;X^\mu)}{\partial x^\rho}\frac{\partial\xi^\mu(x^\mu;X^\mu)}{\partial x^\alpha}]_{x^\mu=X^\mu}\\
&= \Gamma^\rho_{\alpha\lambda}(X^\mu)g_{\rho\beta}(X^\mu)+\Gamma^\rho_{\beta\lambda}(X^\mu)g_{\rho\alpha}(X^\mu).
\end{aligned}$$

Two important points should be explained here.

- The condition $c^2 = -g_{\alpha\beta}(x^\mu)(dx^\alpha/d\tau)(dx^\beta/d\tau)$, with $g_{\alpha\beta}(x^\mu) \equiv g_{\alpha\beta}(x^\mu; X^\mu = x^\mu)$, can be chosen to hold at $\tau = 0$. Eq. (12) ensures that if it holds at $\tau = 0$ it holds also at $\tau > 0$:

$$\begin{aligned}\frac{d}{d\tau}\left[g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right] &= \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} + g_{\alpha\beta}\frac{d^2x^\alpha}{d\tau^2}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d^2x^\beta}{d\tau^2} \\ &= \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} - g_{\alpha\beta}\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^\beta}{d\tau} - g_{\alpha\beta}\Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau} \\ &= \left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu} - g_{\beta\rho}\Gamma^\rho_{\mu\alpha} - g_{\alpha\rho}\Gamma^\rho_{\mu\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau}. \qquad (15)\end{aligned}$$

- The equation of motion (5) can be obtained using eq. (13) by requiring $\int_A^B d\tau$, with τ given by eq. (2), to be stationary.

1.2 Newtonian limit- slow motion in a stationary weak field

Slow motion implies that $dx^i/cd\tau \sim v/c \ll dx^0/cd\tau = dt/d\tau \sim 1$. The equation of motion is therefore approximately given by

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{00}(x^\mu)\left(\frac{dx^0}{d\tau}\right)^2. \qquad (16)$$

Weak field implies $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h \ll 1$. Since the field is stationary we have

$$\Gamma^0_{00} = \frac{1}{2}g^{0\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho}\right) = -\frac{1}{2}g^{0i}\frac{\partial g_{00}}{\partial x^i} = O(h^2), \qquad (17)$$

$$\Gamma^i_{00} = \frac{1}{2}g^{i\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho}\right) = -\frac{1}{2}g^{ij}\frac{\partial g_{00}}{\partial x^j} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} + O(h^2). \qquad (18)$$

Eqs. (17) and (16) imply $dt/d\tau = const.$, which allows replacing τ with t in eq. (16) leading to, using eq. (18),

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}. \qquad (19)$$

In order to obtain the Newtonian limit we must have $h_{00} = -2\Phi/c^2 + const.$. Defining Φ to vanish at infinity (far away from sources) the constant must vanish and we have

$$g_{00} = -\left(1 + \frac{2}{c^2}\Phi\right). \qquad (20)$$

- Discuss meaning of weak field: Earth, Sun.
- Discuss time: For a stationary clock $d\tau = \sqrt{-g_{00}}dt = (1 + \Phi/c^2)dt$. Connect to redshift, Doppler in "elevator". Discuss t as determined by a clock at infinity, with $\Phi = 0$, and "broadcasted" to all other points.

2 The eqs. of physics with gravity

The laws of physics are written in special relativity (SR) as equalities between tensors and their derivatives. Recall vectors and tensors.

- Scalars- unchanged under coordinate transformation. $d\tau$, proper density ρ_0.
- Contra-variant vector- transforms like the coordinates, $dx'^\mu = (\partial x'^\mu/\partial x^\alpha)dx^\alpha$: $V'^\mu = (\partial x'^\mu/\partial x^\alpha)V^\alpha = \Lambda^\mu_\alpha V^\alpha$, example $u^\mu = dx^\mu/d\tau$.
- Covariant vector, $V'_\mu = (\partial x^\alpha/\partial x'_\mu)V_\alpha$, example $\partial f/\partial x^\mu$. If V^μ is contra-variant, $V_\mu = g_{\mu\nu}V^\nu$ is covariant

$$V'_\mu = g'_{\mu\nu}V'^\nu = g'_{\mu\nu}\frac{\partial x'^\nu}{\partial x^\alpha}V^\alpha = g_{\beta\gamma}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\alpha}g^{\alpha\delta}V_\delta = \frac{\partial x^\beta}{\partial x'^\mu}V_\beta. \tag{21}$$

- Tensor $T'^{\alpha\beta}_\gamma = ...$ Examples: $g_{\mu\nu}$, $\partial_\nu V^\mu$ in SR.
- Product, contraction, lowering/rasing of indices of a tensor give a tensor, $g'_{\mu\nu}T'^{\mu\nu\alpha} = (\partial x'^\alpha/\partial x^\beta)g_{\gamma\delta}T^{\gamma\delta\beta}$.

Example: EM. Maxwell's eqs.: $\nabla\mathbf{B} = 0$, $\nabla\mathbf{E} = 4\pi\rho$, $\nabla\times\mathbf{E} = -(1/c)\partial_t\mathbf{B}$, $\nabla\times\mathbf{B} = (1/c)\partial_t\mathbf{E} + (4\pi/c)\mathbf{j}$. We can write $\mathbf{B} = \nabla\times\mathbf{A}$ ($\nabla\mathbf{B}$) and $\mathbf{E} = -(1/c)\partial_t\mathbf{A} - \nabla\Phi$ ($\nabla\times\mathbf{E}$). Defining $A^\mu = \{\Phi, \mathbf{A}\}$ we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{22}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. The eqs. for $\nabla\mathbf{B}$ and $\nabla\times\mathbf{E}$ are

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0. \tag{23}$$

Defining $j^\mu = \{c\rho, \mathbf{j}\} = \{c, \mathbf{v}\}\rho = \{c, \mathbf{v}\}\gamma\rho_0 = u^\mu\rho_0$ where ρ_0 is the proper charge density, the eqs. for $\nabla\mathbf{E}$ and $\nabla\times\mathbf{B}$ are

$$\partial_\nu F^{\nu\alpha} = -\frac{4\pi}{c}j^\alpha. \tag{24}$$

These eqs. define F in any coordinate system. Since j^μ is a vector (and derivation in SR, contraction and lowering/rasing of indices of a tensor give a tensor) F is a tensor- if $F^{\mu\nu}$ is the solution at coordinate system x, then $F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$ is the solution at $x' = \Lambda x$,

$$\partial'_\nu F'^{\nu\alpha} = (\Lambda^{-1})^\gamma_\nu \partial_\gamma \Lambda^\nu_\mu \Lambda^\alpha_\beta F^{\mu\beta} = \Lambda^\alpha_\beta \partial_\mu F^{\mu\beta} = -\frac{4\pi}{c}\Lambda^\alpha_\beta j^\beta = -\frac{4\pi}{c} j'^\alpha . \quad (25)$$

The eqs. for particle acceleration, $d\mathbf{p}/dt = q(\mathbf{E} + c^{-1}\mathbf{v}\times\mathbf{B})$, may be written as

$$m\frac{du^\mu}{d\tau} = f^\mu = \frac{q}{c}\eta_{\nu\alpha}F^{\mu\nu}u^\alpha. \quad (26)$$

This eq. is the same in all systems (related by Lorentz transformations) since it is an equality between tensors and their contractions, multiplications and derivatives. In the particle's rest frame, where $u^\mu = \{c, \mathbf{0}\}$, the rhs is $f^\mu = -qF^{\mu 0} = \{0, q\mathbf{E}\}$ as required.

Covariant derivatives.

In SR, the physics eqs. are the same in all systems (related by Lorentz transformations) since they are expressed as equalities between tensors and their derivatives, and the derivatives of tensors (as well as their contractions) are also tensors. This, however, is no longer true in GR. If V^μ is a vector then

$$\frac{\partial V'^\mu}{\partial x'^\nu} = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\mu}{\partial x^\alpha}V^\alpha\right) = \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha}V^\alpha. \quad (27)$$

The second term of the RHS destroys the tensor behavior (in special relativity the transformations are linear, and this term vanishes). We may obtain the tensor that reduces to the regular derivative in the local inertial frame in the following way. Taking x' to be the inertial frame at X, $\xi(x;X)$, we have

$$\frac{\partial V^{X\mu}}{\partial \xi^\nu} = \frac{\partial x^\beta}{\partial \xi^\nu}\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial \xi^\nu}\frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha}V^\alpha. \quad (28)$$

Multiplying by $(\partial\xi^\nu/\partial x^\gamma)(\partial x^\delta/\partial \xi^\mu)$,

$$\frac{\partial \xi^\nu}{\partial x^\gamma}\frac{\partial x^\delta}{\partial \xi^\mu}\frac{\partial V^{X\mu}}{\partial \xi^\nu} = \frac{\partial V^\delta}{\partial x^\gamma} + \frac{\partial x^\delta}{\partial \xi^\mu}\frac{\partial^2 \xi^\mu}{\partial x^\gamma \partial x^\alpha}V^\alpha, \quad (29)$$

which implies

$$\frac{\partial \xi^\nu}{\partial x^\gamma}\frac{\partial x^\delta}{\partial \xi^\mu}\left[\frac{\partial V^{X\mu}}{\partial \xi^\nu}\right]_{\xi(x=X;X)} = \frac{\partial V^\delta}{\partial x^\gamma} + \Gamma^\delta_{\gamma\alpha}V^\alpha. \quad (30)$$

We therefore define the covariant derivative as

$$\frac{DV^\delta}{Dx^\gamma} \equiv \frac{\partial V^\delta}{\partial x^\gamma} + \Gamma^\delta_{\gamma\alpha}V^\alpha. \quad (31)$$

This derivative reduces to the regular derivative in the local inertial frame, and the derivative of the vector yields a tensor,

$$\frac{DV'^{\delta}}{Dx'^{\gamma}} = \frac{\partial x'^{\delta}}{\partial \xi^{\mu}} \frac{\partial \xi^{\nu}}{\partial x'^{\gamma}} \frac{dV^{X\mu}}{d\xi^{\nu}} = \frac{\partial x'^{\delta}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \xi^{\mu}} \frac{\partial \xi^{\nu}}{\partial x^{\theta}} \frac{\partial x^{\theta}}{\partial x'^{\gamma}} \frac{dV^{X\mu}}{d\xi^{\nu}} = \frac{\partial x'^{\delta}}{\partial x^{\alpha}} \frac{\partial x^{\theta}}{\partial x'^{\gamma}} \frac{DV^{\alpha}}{Dx^{\theta}}. \tag{32}$$

For a covariant vector we have

$$\frac{DV_{\delta}}{Dx^{\gamma}} \equiv \frac{\partial V_{\delta}}{\partial x^{\gamma}} - \Gamma^{\alpha}_{\gamma\delta} V_{\alpha}. \tag{33}$$

This can be shown using a similar derivation as for the contra-variant vector, choosing this time x as the inertial ξ system.

Consider next a vector defined along a path, $V^{\mu}(p)$ where p is a parameter along the path (e.g. particle momentum).

$$\frac{dV'^{\mu}(p)}{dp} = \frac{d}{dp}\left[\frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(p)\right] = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dV^{\alpha}}{dp} + \frac{\partial^2 x'^{\mu}}{\partial x^{\beta} \partial x^{\alpha}} \frac{dx^{\beta}}{dp} V^{\alpha}. \tag{34}$$

Again, choosing the primed system to be the local inertial frame,

$$\frac{dV^{X\mu}(p)}{dp} = \frac{\partial \xi^{\mu}}{\partial x^{\alpha}} \frac{dV^{\alpha}}{dp} + \frac{\partial^2 \xi^{\mu}}{\partial x^{\beta} \partial x^{\alpha}} \frac{dx^{\beta}}{dp} V^{\alpha}, \tag{35}$$

and multiplying by $\partial x^{\gamma}/\partial \xi^{\mu}$,

$$\frac{\partial x^{\gamma}}{\partial \xi^{\mu}} \frac{dV^{X\mu}(p)}{dp} = \frac{dV^{\gamma}}{dp} + \Gamma^{\gamma}_{\alpha\beta} \frac{dx^{\beta}}{dp} V^{\alpha}. \tag{36}$$

We therefore define

$$\frac{DV^{\delta}}{Dp} \equiv \frac{dV^{\delta}}{dp} + \Gamma^{\delta}_{\gamma\alpha} \frac{dx^{\gamma}}{dp} V^{\alpha}. \tag{37}$$

Similarly, we define

$$\frac{DV_{\delta}}{Dp} \equiv \frac{dV_{\delta}}{dp} - \Gamma^{\alpha}_{\gamma\delta} \frac{dx^{\gamma}}{dp} V_{\alpha}. \tag{38}$$

The covariant derivative of the metric vanishes, $D_{\mu} g_{\alpha\beta} = \partial_{\mu} g_{\alpha\beta} - \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu} g_{\nu\alpha} = 0$, since we require the metric derivatives to vanish in the local inertial frame. This implies that contraction (and lowering and raising of indices) and covariant derivation are commutative.

The prescription for writing the eqs. of physics:

1. Write special relativity eqs. in tensor form;
2. Replace ∂ with D and $\eta_{\mu\nu}$ with $g_{\mu\nu}$.

This gives eqs. that are the same in all frames, and reduce to SR in the local inertial frames. For EM we have

$$D_\nu F^{\nu\alpha} = -\frac{4\pi}{c} j^\alpha, \quad m\frac{Du^\mu}{D\tau} = \frac{q}{c} g_{\nu\alpha} F^{\mu\nu} u^\alpha. \tag{39}$$

For the motion of a particle under gravity we have

$$\frac{Du^\mu}{D\tau} = 0, \quad \text{i.e.} \quad \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0. \tag{40}$$

3 Gravity

We define the parallel displacement of a S^μ by $DS^\mu/Dp = 0$,

$$\frac{dS^\mu}{dp} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dp}S^\beta. \tag{41}$$

It is straightforward to show that for a closed infinitesimal path around some point X^μ,

$$\Delta S_\mu = \frac{1}{2}R^\alpha_{\mu\beta\gamma}(X^\mu)\oint dx^\beta x^\gamma S_\alpha, \tag{42}$$

where the Riemann-Christpffel tensor is

$$R^\alpha_{\beta\gamma\delta} \equiv \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} + \Gamma^\alpha_{\delta\mu}\Gamma^\mu_{\beta\gamma} - \Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\beta\delta}. \tag{43}$$

The Riemann-Christoffel tensor is the only tensor that can be formed from the metric and its 1st and 2nd derivatives, and is linear in the second derivatives. The Ricci tensor is $R_{\beta\delta} \equiv g^{\alpha\gamma}R_{\alpha\beta\gamma\delta}$ and the curvature scalar is $R \equiv g^{\alpha\beta}R_{\alpha\beta}$. A useful (Bianchi) identity

$$D_\mu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0. \tag{44}$$

A necessary and sufficient condition for the existence of a flat coordinate system (where the metric is $\eta_{\mu\nu}$) is that $R^\alpha_{\beta\gamma\delta} = 0$ everywhere (and that there is a point where $g_{\mu\nu}$ has 1 negative and 3 positive eigenvalues). This tensor is therefore called the curvature tensor.

The result for weak field, $g_{00} = -(1 + 2\Phi/c^2)$, may be written, recalling $\nabla^2\Phi = 4\pi G\rho$, as

$$\nabla^2 g_{00} = -\frac{2}{c^2}\nabla^2\Phi = -\frac{8\pi G\rho}{c^2} = -\frac{8\pi G}{c^4}T_{00}. \tag{45}$$

[Recall SR energy momentum tensor- what the different components are, the example of an ideal fluid.] We therefore guess that the field eqs. are

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{46}$$

where $G_{\mu\nu}$ is a tensor composed of $g_{\mu\nu}$ and its 1st and 2nd derivatives. The requirements from $G_{\mu\nu}$ are: (i) Symmetric tensor; (ii) Each of its terms contains 2 derivatives in order for the dimensions to be L^{-2} (unless there is a new constant with dimensions of length in the theory); (iii) $g^{\mu\alpha}D_\alpha G_{\mu\nu} = 0$

to conserve energy; (iv) $G_{00} = \nabla^2 g_{00}$ in the Newtonian limit (stationary weak field produced by non-relativistic matter).

Requirement (ii) implies that $G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$. (iii) determines $c_2/c_1 = -1/2$,

$$D_\mu G^{\mu\nu} = c_1 D_\mu R^{\mu\nu} + c_2 g^{\mu\nu} D_\mu R = \left(\frac{1}{2}c_1 + c_2\right) g^{\mu\nu} D_\mu R = 0 \tag{47}$$

($g^{\mu\nu} G_{\mu\nu} = (c_1 + 4c_2)R$ so that $D_\mu R = 0$ everywhere only if $D_\nu T^\mu_\mu = 0$ everywhere, which does not hold for an arbitrary mass distribution). The Newtonian limit determines $c_1 = 1$. In this limit, $|T_{ij}| \ll |T_{00}|$ so that $|G_{ij}| \ll |G_{00}|$ implying $R_{ij} \simeq \frac{1}{2} g_{ij} R \simeq \frac{1}{2}\delta_{ij} R$, and $R \simeq \eta^{\mu\nu} R_{\mu\nu} = -R_{00} + \frac{3}{2}R$, so that $R = 2R_{00}$ and $G_{00} = 2c_1 R_{00}$. For a weak field, $R_{00} = \frac{1}{2}\nabla^2 g_{00}$ hence $c_1 = 1$.

Thus, the field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}. \tag{48}$$

Contraction with $g^{\mu\nu}$ gives

$$R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu}, \tag{49}$$

which may be used to write the field eqs. as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}\right). \tag{50}$$

If we allow a constant with the dimensions of length, we may have $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R - \lambda g_{\mu\nu}$ (where the dimensions of λ are L^{-2}). This would not give the correct Newtonian limit, so λ must be small enough. In this case we have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}. \tag{51}$$

Contraction with $g^{\mu\nu}$ gives

$$R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} - 4\lambda, \tag{52}$$

which may be used to write the field eqs. as

$$R_{\mu\nu} + \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}\right). \tag{53}$$