The cosmic microwave radiation background (CMB)

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I'm following Steven Weinberg's Cosmology.

1. Introduction

Since the Universe expands, in the past it was hotter and denser, so atoms were ionized and free electrons would keep radiation in TE through collisions. The number density of photons is ($\mu = 0$, $g = 2$, $p = h\nu/c$ and BE distribution):

$$n(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{c^3}\frac{1}{\exp\left(\frac{h\nu}{T}\right)-1}.$$

As the Universe expands, the radiation began free expansion. Let's assume that free expansion happened suddenly at a time t_L (L for last scattering). A photon with ν at some time $t > t_L$ had at t_L: $\nu(t_L) = \nu(t)R(t)/R(t_L)$, so

$$\Rightarrow n(\nu,t)d\nu = \left[\frac{R(t_L)}{R(t)}\right]^3 n_{T(t_L)}\left[\frac{\nu R(t)}{R(t_L)}\right] d\left[\frac{\nu R(t)}{R(t_L)}\right],$$

where the first factor on the rhs is due to the cosmic expansion dilution, so we get

$$n(\nu,t)d\nu = \frac{8\pi\nu^2 d\nu}{c^3}\frac{1}{\exp\left(\frac{h\nu}{T(t)}\right)-1} = n_{T(t)}(\nu)d\nu,$$

where $T(t) = T(t_L)R(t_L)/R(t)$. The photons keep the blackbody form even after t_L but with a redshifted temperature (we proved this earlier in general).

As long as collision are elastic (with $\nu = \text{const.}$), the transition from opacity to transparency can be over a finite time interval. We'll show that $T(t_L) \approx 3000\,\text{K}$ with $\Delta\nu/\nu \sim T/m_e c^2 \sim 3\times10^{-7}$ for elastic collisions. We'll show that because of the large photon entropy, even this shift and the inelastic collisions with H atoms had almost no effect of the spectrum.

The CMB was predicted by Gamow (40'). Alper and Herman (50') estimated $T_{\text{CMB}} \approx 5\,\text{K}$, based on BBN. Peebles (65') estimated $T_{\text{CMB}} \approx 10\,\text{K}$. The CMB was discovered by Penzias and Wilson (66') – $T_{\text{CMB}} = 3.5 \pm 1.0\,\text{K}$ at $\lambda = 7.5\,\text{cm}$. Roll and Wilkinson (66') measured $T_{\text{CMB}} = 3.0 \pm 0.5\,\text{K}$ at $\lambda = 3.2\,\text{cm}$. All measurements were for $\lambda > 0.1\,\text{cm}$, which is the wavelength in which the blackbody distribution is maximal $\lambda_{\max} \approx hc/5T \approx 0.1(T/3\,\text{K})^{-1}\,\text{cm}$ (so this is the RJ tail, where $n(\nu)h\nu \approx 8\pi\nu^2 T/c^3$). As a results, the measurements do not prove a blackbody radiation, so measurements at smaller wavelength were required. The problem that for that small wavelength the detector must be above the atmosphere, but ballon-borne and rocket-borne were not accurate enough. This was solved with a space mission – FIRAS on COBE (launched on 89'). COBE measured an exact blackbody over $0.05\,\mathrm{cm} \leq \lambda \leq 0.5\,\mathrm{cm}$ with $T_{\rm CMB} = 2.725 \pm 0.002\,\mathrm{K}$. Further measurements at $3\,\mathrm{cm} \leq \lambda \leq 75\,\mathrm{cm}$ and at $\lambda = 0.03\,\mathrm{cm}$ are all consistent with blackbody.

For $T_{\rm CMB} = 2.725\,\mathrm{K}$ we get

$$\begin{aligned} \rho_{\gamma,0} c^2 &= \bar{a}_B T_{\rm CMB}^4 \Rightarrow \rho_{\gamma,0} \approx 4.64 \times 10^{-34}\,\mathrm{g\,cm^{-3}} \\ &\Rightarrow \Omega_\gamma \approx 2.47 \times 10^{-5} h^{-2}. \end{aligned}$$

We'll show later that there is a contribution to the radiation field from relic neutrinos and anti-neutrinos, such that

$$\begin{aligned} \rho_{R,0} &= \left[1 + 3\left(\frac{7}{8}\right)\left(\frac{4}{11}\right)^{4/3}\right] \rho_{\gamma,0} \approx 7.80 \times 10^{-34}\,\mathrm{g\,cm^{-3}} \\ &\Rightarrow \Omega_R \approx 4.16 \times 10^{-5} h^{-2}. \end{aligned}$$

This justifies our neglection of Ω_R in calculating d_L. The number density of photons is high:

$$n_{\gamma,0} = \frac{30}{\pi^4}\zeta(3)\bar{a}_B T_{\rm CMB}^3 \approx 410\,\mathrm{photons\,cm^{-3}},$$

which is much larger than $n_{b,0}$:

$$n_{b,0} = \frac{3\Omega_b H_0^2}{8\pi G m_p} \approx 1.123 \times 10^{-5}\Omega_b h^2 \approx 2.50 \times 10^{-7}\,\mathrm{baryons\,cm^{-3}}.$$

Since both $n_b \propto R^{-3}$ and $n_\gamma \propto R^{-3}$, the ratio n_γ/n_b has been the same at least while both were travelling freely. We define $\eta \equiv n_b/n_\gamma \approx 2.74 \times 10^{-8}\Omega_b h^2 \approx 6.09 \times 10^{-10}$.

2. The equilibrium era

We know that in free expansion $T_\gamma \propto R^{-1}$ and $T_b \propto R^{-2}$, so in TE who wins? Photons, because $n_\gamma \gg n_b$. To see this, we define $\sigma = s/n_b$, the entropy per baryon, which is conserved, since both entropy and baryon number are separately conserved. From energy conservation:

$$d\sigma = \frac{1}{T}\left[d\left(\frac{e}{n_b}\right) + P d\left(\frac{1}{n_b}\right)\right].$$

We now consider photons and NR particles with a fixed number N of NR particles per baryon:

$$\begin{aligned} e &= \bar{a}_B T^4 + \frac{3}{2} n_b N T + n_b m_p c^2, \\ P &= \frac{1}{3}\bar{a}_B T^4 + n_b N T, \end{aligned}$$

so

$$\begin{aligned}
d\sigma &= \frac{1}{T}\left[d\left(\frac{\bar{a}_B T^4}{n_b}\right) + \frac{3}{2}N dT + \left(\frac{1}{3}\bar{a}_B T^4 + n_b N T\right) d\left(\frac{1}{n_b}\right)\right] \\
&= \frac{1}{T}\left[\frac{4}{3}\bar{a}_B T^4 d\left(\frac{1}{n_b}\right) + \frac{1}{n_B}\frac{4\bar{a}_B T^4}{T} dT + \frac{3}{2}N dT + N T n_b d\left(\frac{1}{n_b}\right)\right] \\
&= \frac{1}{T}\frac{\bar{a}_B T^4}{n_b}\left(-\frac{4}{3}\frac{dn_b}{n_b}\right) + \frac{3}{2}N\frac{dT}{T} - N\frac{dn_b}{n_b} + \frac{1}{n_b}\frac{4\bar{a}_B T^4}{T^2} dT \\
&\Rightarrow \ \sigma = \frac{4\bar{a}_B T^3}{3 n_b} + N \ln\left(\frac{T^{3/2}}{n_b C}\right),
\end{aligned}$$

where C is a constant of integration. σ remains constant in TE, and we know that the first term on the rhs in $> 10^8$ at present. Let's assume that it was large in TE as well. So T^3/n_b was very close to a constant at TE unless $T^{3/2}/n_b$ changed by a huge amount. For example, if the first term on the rhs would change by 0.01%, so to keep σ constant $T^{3/2}/n_b$ should change by $N \ln(T^{3/2}/n_b) = 10^{-4} \times 10^8 \Rightarrow T^{3/2}/n_b = \exp(10^4/N)$, therefore T^3/n_b is essentially constant. If we define C to be equal to $T^{3/2}/n_b$ at some typical time in the TE era, we get

$$\sigma = \frac{4\bar{a}_B T^3}{3 n_b} = \frac{4}{3}\frac{n_{\gamma,0}}{n_{b,0}}\frac{\pi^4}{30\zeta(3)} \approx 3.60\frac{n_{\gamma,0}}{n_{b,0}} \approx 1.31 \times 10^8 \left(\Omega_b h^2\right)^{-1} \approx 5.91 \times 10^9.$$

Since $n_b \propto R^{-3}$, we find $T \propto R^{-1}$. Note that if the first term on the rhs was $< 10^{-8}$ (and not $> 10^8$), then the second term on rhs would dominate and $T \propto n_b^{2/3} \propto R^{-2}$ as expected for NR.

Photons stop exchanging energy with matter when $\Gamma_\gamma < H$, where Γ_γ is the rate at which a photon interchange T energy through scattering on electrons. But Γ_e, the rate at which electrons gain/lose T energy through scattering on photons, is larger from Γ_γ by $n_\gamma/n_b > 10^8$, so when $\Gamma_\gamma < H$ we still have $\Gamma_e \gg H$. This means that the kinetic energy of electrons remains in TE with photons and do not decrease as R^{-2}. Since $T_\gamma \propto R^{-1}$ in TE and for $t > t_L$, the electrons' temperature goes as R^{-1} and the last few exchanges of energy that photons have with electrons do not affect the photons energy distribution.

3. Matter-radiation equilibrium

We have

$$\begin{aligned}
\frac{\rho_R}{\rho_M} &\propto \frac{1}{R} \propto T \Rightarrow \frac{\rho_R}{\rho_M} = \frac{T}{T_{\rm CMB}}\frac{\rho_{R,0}}{\rho_{M,0}} = \frac{T}{T_{\rm CMB}}\frac{\Omega_R}{\Omega_M} \\
&\Rightarrow \ T_{EQ} = \frac{T_{\rm CMB}\Omega_M}{\Omega_R} = \Omega_M h^2 \, 6.56 \times 10^4\,{\rm K} \approx 9,347\,{\rm K}.
\end{aligned}$$

4. Freeze-out time

At high T collisions can change the energy of an individual photon (without changing the distribution of photons). Let's work out when photons stopped exchanging energies of T with electrons. The rate at which photons are scattered on electrons is $\Lambda_\gamma = \sigma_T n_e c$, where $\sigma_T \approx 0.66525\times10^{-24}\,\mathrm{cm}$ is the Thompson cross-section. We'll show later that the mass fractions of Hydrogen and Helium at this era are $X_{\mathrm{H}} \approx 0.76$ and $Y_p \equiv X_{\mathrm{He}} \approx 0.24$, respectively. For $T \gtrsim 20,000\,\mathrm{K}$ the plasma is fully ionized: $n_e/n_b \approx 0.76 + 0.5\times0.24 = 0.88$, so $n_e \approx 0.88 n_b = 0.88 n_{b,0}(T/T_{\mathrm{CMB}})^3$. We find:

$$\Lambda_\gamma \approx 0.88 n_{b,0}\left(\frac{T}{T_{\mathrm{CMB}}}\right)^3 \sigma_T c \approx 1.97\times10^{-19}\Omega_b h^2\left(\frac{T}{T_{\mathrm{CMB}}}\right)^3\ \mathrm{s}^{-1}.$$

The rate for energy transfer is

$$\Gamma_\gamma \approx \left(\frac{T}{m_e c^2}\right)\Lambda_\gamma \approx 9.0\times10^{-29}\Omega_b h^2\left(\frac{T}{T_{\mathrm{CMB}}}\right)^4\ \mathrm{s}^{-1}.$$

For H we'll assume radiation dominated:

$$\begin{aligned} H &= H_0\sqrt{\Omega_R\left(\frac{T}{T_{\mathrm{CMB}}}\right)^4} \approx 2.1\times10^{-20}\left(\frac{T}{T_{\mathrm{CMB}}}\right)^2\ \mathrm{s}^{-1} \\ \Rightarrow\ T_{\mathrm{freeze}} &\approx \sqrt{\frac{2.1\times10^{-20}}{9.0\times10^{-29}}}\left(\Omega_b h^2\right)^{-1/2} T_{\mathrm{CMB}} \approx 4.16\times10^4\left(\Omega_b h^2\right)^{-1/2} \approx 2.8\times10^5\,\mathrm{K}, \end{aligned}$$

so the assumption that the Universe is radiation dominated is justified. For $T \lesssim 10^5\,\mathrm{K}$, photons still have a lot of scattering, since $\Gamma_\gamma \gg H$. For example, at $T = 10^4\,\mathrm{K}$, we have

$$\frac{\Lambda_\gamma}{H} \approx \frac{1.97\times10^{-19}\Omega_b h^2}{2.1\times10^{-20}}\left(\frac{T}{T_{\mathrm{CMB}}}\right) \approx 765,$$

so for $T \ll 10^4\,\mathrm{K}$, when the Universe is matter dominated, we can calculate the temperature of last scattering:

$$\begin{aligned} H &= H_0\sqrt{\Omega_M\left(\frac{T}{T_{\mathrm{CMB}}}\right)^3} \approx 3.2\times10^{-18}\sqrt{\Omega_M h^2}\left(\frac{T}{T_{\mathrm{CMB}}}\right)^{3/2}\ \mathrm{s}^{-1} \\ \Rightarrow\ T &\approx \left(\frac{3.2\times10^{-18}}{1.97\times10^{-19}}\frac{\sqrt{\Omega_M h^2}}{\Omega_b h^2}\right)^{2/3} T_{\mathrm{CMB}} \approx 17.5\frac{\left(\Omega_M h^2\right)^{1/3}}{\left(\Omega_b h^2\right)^{2/3}}\,\mathrm{K} \approx 116\,\mathrm{K}. \end{aligned}$$

This is not what actually happens because of recombination, leading to a sharp drop in Λ_γ at $\sim 3000\,\mathrm{K}$. The different timescales are plotted in Figure 1.

5. Recombination and last scattering

We begin the calculation at early enough times, where p, e, H and He are in TE with T_γ. We consider the Hydrogen atoms in any bound state: $1s$, $2s$, $2p$, etc. Although $\sim 24\%$ of the mass is in the form of Helium, for $T \lesssim 4,400\,\mathrm{K}$ Helium is neutral so does not play a role here. We have $g_p = g_e = 2$ and since for $1s$ there are 2 hyperfine states with $S = 0, 1$ we have $g_{1s} = 1 + 3 = 4$.

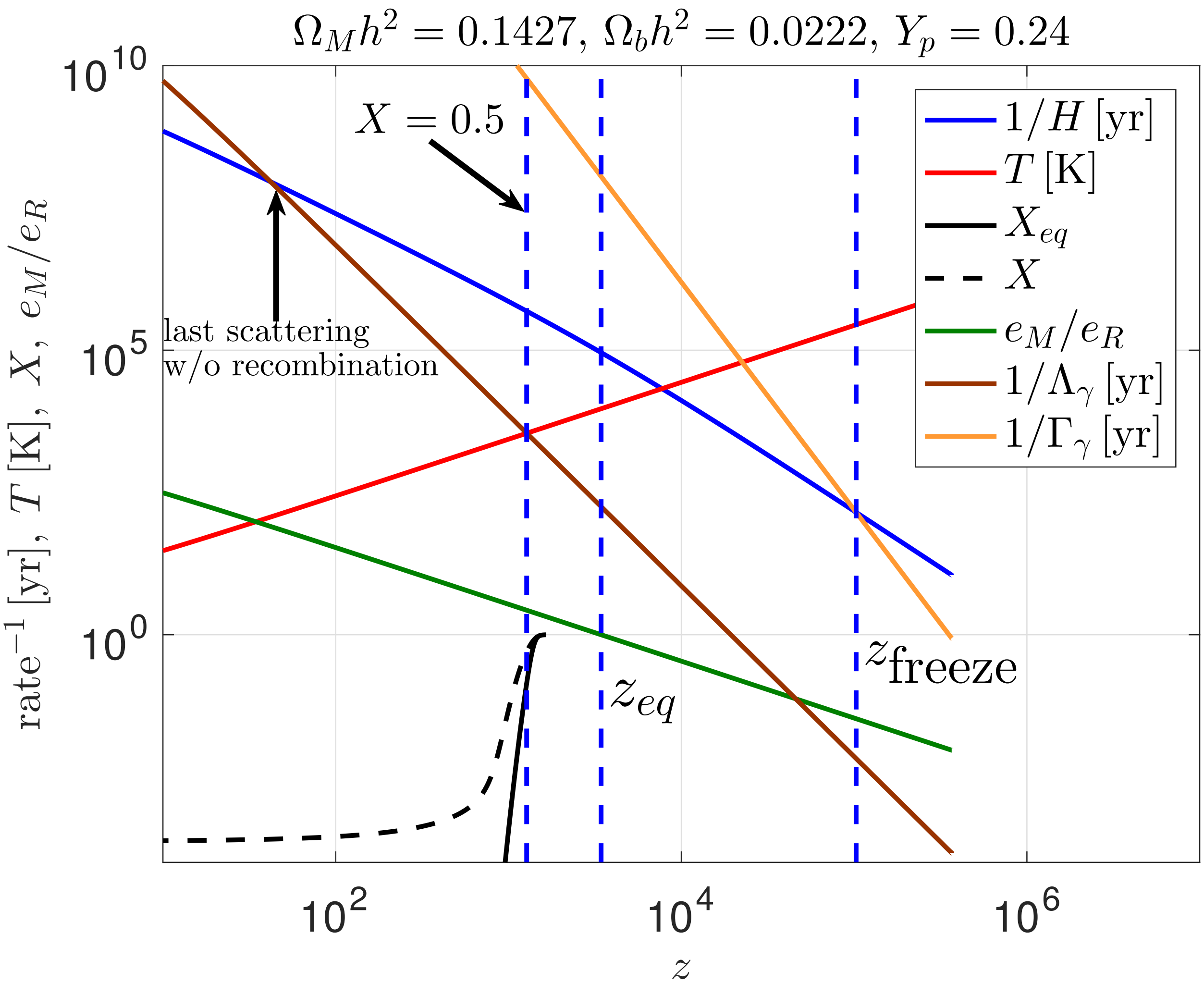

Fig. 1.— $1/H$ (blue), T (red), X_{eq} (solid black), X (dashed black), e_M/e_R (green), $1/\Lambda_\gamma$ (brown) and $1/\Gamma_\gamma$ (orange) as a function of the redshift for $\Omega_M h^2 = 0.1427$, $\Omega_b h^2 = 0.0222$ and $Y_p = 0.24$. The freeze-out time ($\Gamma_\gamma = H$), the equilibrium time ($e_M = e_R$) and the time of last scattering without recombination ($\Lambda_\gamma = H$) are indicated.

5.1. Saha's equation

At first $p+e \leftrightarrow 1s$ happens rapidly through cascades of radiative transfers from excited states, so $\mu_p + \mu_e = \mu_{1s}$. Using

$$\mu_i = T\ln\left[\frac{n_i}{g_i}\left(\frac{2\pi m_i T}{h^2}\right)^{-3/2}\right] + m_i c^2,$$

we get

$$\frac{n_p}{2}\left(\frac{2\pi m_p T}{h^2}\right)^{-3/2}\exp\left(\frac{m_p c^2}{T}\right)\frac{n_e}{2}\left(\frac{2\pi m_e T}{h^2}\right)^{-3/2}\exp\left(\frac{m_e c^2}{T}\right) = \frac{n_{1s}}{4}\left(\frac{2\pi m_H T}{h^2}\right)^{-3/2}\exp\left(\frac{m_H c^2}{T}\right)$$

$$\Rightarrow \quad \frac{n_{1s}}{n_p n_e} = \left(\frac{2\pi m_e T}{h^2}\right)^{-3/2}\exp\left(\frac{B_1}{T}\right), \tag{1}$$

where we took $m_p \simeq m_H$ outside the exponent and $B_1 = (m_p + m_e - m_H)c^2 \approx 13.6\,\mathrm{eV}$. Because of charge neutrality $n_p = n_e$. The number density of excited states is less from n_{1s} by $\exp(-\Delta E/T)$, where ΔE is the excitation energy ($\Delta E \geq B_1(1-1/4) \simeq 10.2\,\mathrm{eV}$). For $T < 4,200\,\mathrm{K}$ this exponent is $< 6\times 10^{-13}$ so we can neglect the excited states as long as there is TE (note that excited states have the same μ, since the atom can go between states by emitting or absorbing photons). Since we have $n_p + n_{1s} \simeq 0.76 n_b$ and $X = n_p/(n_p + n_{1s})$ satisfies

$$X\left[1 + X\frac{n_{1s}}{n_p^2}\left(n_p + n_{1s}\right)\right] = 1,$$

we get $X(1+XS) = 1$, where

$$S = \frac{n_{1s}}{n_p^2}\left(n_p + n_{1s}\right) \approx 0.76 n_b\left(\frac{2\pi m_e T}{h^2}\right)^{-3/2}\exp\left(B_1/T\right).$$

The pre-factor of the exponent is a small number:

$$\begin{aligned}
S &\approx 0.76\frac{n_b}{n_\gamma}16\pi\zeta(3)\left(\frac{T}{hc}\right)^3\left(\frac{2\pi m_e T}{h^2}\right)^{-3/2}\exp\left(B_1/T\right) \\
&= 0.76\frac{n_b}{n_\gamma}\zeta(3)\frac{8}{\sqrt{2\pi}}\left(\frac{T}{m_e c^2}\right)^{3/2}\exp\left(B_1/T\right) \\
&\approx 2.92\eta\left(\frac{T}{m_e c^2}\right)^{3/2}\exp\left(B_1/T\right) \\
&\approx 8.0\times 10^{-8}\left(\frac{T}{m_e c^2}\right)^{3/2}\exp\left(B_1/T\right)\Omega_b h^2.
\end{aligned}$$

Recombination happens where S is of order unity, which requires $T < B_1$ to compensate for the small pre-factor (we get the transition at $T \approx 0.3 - 0.35\,\mathrm{eV}$, see Figure 2).

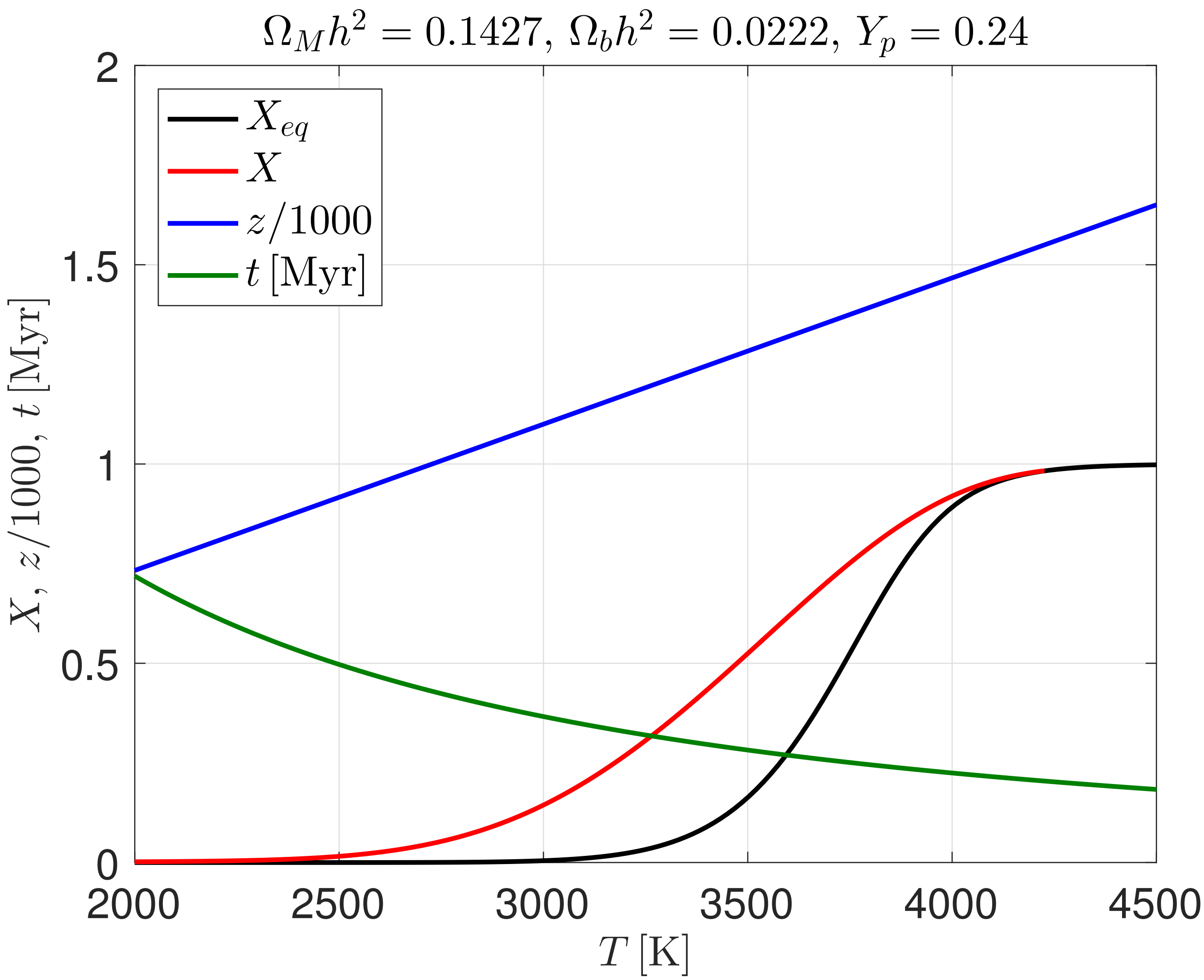

Fig. 2.— The solution of the Saha's equation (X_{eq}, black), the more accurate values of the ionization fraction (X, red), the redshift (blue) and the time to drop from 10^6 K as a function of the temperature for $\Omega_M h^2 = 0.1427$, $\Omega_b h^2 = 0.0222$ and $Y_p = 0.24$.

5.2. A more accurate calculation

The result from the previous section is not correct in detail, because equilibrium was not maintained for low ionization levels. For example, a photon emitted after electron capture to the ground level can easily ionize another Hydrogen atom (recall that at this era photons almost do not lose energy during scatterings), such that there is no net change in ionization. Similarly, a photon from $n \geq 3$ decay to the ground level can excite another atom from $n = 1$ to $n = 2$ so no net change in atoms at the ground level. However, a photon from $2p \to 1s$ transition (Lyman α photon) can excite another atom only until it is redshifted away from the absorption resonance. Nevertheless, this is so inefficient that we must consider the transition $2s \to 1s + 2\gamma$ as well (here the energy of each photon is low, so they cannot excite from $1s$).

We make the following assumptions:

1. Collisions between Hydrogen atoms and radiative transitions between Hydrogen levels are rapid, do they are in TE with T_γ, except the $1s$ level, which is reached only by slow or inefficient processes. The other levels satisfy $n_{nl} = (2l+1)n_{2s}\exp[(B_2 - B_n)/T]$, where $B_n = B_1/n^2$ is the binding energy of the nth level (this is true for n not too large, where the radii of the atoms become large).

2. The net rate of change of n_{1s} is given by radiative decays from $2s$ and $2p$ minus the inverse rates. All other processes are assumed to be cancelled by reionization or reexcitation of other atoms by the emitted photons. Recombination decreases the number $n_e R^3$ in a comoving volume R^3 at a rate $\alpha(T)n_p n_e R^3$, where $\alpha(T)$ do not include recombination directly to $1s$ - "Case B recombination coefficient"). Ionization from excited states increase $n_e R^3$ by a sum of terms proportional to $n_{nl}R^3$ ($n > 1$). Since $n_{nl} \propto n_{2s}$, the ionization increases $n_e R^3$ by a rate $\beta(T)n_{2s}R^3$:

$$\frac{d}{dt}\left(n_e R^3\right) = -\alpha n_e^2 R^3 + \beta n_{2s} R^3.$$

Dividing by the constant nR^3, where $n = n_p + n_H = n_p + \sum_{nl} n_{nl} = 0.76 n_b$, we get

$$\frac{d}{dt}\left(\frac{n_e}{n}\right) = -\alpha\frac{n_e^2}{n} + \beta\frac{n_{2s}}{n}. \tag{2}$$

A relation between the forward and backward rates can always be obtained by considering equilibrium (where the time derivative in the lhs is zero). So in equilibrium of $e + p \leftrightarrow 2s$ we have

$$\frac{n_{2s}}{n_e^2} = \left(\frac{2\pi m_e T}{h^2}\right)^{-3/2} \exp\left(\frac{B_2}{T}\right)$$

(same derivation as Equation (1)), and we get

$$\frac{\beta}{\alpha} = \left(\frac{n_e^2}{n_{2s}}\right)_{eq} = \left(\frac{2\pi m_e T}{h^2}\right)^{3/2} \exp\left(-\frac{B_2}{T}\right).$$

A fit to numerical calculations of $\alpha(T)$ is given by

$$\alpha(T) = \frac{1.4377\times 10^{-10}\,(T\,[\mathrm{K}])^{-0.6106}}{1+5.085\times 10^{-3}\,(T\,[\mathrm{K}])^{0.5300}}\,\mathrm{cm^3\,s^{-1}}.$$

3. The total number of excited Hydrogen atoms in a comoving volume, $1/n$, changes slower than individual radiative processes, such that the net increase in this number by recombination and ionization is balanced by the net decrease by transition to and from $1s$:

$$\begin{aligned} \alpha n_e^2 - \beta n_{2s} &= (\Gamma_{2s}+3P\Gamma_{2p})\,n_{2s} - \varepsilon n_{1s} \\ &\Rightarrow\ n_{2s} = \frac{\alpha n_e^2 + \varepsilon n_{1s}}{\Gamma_{2s}+3P\Gamma_{2p}+\beta}, \end{aligned} \tag{3}$$

where $\Gamma_{2s}\approx 8.22458\,\mathrm{s^{-1}}$ and $\Gamma_{2p}\approx 4.699\times 10^8\,\mathrm{s^{-1}}$ are the rates for the radiative decay processes $2s\to 1s+2\gamma$ and $2p\to 1s+\gamma$, respectively ($2p\to 1s+2\gamma$ can be neglected), the factor 3 is because $n_{2p}=3n_{2s}$, P is the probability that a Lyman α photon will escape without exciting $1s$ to $2p$, and ε is the rate at which $1s\to 2s$ or $1s\to 2p$, not including $1s+\gamma\to 2p$ with a Lyman α photon from $2p\to 1s+\gamma$, which is included in P. We consider $T \ll (B_2-B_3)\approx 2.2\times 10^4\,\mathrm{K}$, so all n_{nl} with $n>2$ are $\ll n_{2s}$ such that $n_H\approx n_{1s}+n_{2s}+n_{2p} = n_{1s}+4n_{2s}$. Using this in Equation (3) we get

$$n_{2s} = \frac{\alpha n_e^2 + \varepsilon n_H}{\Gamma_{2s}+3P\Gamma_{2p}+\beta+4\varepsilon}. \tag{4}$$

At equilibrium the rhs of the first line in Equation (3) is zero:

$$\frac{\varepsilon}{\Gamma_{2s}+3P\Gamma_{2p}} = \left(\frac{n_{2s}}{n_{1s}}\right)_{eq} = \exp\left(-\frac{B_1-B_2}{T}\right) \equiv \mathcal{E}. \tag{5}$$

Using Equations (4) and (5) in Equation (2) with the definitions $\Gamma \equiv \Gamma_{2s}+3P\Gamma_{2p}$, $\varepsilon = \Gamma\exp[-(B_1-B_2)/T] = \Gamma\mathcal{E}$ we get:

$$\begin{aligned} \frac{d}{dt}\left(\frac{n_e}{n}\right) &= -\alpha\frac{n_e^2}{n} + \frac{\beta}{n}\left(\frac{\alpha n_e^2+\varepsilon n_H}{\Gamma+\beta+4\varepsilon}\right) \\ &= \alpha\frac{n_e^2}{n}\left(\frac{\beta}{\Gamma+\beta+4\varepsilon}-1\right) + \frac{\varepsilon\beta n_H}{n\,(\Gamma+\beta+4\varepsilon)} \\ &= -\alpha\frac{n_e^2}{n}\frac{\Gamma+4\varepsilon}{\Gamma+\beta+4\varepsilon} + \frac{\varepsilon\beta n_H}{n\,(\Gamma+\beta+4\varepsilon)} \\ &= \frac{1}{n\,(\Gamma+\beta+4\Gamma\mathcal{E})}\left[-\alpha n_e^2\,(\Gamma+4\Gamma\mathcal{E}) + \Gamma\mathcal{E}\beta n_H\right] \\ &= \frac{\Gamma}{\Gamma(1+4\mathcal{E})+\beta}\left[-(1+4\mathcal{E})\,\frac{\alpha n_e^2}{n} + \mathcal{E}\frac{\beta n_H}{n}\right] \end{aligned}$$

Since $1+4\mathcal{E}\approx 1$ for the temperature range that we consider and

$$\mathcal{E}\frac{\beta}{\alpha n} = \mathcal{E}\left(\frac{n_e^2}{n_{2s}}\right)_{eq}\frac{1}{n} = \left(\frac{n_e^2}{n_{1s}}\right)_{eq}\frac{1}{n} = \frac{1}{S},$$

we get

$$\frac{dX}{dt} = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta}\alpha n \left[-X^2 + \frac{1-X}{S}\right], \tag{6}$$

where, as a reminder, $X = n_e/n = n_p/n = 1 - n_H/n$. Note that for a constant temperature, any solution of the Saha's equation, X_{eq}, will satisfy Equation (6). In fact, it turns out that always $X > X_{eq}$, so we always get $dX/dt < 0$. The first term on the rhs of Equation (6) is the suppression of the recombination rates, since the transitions $2s, 2p \to 1s$ are slower than the ionization.

We still need to calculate P:

$$P(t) = \int_{-\infty}^{\infty} d\omega \mathcal{P}(\omega) \exp\left[-\int_t^{\infty} dt' n_{1s}(t') c\sigma\left(\frac{\omega R(t)}{R(t')}\right)\right],$$

where $\mathcal{P}(\omega)d\omega$ is the probability that a photon from the $2p \to 1s$ transition has an energy between $\hbar\omega$ and $\hbar(\omega + d\omega)$, normalized such that $\int \mathcal{P}(\omega)d\omega = 1$:

$$\mathcal{P}(\omega) = \frac{\Gamma_{2p}}{2\pi}\frac{1}{(\omega-\omega_\alpha)^2 + \frac{\Gamma_{2p}^2}{4}},$$

$\omega_\alpha = ck_\alpha$, $k_\alpha = (B_1 - B_2)/\hbar c$, $\sigma(\omega)$ is the cross-section for the transition $1s \to 2p$ by a photon with an energy $\hbar\omega$:

$$\sigma(\omega) = \frac{3}{2}\frac{2\pi^2\Gamma_{2p}}{k_\alpha^2}\mathcal{P}(\omega)$$

(Breit-Wigner formula) and $R(t)/R(t')$ is to take care of the redshift. If a photon did not manage to escape, then the capture must be at a time much less than the expansion time, so $n_{1s}(t') \approx n_{1s}(t)$ and we can also approximate $R(t)/R(t') \approx 1 - H(t)(t'-t)$. Now we change variables to $\omega' = [1 - H(t)(t'-t)]\omega \Rightarrow d\omega' = -H(t)\omega dt'$ to get

$$P(t) = \int_{-\infty}^{\infty} d\omega \mathcal{P}(\omega) \exp\left[-\frac{3\pi^2\Gamma_{2p} n_{1s}(t)c}{\omega H(t) k_\alpha^2}\int_{-\infty}^{\omega} d\omega' \mathcal{P}(\omega')\right].$$

$\mathcal{P}(\omega)$ is negligible expect near ω_α, so we can change $\omega \to \omega_\alpha$ in the $1/\omega$ pre-factor to get

$$\begin{aligned} P(t) &= \int_{-\infty}^{\infty} d\omega \mathcal{P}(\omega) \exp\left[-A\int_{-\infty}^{\omega} d\omega' \mathcal{P}(\omega')\right] \\ &= -\frac{1}{A}\int_{-\infty}^{\infty} d\omega \frac{d}{d\omega}\left\{\exp\left[-A\int_{-\infty}^{\omega} d\omega' \mathcal{P}(\omega')\right]\right\} \\ &= -\frac{1}{A}\exp\left[-A\int_{-\infty}^{\omega} d\omega' \mathcal{P}(\omega')\right]\Bigg|_{-\infty}^{\infty} = \frac{1-\exp(-A)}{A}, \end{aligned}$$

where $A = 3\pi^2\Gamma_{2p} n_{1s}(t)c/[\omega_\alpha H(t) k_\alpha^2]$. This can be written as

$$P(t) = F\left(\frac{3\pi^2\Gamma_{2p} n_{1s}(t)c}{\omega_\alpha H(t) k_\alpha^2}\right), \quad F(x) = \frac{1-\exp(-x)}{x}.$$

It turns out that the argument of F is large, so that

$$P \approx \frac{\omega_\alpha H(t) k_\alpha^2}{3\pi^2 \Gamma_{2p} n_{1s}(t) c} = \frac{8\pi H(t)}{3\lambda_\alpha^3 \Gamma_{2p} n_{1s}(t)},$$

where $\lambda_\alpha \approx 1215.682 \times 10^{-8}$ cm is the Lyman α wavelength. Using $3P\Gamma_{2p} = 8\pi H/(\lambda_\alpha^3 n_{1s})$ we get from Equation (6)

$$\frac{dX}{dt} = \frac{\Gamma_{2s} + \frac{8\pi H}{\lambda_\alpha^3 n_{1s}}}{\Gamma_{2s} + \frac{8\pi H}{\lambda_\alpha^3 n_{1s}} + \beta} \alpha n \left[-X^2 + \frac{1-X}{S} \right].$$

It also turns out that $n_{1s}/n_{2s} \gg 1$ (although not as large as would be in equilibrium), so $n_{1s} \approx n_H = (1-X)n$ and we use $dt/dT = -1/(HT)$ (note that $T \propto R^{-1} \Rightarrow \dot{T}/T = -\dot{R}/R = -H$) to finally get

$$\frac{dX}{dT} = -\frac{\alpha n}{HT} \left[1 + \frac{\beta}{\Gamma_{2s} + \frac{8\pi H}{\lambda_\alpha^3 n(1-X)}} \right]^{-1} \left[-X^2 + \frac{1-X}{S} \right]. \quad (7)$$

For H it is justified to take $H = H_0 \sqrt{\Omega_M (T/T_{\rm CMB})^3 + \Omega_R (T/T_{\rm CMB})^4}$. Equation (7) can be integrated from high enough temperature, such that X_{eq} is a good approximation to X (since $X_{eq} < X < 1$ and for high enough temperatures $X_{eq} \approx 1$) but low enough such that all Helium is neutral (all Helium is doubly ionized until $T \sim 20,000$ K and there is still some single ionized Helium for 4,400 K). A good choice is $T = 4,260$ K ($z = 1550$). The result is plotted in Figure 2. Large deviations from X_{eq} begin as soon as X_{eq} is dropping significantly below 1. In particular, X has as asymptotic value ($X \approx 2.40 \times 10^{-4}$ at $z = 10$), which plays an important role in the formation of the first stars. The treatment here is quite accurate, expect for $T > 4,300$ K, where contribution of electrons from Helium cannot be ignored.

6. Opacity

The probability that a photon present at a time $t(T)$ will undergo at least one more scattering before the present is given by

$$\mathcal{O}(T) = 1 - \exp\left[-\int_{t(T)}^{t_0} c\sigma_T n_e(t) dt \right].$$

This rises from 0 at low T to near 1 at high T. We can use $dt/dT = -1/(HT)$ to write

$$\mathcal{O}(T) = 1 - \exp\left[-c\sigma_T \int_{T_{\rm CMB}}^{T} n_e(T') \frac{dT'}{T' H(T')} \right].$$

When analysing CMB anisotropies, we're interested in when photons observed today were last scattered. The probability that the last scattering of a photon was before the temperature dropped to T is $1 - \mathcal{O}(T)$ and the probability that the last scattering was after the temperature dropped further to $T - dT$ is $\mathcal{O}(T - dT)$, so the probability that the last scattering was in $[T, T - dT]$ is $1 - [1 - \mathcal{O}(T)] - \mathcal{O}(T - dT) = \mathcal{O}'(T)dT$. Since $\mathcal{O}(T)$ increases monotonically from $\mathcal{O} = 0$ at $T = T_{\mathrm{CMB}}$ to $\mathcal{O} = 1$ at $T = \infty$, then $\mathcal{O}'(T)$ is a positive normalized probability distribution with a unit integral. If we write $\mathcal{O}(T) = 1 - \exp(-\tau)$ then $\mathcal{O}'(T) = \tau' \exp(-\tau)$. The distribution $\mathcal{O}'(T)$ is peaked around $T_L \approx 2,945\,\mathrm{K}$ with $\sigma \approx 250\,\mathrm{K}$.