Galaxy clusters and the Sunyaev-Zel'dovich effect

Doron Kushnir

1. Galaxy clusters

Clusters of galaxies (GC) are configurations containing typically hundreds of galaxies in a region about $r \sim$ Mpc in size. The radial velocity dispersion of the galaxies is $\sigma_r \sim 10^3\,\mathrm{km\,s^{-1}}$, suggesting that GC are gravitationally bound, otherwise they would disperse on a crossing timescale

$$t_c \sim \frac{r}{v_r} \sim \left(\frac{r}{\mathrm{Mpc}}\right)\left(\frac{\sigma_r}{10^3\,\mathrm{km\,s^{-1}}}\right)^{-1}\,\mathrm{Gyr} \ll t_H \approx 14\,\mathrm{Gyr}, \tag{1}$$

where σ_r has been substituted for the radial velocity, v_r. The required mass to keep GC bound is (the binding energy is twice the kinetic energy from the virial theorem):

$$\begin{aligned} \frac{GM}{r} &\sim \langle v^2\rangle \approx 3\sigma_r^2 \\ &\Rightarrow M \sim 3\frac{r\sigma_r^2}{G} \sim 7\times 10^{14}\left(\frac{\sigma_r}{10^3\,\mathrm{km\,s^{-1}}}\right)^2\left(\frac{r}{\mathrm{Mpc}}\right) M_\odot, \end{aligned} \tag{2}$$

where we assumed that the galaxy velocity vector orientations are uncorrelated, $\langle v^2\rangle = 3\sigma_r^2$. The total luminosity from a GC is $L \sim 10^{13} L_\odot$ (that is ~ 1000 galaxies with a typical galaxy luminosity of $\sim 10^{10} L_\odot$), such that the mass-to-light ratio is $M/L \sim 70\,M_\odot/L_\odot$. Careful studies that also take into account the mass profile with the GC suggest $M/L \sim 300\,M_\odot/L_\odot$. This can be compared with the mass-to-light ratio in galaxies, $M/L \sim (1-10)\,M_\odot/L_\odot$, such that only $\sim 10\%$ of the mass in GC can be accounted for by the galaxies (the "missing mass" problem). The missing mass is naturally explained in the ΛCDM model, where the mass of the GC is dominated by DM.

Observation by the X-ray satellite *Uhuru* (70s) established that

- GC are the most common bright extragalactic X-ray sources, with luminosities $L_X \sim 10^{43} - 10^{45}\,\mathrm{erg\,s^{-1}}$.
- The X-ray sources associated with GC are extended with sized $0.2 - 3$ Mpc.
- GC have X-ray spectra that show no strong evidence for low-energy photoabsorption, unlike the spectra of compact sources.
- The X-ray emission from GC is not time variable, as is the emission from many point sources of X-rays.

These finding suggest that the emission from GC is diffuse, and not the result of one or many compact sources. Later spectral X-ray observations with *OSO*-8 and *Ariel*-5 established that the primary X-ray emission mechanism is thermal emission from diffuse hot intra-cluster gas (ICM). To appreciate this, we next discuss *bremsstrahlung* emission.

2. Bremsstrahlung

Radiation due to the acceleration of charge in the Coulomb field of another charge is called bremsstrahlung of *free-free emission*. The ICM should have temperatures such that the thermal velocity of the protons, $\sim \sqrt{T/m_p}$, is comparable to the velocity of the galaxies in the GC, as both are bound by the same gravitational potential. We find $T \sim m_p\sigma_r^2 = m_p c^2(\sigma_r/c)^2 \approx \mathrm{Gev}(10^{-2}/3)^2 \approx 10\,\mathrm{keV}$, in the X-ray range. The electrons are non-relativistic with $T/(m_e c^2) \sim 10/500 = 0.02 \ll 1$, so we focus on non-relativistic bremsstrahlung. Although quantum treatment is required, as the photons with energies comparable to that of the electrons can be produced, we will use classical treatment and state the quantum results as corrections (*Gaunt factors*).

Consider an electron with charge $-e$ that moves in a fixed Coulomb field of an ion with charge Ze located et the origin (the relative accelerations are inversly proportional to the mass, so we neglect the acceleration of the ion). We neglect the deviation of the electron's path from a straight line (*small-angle scattering* regime), which has an impact parameter b with the ion. The dipole moment is $\vec{d} = -e\vec{R}$, where $\vec{R}$ is the position of the electron. The second derivative is $\ddot{\vec{d}} = -e\dot{\vec{v}}$, where $\vec{v}$ is the velocity of the electron. We state a few results from the non-relativistic dipole approximation for the power emitted into solid angle $d\Omega$, the total power emitted, and the total energy per frequency range:

$$
\begin{aligned}
\frac{dP}{d\Omega} &= \frac{(\ddot{\vec{d}})^2}{4\pi c^3}\sin^2\Theta \\
P &= \frac{2(\ddot{\vec{d}})^2}{3c^3} \\
\frac{dW}{d\omega} &= \frac{8\pi\omega^4}{3c^3}\left|\hat{d}(\omega)\right|^2,
\end{aligned}
\tag{3}
$$

where Θ is the angle with the direction of the acceleration (no radiation is emitted along the direction of the acceleration and the maximum is emitted perpendicular to the acceleration), $\hat{d}(\omega)$ is the Fourier transform of $d(t)$, $d(t) = \int_{-\infty}^{\infty} \exp^{-i\omega t}\hat{d}(\omega)d\omega$. Note that the dipole moment in an electron-electron system or in an ion-ion system is zero, justifying out focus on the electron-ion system. Taking the Fourier transform of $\ddot{\vec{d}} = -e\dot{\vec{v}}$, we find

$$
-\omega^2\hat{\vec{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\vec{v}}\exp^{-i\omega t}dt. \tag{4}
$$

The electron is in close interaction with the ion over a time interval $\tau = b/v$ (the *collision time*). For $\omega\tau \gg 1$ the exponential oscillates rapidly and the integral is small. For $\omega\tau \ll 1$ the exponential is unity, so we can write

$$
\hat{\vec{d}}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v}, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1, \end{cases} \tag{5}
$$

where $\vec{\Delta v}$ is the change of velocity during the collision. Using Equation (3) we find

$$\frac{dW}{d\omega} \sim \begin{cases} \frac{2e^2}{3\pi c^3}\left|\vec{\Delta v}\right|^2, & \omega\tau \ll 1 \\ 0, & \omega\tau \gg 1, \end{cases} \tag{6}$$

$\vec{\Delta v}$ is almost normal to the path, so we can estimate

$$\Delta v = \frac{Ze^2}{m_e}\int_{-\infty}^{\infty}\frac{bdt}{(b^2+v^2t^2)^{3/2}} = \frac{Ze^2}{m_e b}\frac{t}{\sqrt{b^2+v^2t^2}}\Big|_{-\infty}^{\infty} = \frac{2Ze^2}{m_e bv}. \tag{7}$$

The emission from a single electron is, therefore,

$$\frac{dW(b)}{d\omega} \sim \begin{cases} \frac{8Z^2e^6}{3\pi c^3 m_e^2 v^2 b^2}\left|\vec{\Delta v}\right|^2, & b \ll v/\omega \\ 0, & b \gg v/\omega. \end{cases} \tag{8}$$

For ion density n_i, electron density n_e, and a fixed electron speed v, the flux of electrons per unit area per unit time on one ion is $n_e v$ with the area element of $2\pi bdb$. The total emission per unit time per unit volume per unit frequency range is then

$$\frac{dW}{d\omega dVdt} = n_e n_i 2\pi v \int_{b_{\rm min}}^{\infty}\frac{dW(b)}{d\omega}bdb, \tag{9}$$

where $b_{\rm min}$ discussed below. We use (and later justify) the low frequency limit of Equation (8) in Equation (10) to find

$$\frac{dW}{d\omega dVdt} = \frac{16e^6}{3c^3m_e^2v}n_e n_i Z^2\int_{b_{\rm min}}^{b_{\rm max}}\frac{db}{b} = \frac{16e^6}{3c^3m_e^2v}n_e n_i Z^2 \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right), \tag{10}$$

where $b_{\rm max}$ is where larger b values satisfy $b \ll v/\omega$ and the contribution to the integral is negligible. Because of the logarithmic dependence we can just take $b_{\rm max} = v/\omega$, since for most logarithmic intervals the low frequency asymptotic limit is applicable. The value of $b_{\rm min}$ can be estimated in two ways. The small-angle scattering approximation is invalid when $\Delta v \sim v$, so we define $b^{(1)}_{\rm min} = 4Ze^2/\pi m_e v^2$. The classical treatment is invalid where $bm_e v \sim \hbar$, so we define $b^{(2)}_{\rm min} = h/m_e v$. The transition between $b^{(1)}_{\rm min}$ and $b^{(2)}_{\rm min}$ happens when $mv^2/2 \sim Z^2 Ry$, where $Ry = \alpha^2 m_e c^2/2 = me^4/2\hbar^2 \approx 13.6\,{\rm eV}$ is the Rydberg energy for the hydrogen atom.

The exact results are tabulated in terms of the Gaunt factor, $g_{ff}(v,\omega)$:

$$\frac{dW}{d\omega dVdt} = \frac{16\pi e^6}{3\sqrt{3}c^3m_e^2v}n_e n_i Z^2 g_{ff}(v,\omega), \tag{11}$$

where

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right). \tag{12}$$

For the thermal emission we need to integrate over the probability than en electron has a speed in the speed range dv:

$$d\mathcal{P} \propto v^2 \exp\left(-\frac{m_e v^2}{2T}\right) dv. \tag{13}$$

We find

$$\frac{dW(T,\omega)}{d\omega dV dt} = \frac{\int_{v_{\rm min}}^{\infty} \frac{dW(v,\omega)}{d\omega dV dt} v^2 \exp\left(-\frac{m_e v^2}{2T}\right) dv}{\int_0^\infty v^2 \exp\left(-\frac{m_e v^2}{2T}\right) dv}, \tag{14}$$

where $v_{\rm min}$ is set by the condition that at least one photon can be emitted (*photon discreteness effect*), $h\nu \le m_e v^2/2$, or $v_{\rm min} = \sqrt{2h\nu/m_e}$. Evaluating Equation (14), we find

$$\begin{aligned}
\frac{dW(T,\nu)}{d\nu dV dt} &= 2\pi \frac{dW(T,\omega)}{d\nu dV dt} = \frac{2\pi \int_{v_{\rm min}}^{\infty} \frac{16\pi e^6}{3\sqrt{3}c^3 m_e^2} n_e n_i Z^2 g_{ff}(v,\omega) v \exp\left(-\frac{m_e v^2}{2T}\right) dv}{\int_0^\infty v^2 \exp\left(-\frac{m_e v^2}{2T}\right) dv} \\
&= \frac{32\pi^2 e^6}{3\sqrt{3}c^3 m_e^2} \sqrt{\frac{m_e}{2T}} n_e n_i Z^2 \frac{\int_{\sqrt{m/2T} v_{\rm min}}^{\infty} g_{ff}(v,\omega) x \exp\left(-x^2\right) dx}{\int_0^\infty x^2 \exp\left(-x^2\right) dx} \\
&= \frac{2^5 \pi e^6}{3c^3 m_e} \sqrt{\frac{2\pi}{3m_e}} T^{-1/2} n_e n_i Z^2 \int_{\sqrt{h\nu/T}}^{\infty} g_{ff}(v,\omega) x \exp\left(-x^2\right) dx \\
&= \frac{2^5 \pi e^6}{3c^3 m_e} \sqrt{\frac{2\pi}{3m_e}} T^{-1/2} n_e n_i Z^2 \exp(-h\nu/T) \bar{g}_{ff},
\end{aligned} \tag{15}$$

where $\bar{g}_{ff}(T,\nu)$ is the *velocity averaged Gaunt factor*. We see the $T^{-1/2}$ dependence and that the spectrum is flat in log-log up to its exponential cut-off at $h\nu \sim T$. The detection of such spectrum from GC (together with line emission of highly ionized iron) established that the X-ray emission is thermal with $T \sim 10\,{\rm keV}$ (we are actually assuming here optically thin plasma, see below). The values of $\bar{g}_{ff}$ for $h\nu/T \gg 1$ are unimportant because of the exponential cut-off. $\bar{g}_{ff}$ is of order unity for $h\nu/T \sim 1$ and it is in the range 1 to 5 for $10^{-4} < h\nu/T < 1$, such that setting $\bar{g}_{ff} = 1$ provides a good order of magnitude estimate. Writing Equation (15) in CGS units, we find:

$$\varepsilon_\nu^{ff} \equiv \frac{dW}{d\nu dV dt} = 6.8 \times 10^{-38} T^{-1/2} n_e n_i Z^2 \exp(-h\nu/T) \bar{g}_{ff}\, {\rm erg\, s^{-1}\, cm^{-3}\, Hz^{-1}}, \tag{16}$$

where T is in K.

The total power per unit volume emitted by thermal bremsstrahlung can be obtained by integrating Equation (15) over frequency:

$$\begin{aligned}
\frac{dW(T)}{dV dt} &= \frac{2^5 \pi e^6}{3c^3 m_e} \sqrt{\frac{2\pi}{3m_e}} T^{-1/2} n_e n_i Z^2 \int_0^\infty \exp(-h\nu/T) \bar{g}_{ff} d\nu \\
&= \frac{2^5 \pi e^6}{3hc^3 m_e} \sqrt{\frac{2\pi T}{3m_e}} n_e n_i Z^2 \bar{g}_B,
\end{aligned} \tag{17}$$

where $\bar{g}_B(T)$ is a frequency average of the velocity averaged Gaunt factor, which is in the range 1.1 to 1.5. Choosing a values of 1.2 will give an accuracy to within about 20%. A more straight forward way to write Equation (18) is (recall $\sigma_T = (8\pi/3)r_0^2$, $r_0 = e^2/m_e c$):

$$\begin{aligned} \frac{dW(T)}{dVdt} &= \sqrt{\frac{8}{6\pi}}\sigma_T\sqrt{\frac{2T}{m_e}}\alpha m_e c^2 n_e n_i Z^2 \bar{g}_B, \\ &\sim t_{\rm coll}^{-1} n_e \alpha m_e c^2 Z^2, \end{aligned} \tag{18}$$

where the time between collisions is $t_{\rm coll} \sim (\sigma_T v_{th} n_i)^{-1}$ with the thermal velocity $v_{th} \sim \sqrt{2T/m_e}$. We see that in each collision the electron emits $\alpha m_e c^2 Z^2$. Equation (18) in CGS is

$$\varepsilon^{ff} \equiv \frac{dW}{dVdt} = 1.4 \times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B \, {\rm erg\,s^{-1}\,cm^{-3}}, \tag{19}$$

where T is in K. We can now estimate the ICM density, assuming it is ionized hydrogen:

$$L_X \sim \varepsilon^{ff} V \sim 1.4 \times 10^{-27} (10^8\,{\rm K})^{1/2} n^2 \times 1.2 \frac{4\pi}{3} ({\rm Mpc})^3 \approx 2 \times 10^{51} n^2 \, {\rm erg\,s^{-1}}. \tag{20}$$

For $L_X = 10^{45}\,{\rm erg\,s^{-1}}$ we find $n \sim 10^{-3}\,{\rm cm^{-3}}$. The total ICM mass is $M_g \sim (4\pi/3){\rm Mpc}^3 n m_p \approx 10^{14} M_\odot$, which is comparable to the mass in the galaxies, but is still $10 - 20\%$ of the total GC mass. This is explained with $\Omega_m/\Omega_b \approx 6.5$

We assumed so far that the plasma is optically thin. The thermal free-free absorption coefficient, α_ν^{ff}, is related to the emission by Kirchhoff's law:

$$\alpha_\nu^{ff} = \frac{\varepsilon_\nu^{ff}}{4\pi B_\nu(T)}, \tag{21}$$

with

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/T) - 1} \tag{22}$$

the Planck function. We find

$$\alpha_\nu^{ff} = \frac{4e^6}{3hcm_e}\sqrt{\frac{2\pi}{3m_e}} T^{-1/2} n_e n_i Z^2 \nu^{-3} \left(1 - \exp(-h\nu/T)\right) \bar{g}_{ff}. \tag{23}$$

In CGS units we get

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-1/2} n_e n_i Z^2 \nu^{-3} \left(1 - \exp(-h\nu/T)\right) \bar{g}_{ff} \, {\rm cm^{-1}}, \tag{24}$$

where T is in K. For any ν value of interest $1/\alpha_\nu^{ff} \gg$ Mpc, justifying the thin plasma approximation.

3. Compton scattering

We next would like to calculate the effect of the scattering of the CMB photons by the ICM. The electrons are non-relativistic and the energy of the CMB photons is much below the ICM temperature, $h\nu \ll T$. The optical depth for scattering in the ICM is $n\sigma_T r \sim 1e-3 \times 6 \times 10^{-25} \times$ Mpc $\sim 10^{-3}$, so we expect a small effect.

We begin by considering Thomson scattering, where the incident photon is described as an electromagnetic wave. For non-relativistic velocities of the electron we can neglect magnetic fields (recall that $E = B$ for an electromagnetic wave), such that the equation of motion due to a linearly polarized wave is $m_e\ddot{\vec{r}} = e\hat{\epsilon}E_0 \sin\omega_0 t$, where $\hat{\epsilon}$ is the E-field direction. For the dipole moment we get $m\ddot{\vec{d}} = e^2\hat{\epsilon}E_0 \sin\omega_0 t$. From Equation (3) we find

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{e^4E_0^2}{8\pi m_e^2c^3}\sin^2\Theta, \\ P &= \frac{e^4E_0^2}{3m_e^2c^3}, \end{aligned} \tag{25}$$

where the time average of $\sin^2\omega_0 t$ gives a factor 1/2. Defining the *differential cross section* $d\sigma$ for scattering into $d\Omega$ by

$$\frac{dP}{d\Omega} = \langle S\rangle\frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi}\frac{d\sigma}{d\Omega}, \tag{26}$$

where $\langle S\rangle$ is the incident flux, we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm polarized} = \frac{e^4}{m_e^2c^4}\sin^2\Theta = r_0^2\sin^2\Theta. \tag{27}$$

For the total cross-section we integrate over solid angle:

$$\sigma = \int\frac{d\sigma}{d\Omega} = 2\pi r_0^2\int_{-1}^{1}(1-\mu^2)d\mu = \frac{8\pi}{3}r_0^2. \tag{28}$$

(Alternatively, we can use $P = \langle S\rangle\sigma$).

Note that the scattered radiation is linearly polarized in the plane of $\hat{\epsilon}$ and the direction of the scattering $\hat{n}$. The cross section for scattering of unpolarized light can be obtained by writing the unpolarized beam as an independent superposition of two linear-polarized beams with perpendicular axes ($\hat{\epsilon}_1$ in the plane of the incident and scattered direction and $\hat{\epsilon}_2$, perpendicular to this plane). As before, Θ is the angle between $\hat{\epsilon}_1$ and $\hat{n}$ (the angle between $\hat{\epsilon}_2$ and $\hat{n}$ is $\pi/2$), and we define $\theta = \pi/2 - \Theta$ (the angle between the direction of the beam, $\hat{\epsilon}_1 \times \hat{\epsilon}_2$, and $\hat{n}$). The unpolarized differential cross section is the average of the linear-polarized cross-sections through angles Θ and $\pi/2$:

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\rm unpolarized} &= \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\rm polarized} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\rm polarized}\right] \\ &= \frac{1}{2}r_0^2(1+\sin^2\Theta) = \frac{1}{2}r_0^2(1+\cos^2\theta), \end{aligned} \tag{29}$$

which depends only on the angle between the incident and scattered direction, as should be for unpolarized radiation. It is also symmetric under reflection ($\theta \to -\theta$) (*forward-backward symmetry*). The total cross-section is obtained by integrating over solid angle (note that $\cos\theta$ has a flat distribution), to obtain $\sigma_{\rm unpolarized} = \sigma_{\rm polarized} = (8\pi/3)r_0^2$. We obtain the same total cross-section since an electron at rest has no net direction intrinsically defined. Finally, the polarized intensities in the plane normal to $\hat{n}$ and perpendicular to the plane are in the ratio $\cos^2\theta : 1$, such that the scattered wave has $\Pi = (1-\cos^2\theta)/(1+\cos^2\theta) \geq 0$ degree of polarization. So an electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization (no net polarization for $\theta = 0$ and 100% polarization for $\theta = \pi/2$).

The cross sections above are frequency independent and the energy of the photon is unchanged after scattering, $\epsilon = \epsilon_1$, so the scattering is *coherent* or *elastic*. These results are only valid for low frequency where we can use the classical description. As $h\nu$ becomes comparable to $m_e c^2$ (X-rays) we must use the quantum mechanical cross sections. We next consider the quantum effects (Compton scattering), which affect both the kinematics and the cross sections.

The kinematics are altered because a photon has a momentum $h\nu/c$ and an energy $h\nu$, so the scattering is no longer elastic ($\epsilon_1 \neq \epsilon$) because of the recoil of the charge. The initial and final four-momenta of the photon are $P_{\gamma i} = (\epsilon/c)(1, \hat{n}_i)$ and $P_{\gamma f} = (\epsilon_1/c)(1, \hat{n}_f)$ with an angle θ between $\hat{n}_i$ and $\hat{n}_f$. The initial and final four-momenta of the electron are $P_{ei} = (m_e c, 0)$ and $P_{ef} = (E/c, \vec{p})$. Conservation of momentum and energy is $P_{\gamma i} + P_{ei} = P_{\gamma f} + P_{ef}$, so we get:

$$
\begin{aligned}
|P_{ef}|^2 &= |P_{\gamma i} + P_{ei} - P_{\gamma f}|^2 \\
\Rightarrow |(E/c, \vec{p})|^2 &= |(mc + \epsilon/c - \epsilon_1/c, \epsilon\hat{n}_i/c - \epsilon_1\hat{n}_f/c))|^2 \\
\Rightarrow -E^2/c^2 + \vec{p}^2 &= -(m_e^2c^2 + \epsilon^2/c^2 + \epsilon_1^2/c^2 + 2m_e\epsilon - 2m_e\epsilon_1 - 2\epsilon\epsilon_1/c^2) \\
&+ \epsilon^2/c^2 + \epsilon_1^2/c^2 - 2\epsilon\epsilon_1\cos\theta/c^2 \\
\Rightarrow \frac{2\epsilon\epsilon_1}{c^2}(\cos\theta - 1) &= 2m_e\epsilon_1 - 2m_e\epsilon \\
\Rightarrow \epsilon_1 &= \frac{\epsilon}{1 + \frac{\epsilon}{m_e c^2}(1-\cos\theta)}.
\end{aligned}
\tag{30}
$$

In terms of wavelength, $\lambda = hc/\epsilon$, we get

$$
\lambda_1 - \lambda = \lambda_c(1 - \cos\theta), \tag{31}
$$

where $\lambda_c \equiv h/m_e c \approx 0.02426\,\mathring{A}$ is the *Compton wavelength*. So there is a wavelength change of order λ_c in scattering. For $\lambda \gg \lambda_c$ ($h\nu \ll m_e c^2$) the scattering is elastic with no net change of the photon energy in the rest frame of the electron.

The quantum effect on the cross section for unpolarized radiation is given by the *Klein-Nishina* formula (given here without a proof)

$$
\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_1^2}{\epsilon^2}\left(\frac{\epsilon}{\epsilon_1} + \frac{\epsilon_1}{\epsilon} - \sin^2\theta\right). \tag{32}
$$

For $\epsilon \sim \epsilon_1$ we get the classical expression. The effect is a reduction in the cross section for high energies, making Compton scattering less efficient. The total cross section can be shown to be

$$\sigma = \sigma_T \frac{3}{4}\left[\frac{1+x}{x^2}\left(\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right) + \frac{1}{2x}\ln(1+2x) - \frac{1+3x}{(1+2x)^2}\right], \tag{33}$$

where $x = h\nu/m_e c^2$. In the non-relativistic regime ($x \ll 1$) we get $\sigma \approx \sigma_T(1 - 2x + 26x^2/5 + ...)$ and in the ultra relativistic regime ($x \gg 1$) we have $\sigma = (3/8)\sigma_T(\ln 2x + 1/2)/x$.

We next consider the case that the electron is in motion, with some β (and Lorentz factor γ). We assume that in the rest frame of the electron $h\nu \ll m_e c^2$, so that we can neglect the relativistic correction in the Klein-Nishina formula. This is obviously the case for the CMB photons and the thermal electrons in the ICM, but even for relativistic electrons in the ICM the condition is fulfilled for $\gamma < m_e c^2/T_{\rm CMB} \sim 10^9$, which is larger than expected for the relativistic electron population in the ICM. We get in the rest-frame of the electron

$$\epsilon_1' = \frac{\epsilon'}{1 + \frac{\epsilon'}{m_e c^2}(1-\cos\Theta)} \approx \epsilon'\left[1 - \frac{\epsilon'}{m_e c^2}(1-\cos\Theta)\right], \tag{34}$$

where Θ is related to the angels in the lab frame (see below). Note that for relativistic electrons, the energies of the photon before scattering, in the rest frame of the electron, and after scattering are in the approximate ratios $1 : \gamma : \gamma^2$, allowing to convert a low-energy photon to a high-energy one by a factor of γ^2 (limited by $\gamma m_e c^2$ from energy conservation). In such a case the scattering process is called *inverse Compton*. Before the availability of GC X-ray spectrum observations, it was suggested that the X-ray emission is inverse Compton from a population of relativistic electrons. The energy density of relativistic electrons in the ICM is in fact much smaller than the thermal energy density, and can be probed with radio (and possibly γ-ray) observations. In what follows, we focus on the non-relativistic electrons in the ICM.

4. The Kompaneets equation

Consider the isotropic photon phase space density, $n(\omega)$, due to scattering from electrons. The *Boltzman equation* for $n(\omega)$ is

$$\frac{\partial n(\omega)}{\partial t} = c\int d^3p \int \frac{d\sigma}{d\Omega} d\Omega \left[f_e(\vec{p}_1)n(\omega_1)(1+n(\omega)) - f_e(\vec{p})n(\omega)(1+n(\omega_1))\right], \tag{35}$$

where we consider the scattering events $p + \omega \leftrightarrow p_1 + \omega_1$ over electrons, with the phase density of momentum $\vec{p}$ given by $f_e(\vec{p})$, including the effect of stimulated scattering with the $1+n$ factors. We next expand the Boltzman equation to second order in the small fractional energy transfer per scattering, yielding an approximation called the *Focker-Planck equation.* For photons scattering off a non-relativistic, thermal distribution of electrons with temperature T, the Focker-Planck equation is known as the Kompaneets equation. We define the dimensionless energy transfer to the photons as

$$\Delta \equiv \frac{\hbar(\omega_1 - \omega)}{T} \ll 1, \tag{36}$$

and expand $n(\omega_1)$ and $f_e(E_1)$, where $f_e(E) = n_e(2\pi m_e T)^{-3/2}\exp(-E/T)$, $E = p^2/2m_e$. The second order expansion for $n(\omega_1)$ is

$$\begin{aligned} n(\omega_1) &= n(\omega) + (\omega_1 - \omega)\frac{\partial n(\omega)}{\partial \omega} + \frac{1}{2}(\omega_1 - \omega)^2 \frac{\partial^2 n(\omega)}{\partial \omega^2} + ... \\ &= n + \Delta n' + \frac{1}{2}\Delta^2 n'' + ..., \end{aligned} \tag{37}$$

where $n' = \partial n/\partial x$, $n'' = \partial^2 n/\partial x^2$, $x = \hbar\omega/T$. The second order expansion for $f_e(E_1)$ is

$$\begin{aligned} f_e(E_1) &= f_e(E) - \frac{E_1 - E}{T} f_e(E) + \frac{1}{2}\frac{(E_1 - E)^2}{T^2} f_e(E) + ... \\ &= f_e + \Delta f_e + \frac{1}{2}\Delta^2 f_e. \end{aligned} \tag{38}$$

Using these expansion in Equation (35) we find:

$$\begin{aligned} \frac{1}{c}\frac{\partial n(\omega)}{\partial t} &\approx \int\int d^3p \frac{d\sigma}{d\Omega} d\Omega f_e \left[(1 + \Delta + \frac{1}{2}\Delta^2)(n + \Delta n' + \frac{1}{2}\Delta^2 n'')(1+n) - n(1 + n + \Delta n' + \frac{1}{2}\Delta^2 n'')\right] \\ &\approx (n'(1+n) + n(1+n) - nn') \int\int d^3p \frac{d\sigma}{d\Omega} d\Omega f_e \Delta \\ &+ \left(\frac{1}{2}n''(1+n) + \frac{1}{2}n(1+n) + n'(1+n) - \frac{1}{2}nn''\right) \int\int d^3p \frac{d\sigma}{d\Omega} d\Omega f_e \Delta^2 \\ &\equiv (n' + n(1+n))I_1 + \left(\frac{1}{2}n'' + \frac{1}{2}n(1+n) + n'(1+n)\right) I_2. \end{aligned} \tag{39}$$

The I_1 term gives the "secular" shift in energy, and the I_2 term gives the "random walk" change in energy. We next calculate these terms.

Suppose the electron is traveling in the z-direction with four-momentum $P_{ei} = (E/c, 0, 0, p)$ and the photon is moving with polar and azimutal angles η and ϕ, respectively, such that $P_{\gamma i} = \epsilon/c(1, \sin\eta\cos\phi, \sin\eta\sin\phi, \cos\eta)$, and after the scattering $P_{\gamma f} = \epsilon_1/c(1, \sin\eta_1\cos\phi_1, \sin\eta_1\sin\phi_1, \cos\eta_1)$. We perform a Lorentz transformation to a frame of reference in which the electron is initially at rest, in which the cross section takes the simple form. In the electron rest-frame, the initial and final photon four-momenta are $LP_{\gamma i}$ and $LP_{\gamma f}$, where L^μ_ν is the Lorentz transformation

$$L = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{40}$$

where $\beta = cp/E$ and $\gamma = (1-\beta^2)^{-1/2}$. We define $LP_{\gamma i} = \epsilon'/c(1, \sin\eta'\cos\phi', \sin\eta'\sin\phi', \cos\eta')$ and $LP_{\gamma f} = \epsilon'_1/c(1, \sin\eta'_1\cos\phi'_1, \sin\eta'_1\sin\phi'_1, \cos\eta'_1)$, and we find

$$\begin{aligned}
\epsilon' &= \epsilon(\gamma - \beta\gamma\cos\eta), \\
\epsilon'_1 &= \epsilon_1(\gamma - \beta\gamma\cos\eta_1), \\
\cos\eta' &= (-\beta\gamma + \gamma\cos\eta)\frac{\epsilon}{\epsilon'} = \frac{\cos\eta - \beta}{1 - \beta\cos\eta}, \\
\cos\eta &= \frac{\cos\eta' + \beta}{1 + \beta\cos\eta'}, \\
\cos\eta'_1 &= \frac{\cos\eta_1 - \beta}{1 - \beta\cos\eta_1},\ \cos\eta_1 = \frac{\cos\eta'_1 + \beta}{1 + \beta\cos\eta'_1}.
\end{aligned} \tag{41}$$

From symmetry, the azimuthal angles do not change, but also easy to directly verify:

$$\begin{aligned}
\sin\eta'\cos\phi' &= \sin\eta\cos\phi\frac{\epsilon}{\epsilon'} \\
\Rightarrow \cos\phi' &= \cos\phi\frac{\epsilon\sin\eta}{\epsilon'\sin\eta'} = \frac{\cos\phi\sin\eta}{\sqrt{(\gamma - \beta\gamma\cos\eta)^2 - (-\beta\gamma + \gamma\cos\eta)^2}} \\
&= \frac{\cos\phi\sin\eta}{\sqrt{\gamma^2(1-\beta^2) - \gamma^2(1-\beta^2)\cos^2\eta}} = \cos\phi.
\end{aligned} \tag{42}$$

The angle $\cos\Theta$ in the rest frame of the electron (Equation (34)) is given by:

$$\begin{aligned}
\cos\Theta &= \widehat{LP_{\gamma i}} \cdot \widehat{LP_{\gamma i}} = \cos\eta'\cos\eta'_1 + \sin\eta'\sin\eta'_1(\cos\phi'\cos\phi'_1 + \sin\phi'\sin\phi'_1) \\
&= \cos\eta'\cos\eta'_1 + \sin\eta'\sin\eta'_1\cos(\phi' - \phi'_1).
\end{aligned} \tag{43}$$

The fractional change in the photon energy in the original frame of reference can then be expressed as

$$\begin{aligned}
\frac{\epsilon_1 - \epsilon}{\epsilon} &= \frac{\epsilon'_1}{\epsilon'}\frac{\gamma - \beta\gamma\cos\eta}{\gamma - \beta\gamma\cos\eta_1} - 1 = \frac{\epsilon'_1}{\epsilon'}\frac{1 - \beta\frac{\cos\eta' + \beta}{1+\beta\cos\eta'}}{1 - \beta\frac{\cos\eta'_1 + \beta}{1+\beta\cos\eta'_1}} - 1 \\
&= \frac{\epsilon'_1}{\epsilon'}\frac{1 + \beta\cos\eta'_1}{1 + \beta\cos\eta'} - 1 = \frac{1}{1 + \beta\cos\eta'}\left[\frac{\epsilon'_1}{\epsilon'}(1 + \beta\cos\eta'_1) - 1 - \beta\cos\eta'\right] \\
&\approx -\frac{\epsilon'}{m_e c^2}(1 - \cos\Theta) + \frac{\beta(\cos\eta'_1 - \cos\eta')}{1 + \beta\cos\eta'},
\end{aligned} \tag{44}$$

where we neglected $\epsilon'\beta$ terms or higher.

We next integrate over solid angle in the rest-frame of the electron, using Thomson cross-section

(Equation (29)) :

$$
\begin{aligned}
\langle\epsilon_1-\epsilon\rangle &= \frac{3\sigma_T}{16\pi}\int_{-1}^{1}d(\cos\eta_1')\int_0^{2\pi}d\phi_1'(\epsilon_1-\epsilon)(1+\cos^2\Theta)\\
&= \frac{3\sigma_T}{16\pi}\int_{-1}^{1}d(\cos\eta_1')\int_0^{2\pi}d\phi_1'\left[-\frac{\epsilon\epsilon'}{m_ec^2}(1-\cos\Theta)+\frac{\epsilon\beta(\cos\eta_1'-\cos\eta')}{1+\beta\cos\eta'}\right](1+\cos^2\Theta)\\
&= -\frac{\epsilon\beta\cos\eta'}{1+\beta\cos\eta'}\frac{3\sigma_T}{16\pi}\int_{-1}^{1}d(\cos\eta_1')\int_0^{2\pi}d\phi_1'(1+\cos^2\Theta)\\
&- \frac{\epsilon\epsilon'}{m_ec^2}\frac{3\sigma_T}{16\pi}\int_{-1}^{1}d(\cos\eta_1')\int_0^{2\pi}d\phi_1'(1-\cos\Theta)(1+\cos^2\Theta)\\
&= -\frac{\sigma_T\epsilon\beta\cos\eta'}{1+\beta\cos\eta'}-\frac{\sigma_T\epsilon\epsilon'}{m_ec^2},
\end{aligned}
\tag{45}
$$

where in the transition from the second to third line we used that the $\cos\eta_1'$ term is multiplying an even function so the integral over this term is zero, and the geometrical integrals in the third and forth line are analytical.

We next need to integrate over the electron direction of motion (or over η'). Note that $\cos\eta$ has a flat distribution (and not $\cos\eta'$). Also, we must take into account the relative velocity between the photon and the electron in the lab frame, which changes the rate of collisions:

$$
u=c^2\frac{|P_{ei}P_{\gamma i}|}{E\epsilon}=c\frac{\epsilon(E/c-p\cos\eta)}{E\epsilon}=1-\frac{cp}{E}\cos\eta=1-\beta\cos\eta=\frac{1-\beta^2}{1+\beta\cos\eta'}. \tag{46}
$$

Hence the average energy change is

$$
\langle\langle\epsilon_1-\epsilon\rangle\rangle = \frac{\int_{-1}^{1}\langle\epsilon_1-\epsilon\rangle u\,d\cos\eta}{\int_{-1}^{1}u\,d\cos\eta}. \tag{47}
$$

Since $d\cos\eta=d\cos\eta'(1-\beta^2)/(1+\beta\cos\eta')^2$, the denominator is

$$
\int_{-1}^{1}\frac{(1-\beta^2)^2}{(1+\beta\cos\eta')^3}d\cos\eta'=2, \tag{48}
$$

so we get

$$
\langle\langle\epsilon_1-\epsilon\rangle\rangle = \frac{1}{2}\int_{-1}^{1}\frac{(1-\beta^2)^2}{(1+\beta\cos\eta')^3}\left(-\frac{\sigma_T\epsilon\beta\cos\eta'}{1+\beta\cos\eta'}-\frac{\sigma_T\epsilon\epsilon'}{m_ec^2}\right)d\cos\eta'. \tag{49}
$$

The first order terms in β make no contribution to the integral over η'. so the leading terms are second order in β or first order in $\epsilon/(m_ec^2)$:

$$
\begin{aligned}
\langle\langle\epsilon_1-\epsilon\rangle\rangle &\approx -\frac{\sigma_T\epsilon\epsilon'}{m_ec^2}-\frac{1}{2}\sigma_T\epsilon\beta(1-\beta^2)^2\int_{-1}^{1}\frac{\cos\eta'}{(1+\beta\cos\eta')^4}d\cos\eta'\\
&\approx -\frac{\sigma_T\epsilon\epsilon'}{m_ec^2}+\frac{4}{3}\sigma_T\epsilon\beta^2\approx-\frac{\sigma_T\epsilon^2}{m_ec^2}+\frac{4}{3}\sigma_T\epsilon\beta^2.
\end{aligned}
\tag{50}
$$

Since the mean photon fractional energy change contains terms of order β^2 and $\epsilon/m_e c^2$, we are interested in any terms in the average squared fractional energy change of the same order. Inspection of Equation (44) shows that to this order, we have:

$$\frac{(\epsilon_1-\epsilon)^2}{\epsilon^2} \approx \beta^2 \left(\cos\eta_1' - \cos\eta'\right)^2 . \tag{51}$$

To this order we can neglect the difference between u and unity and between $d\cos\eta$ and $d\cos\eta'$, so that

$$\begin{aligned} \langle\langle(\epsilon_1-\epsilon)^2\rangle\rangle &\approx \frac{3\sigma_T\beta^2\epsilon^2}{32\pi}\int_{-1}^{1} d\cos\eta' \int_{-1}^{1} d\cos\eta_1' \int_0^{2\pi} d\phi_1' \left(\cos\eta_1' - \cos\eta'\right)^2 \left(1+\cos^2\Theta\right)^2 \\ &= \frac{2\sigma_T}{3}\beta^2\epsilon^2 \end{aligned} \tag{52}$$

The last step is integrating over f_e, so we get

$$\begin{aligned} I_2 &= 4\pi n_e\sigma_T\frac{\epsilon^2}{T^2}(2\pi m_e T)^{-3/2}\frac{2}{3}\int_0^\infty p^2 dp\exp(-p^2/2m_eT)\beta^2 \\ &= \frac{8\pi}{3m_e^2c^2}n_e\sigma_T x^2(2\pi m_eT)^{-3/2}\int_0^\infty p^4 dp\exp(-p^2/2m_eT) \\ &= \frac{16T}{3m_ec^2\sqrt{\pi}}n_e\sigma_T x^2\int_0^\infty y^4\exp(-y^2)dy \\ &= 2x^2 n_e\sigma_T\frac{T}{m_ec^2} \end{aligned} \tag{53}$$

and

$$\begin{aligned} I_1 &= 4\pi n_e\sigma_T\frac{\epsilon}{T}(2\pi m_eT)^{-3/2}\int_0^\infty p^2dp\exp(-p^2/2m_eT)\left(\frac{4}{3}\beta^2 - \frac{\epsilon}{m_ec^2}\right) \\ &= \frac{2I_2}{x} - \frac{4}{\sqrt{\pi}}n_e\sigma_T x\frac{\epsilon}{m_ec^2}\int_0^\infty y^2dy\exp(-y^2) \\ &= 4xn_e\sigma_T\frac{T}{m_ec^2} - xn_e\sigma_T\frac{\epsilon}{m_ec^2} = xn_e\sigma_T(4-x)\frac{T}{m_ec^2}. \end{aligned} \tag{54}$$

We see that energy is gained or lost depending on the sign of $4-x$. We substitute these into Equation (39) to finally find the Kompaneets equation:

$$\begin{aligned} \frac{\partial n}{\partial t_c} &= (n'+n+n^2)x(4-x)\frac{T}{m_ec^2} + 2\left(\frac{1}{2}n'' + \frac{1}{2}n(1+n) + n'(1+n)\right)x^2\frac{T}{m_ec^2} \\ &= \frac{T}{m_ec^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4(n'+n+n^2)\right], \end{aligned} \tag{55}$$

where $t_c \equiv (n_e\sigma_T c)t$ is the time measured in units of mean time between scatterings.

The steady-state solution of the Kompaneets equation is given by

$$
\begin{aligned}
0 &= n' + n + n^2 \\
&\Rightarrow \frac{dn}{dx} = -n - n^2 \\
&\Rightarrow \frac{dx}{dn} = -\frac{1}{n(1+n)} = -\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&\Rightarrow x - x_0 = \ln\left(\frac{1+n}{n}\right) \\
&\Rightarrow 1 + \frac{1}{n} = \exp(x - x_0) \\
&\Rightarrow n = \frac{1}{\exp(x-x_0) - 1} = \frac{1}{\exp(\frac{\hbar\omega - \mu}{T}) - 1}.
\end{aligned} \tag{56}
$$

We see that the spectrum reaches equilibrium after photons have been scattered up to energies forming the Bose-Einstein distribution with finite chemical potential, Equation (56). We have that $\mu = 0$ for photons only when we can create and destroy photons, but here $\mu \neq 0$ because the number of photons is conserved when only Compton scattering is allowed. The steady state, "saturated" spectrum is approximated by *Wien law*, $n(x) \propto \exp(-x)$, when the occupation number is small ($-\mu/T \gg 1$). The number of photons conservation is directly obtaine from the Kompaneets equation:

$$
\begin{aligned}
\frac{d}{dt}N &\propto \frac{d}{dt}\int_0^\infty \omega^2 n(\omega) d\omega = \int_0^\infty \omega^2 \frac{\partial n(\omega)}{\partial t} d\omega \propto \int_0^\infty \frac{\omega^2}{\omega^2}\frac{\partial}{\partial \omega}\left(\omega^4(n' + n + n^2)\right) d\omega \\
&= \omega^4(n' + n + n^2)\Big|_0^\infty = 0,
\end{aligned} \tag{57}
$$

as long as $\omega^4 n$ and $\omega^4 n'$ vanish at 0 and at ∞.

5. The Compton y parameter

A convenient way to measure the importance of repeated Compton scattering is with the Compton y parameter, defined by $y \equiv$ (average fractional energy change per scattering) $\times$ (mean number of scatterings). For $y \gtrsim 1$ the total photon energy and spectrum will be significantly altered; whereas for $y \ll 1$, the total energy is not much changed. The average energy change per scattering is given by $I_1 T/n_e\sigma_T$ (remember that I_1 is the average of $\Delta = \Delta\epsilon/T$), so we find for the right parentheses $(4T - \epsilon)/m_e c^2$. Energy is gained or lost depending on the sign of $4T - \epsilon$. The mean number of scattering is $N_s \approx \max(\tau_s, \tau_s^2) \approx \tau_s(1+\tau_s)$, where $\tau_s \sim n_e\sigma_T r$ ($\sim 10^{-3}$ for GC, as we showed above). The τ_s term described the chance to collide, while the τ_s^2 is diffusion-like term. The final expression is

$$
y \approx \frac{4T - \epsilon}{m_e c^2}\tau_s(1+\tau_s) \approx \frac{4T}{m_e c^2}\tau_s(1+\tau_s), \tag{58}
$$

where the last approximation is when the energy transfer in the rest frame of the electron is negligible ($T \gg \epsilon$). Note that for $\tau_s \ll 1$, we have $y \propto \int n_e T dr \propto \int P dr$, so an observation of y allows to estimate the column pressure.

For a given y parameter, the final energy of a photon ϵ_f is given by

$$\epsilon_f = \Pi_{i=1}^{N_s} \left(1 + \frac{\Delta\epsilon}{\epsilon}\right) \epsilon_i, \tag{59}$$

where ϵ_i is the initial photon energy. For $T \gg \epsilon$ we find $\epsilon_f = \epsilon_i (1 + \Delta\epsilon/\epsilon)^{N_s} = \epsilon_i \exp(y)$. For the total energy of the photos:

$$\begin{aligned} \frac{de}{dt} &= n_e \sigma_T c \frac{d}{dt_c} A \int_0^\infty \omega^3 n(\omega) d\omega \\ &= n_e \sigma_T c A \int_0^\infty \omega^3 \frac{\partial n(\omega)}{\partial t_c} d\omega = \frac{n_e \sigma_T c \hbar}{m_e c^2} A \int_0^\infty \frac{\omega^3}{\omega^2} \frac{\partial}{\partial \omega} \left(\omega^4 (n' + n + n^2)\right) d\omega. \end{aligned} \tag{60}$$

We can integrate by parts to find

$$\begin{aligned} \int_0^\infty \omega \frac{\partial}{\partial \omega} \left(\omega^4 (n' + n + n^2)\right) d\omega &= \omega^5 (n' + n + n^2)\Big|_0^\infty - \int_0^\infty \omega^4 (n' + n + n^2) d\omega \\ &\approx -\int_0^\infty \omega^4 n' d\omega = -\frac{T}{\hbar} \int_0^\infty \omega^4 \frac{\partial n}{\partial \omega} d\omega \\ &= -\frac{T}{\hbar} \omega^4 n \Big|_0^\infty + \frac{4T}{\hbar} \int_0^\infty \omega^3 n d\omega \\ &, \end{aligned} \tag{61}$$

where we assumed $n, n^2 \ll n'$ ($T \gg \epsilon$). We find

$$\begin{aligned} \frac{de}{dt} &= \frac{n_e \sigma_T c \hbar}{m_e c^2} A \frac{4T}{\hbar} \int_0^\infty \omega^3 n d\omega = \frac{4T n_e \sigma_T c}{m_e c^2} e \\ &\Rightarrow \quad e \propto \exp\left(\frac{4T n_e \sigma_T c}{m_e c^2} t\right) = \exp(y). \end{aligned} \tag{62}$$

6. The Sunyaev-Zel'dovich effect

We are interested in the change of the CMB due to scattering by the ICM (The SZ effect). We rewrite the Kompaneets equation as:

$$\begin{aligned} \frac{\partial n(\omega, l)}{\partial l} &= \frac{n_e(l) \sigma_T T}{m_e c^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 (n' + n + n^2)\right] \\ &\approx \frac{n_e(l) \sigma_T T}{m_e c^2} \frac{1}{x^2} \frac{\partial}{\partial x} (x^4 n'), \end{aligned} \tag{63}$$

where l is the proper distance coordinate along the line of sight through the ICM, and we used $n, n^2 \ll n'$ ($T \gg \epsilon$). For $y \ll 1$ we get

$$\Delta n(\omega) = \frac{\tilde{y}}{x^2} \frac{\partial}{\partial x} (x^4 n') = \frac{\tilde{y}}{\omega^2} \frac{\partial}{\partial \omega} \left(\omega^4 \frac{\partial n(\omega)}{\partial \omega}\right), \tag{64}$$

where $\tilde{y} = y/4$. For a black-body with temperature $T_{\rm CMB}$, $n = (\exp(\hbar\omega/T_{\rm CMB}) - 1)^{-1}$, we find

$$\Delta n(\omega) \quad = \quad \tilde{y} \left(\frac{-\tilde{x} + (\tilde{x}^2/4)\coth(\tilde{x}/2)}{\sinh^2(\tilde{x}/2)} \right), \qquad (65)$$

where $\tilde{x} = \hbar\omega/T_{\rm CMB}$. The characteristic dependance on ω makes it possible to distinguish between CMB anisotropies due to the SZ effect and the primary anisotropies discussed before.

In the Rayleigh-Jeans part ($\tilde{x} \ll 1$), we find $\Delta n \to -2\tilde{y}/\tilde{x}$ and $n \to 1/x$, such that $\Delta T_{\rm CMB}/T_{\rm CMB} = -2\tilde{y}$. The "reduction" in temperature in the Rayleigh-Jeans part is compensated by the part of the spectrum with $\tilde{x} > 3.830$, where $\Delta n > 0$, and the asymptotic behavior $\Delta n/n \to \tilde{y}\tilde{x}^2$. The scatterings with the hot electron gas transfer low photon energy to high energy (recall that the number of photons is conserved). The change of the spectrum is plotted in Figure 1.

A measurement of the y parameter constrain $n_e l$. Since for l we would need the angular distance of the GC, the constrain on n_e would scale like H_0. This can be compared to the bremsstrahlung emission that constrain $n_e^2 dl$, such that the constrain on n_e would scale like $\sqrt{H_0}$. Requiring the same value of n_e from both observation allows to estimate H_0. In practice, the assumptions involve in this process for the shape of the GC profiles add a too large systematic uncertainty for a useful determination of H_0. The SZ effect adds correlations to the CMB fluctuations at very small angular separation. If combined with a model of structure formation, they can provide useful cosmological information.

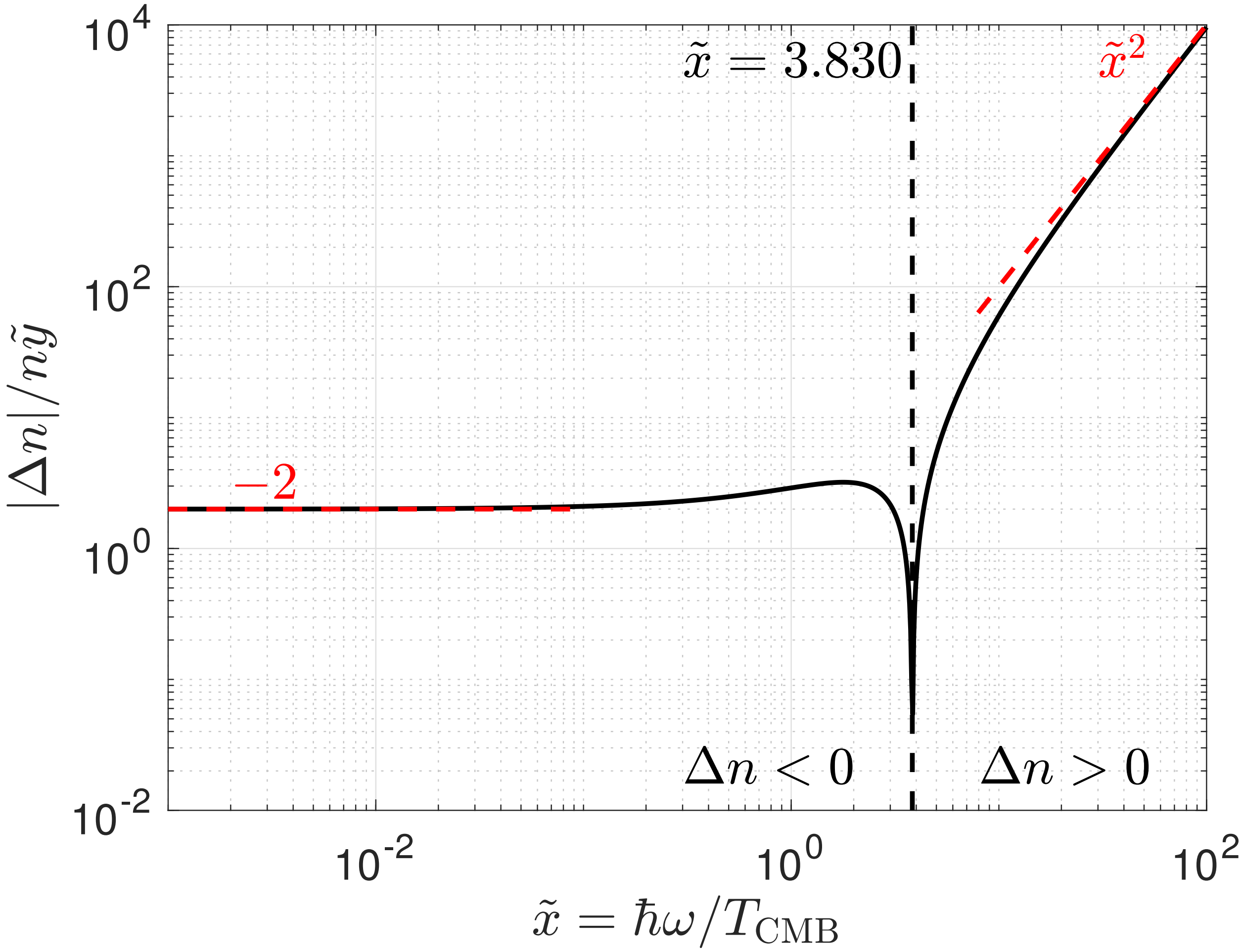

Fig. 1.— $|\Delta n|/n\tilde{y}$ from Equation (65) as a function of $\tilde{x} = \hbar\omega/T_{\rm CMB}$.