Distribution functions in an expanding Universe

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I'm loosely following Steven Weinberg's Cosmology and Yossi Nir's notes.

1. Distribution functions

Particles are in a thermal equilibrium (TE) if the interaction rate is much larger than the expansion rate of the Universe $\Gamma(T) \gg H(T)$. Particle of specie A that interact with photons, with $\Gamma_{A\gamma}(T) \gg H(T)$, have the same temperature as the photons, $T_A = T_\gamma$, so T_γ is called the temperature of the Universe, T. We can follow the evolution of each specie that once was in TE, either if it is in TE or in a free expansion. The distribution function of a specie A is $f_A(\vec{x}, \vec{p}, t)$ such that the number of particles is $dN_A = f_A(\vec{x}, \vec{p}, t) dV d^3p = f_A(\vec{p}, t) dV d^3p$ because of homogeneity.

For a small expansion of the Universe $t \to t + \delta t$ we have (recall $p \propto 1/R$)

$$\begin{aligned} dV &\to dV \left[\frac{R(t+\delta t)}{R(t)}\right]^3, \ [p, p+\delta p] \to \left[p\frac{R(t)}{R(t+\delta t)}, (p+\delta p)\frac{R(t)}{R(t+\delta t)}\right] \\ &\Rightarrow d^3p \to d^3p \left[\frac{R(t)}{R(t+\delta t)}\right]^3. \end{aligned}$$

if dN_A is a constant during the expansion, then the shape of f_A is conserved. Assume that the specie A decoupled at some time t_D with T_D and R_D (when $\Gamma_A = H$), then for $t < t_D$ we have $f_A = f_{A,eq}(\vec{p}, t)$ and for $t > t_D$ we have

$$f_A = f_{A,dec}(\vec{p}, t > t_D) = f_{A,eq}\left[\vec{p}\frac{R(t)}{R(t_D)}\right]. \tag{1}$$

At TE the distribution function is

$$f_A(\vec{p}, t) = \frac{g_A}{h^3}\frac{1}{\exp\{\beta[\varepsilon_A(p) - \mu_A]\} \pm 1},$$

where g_A is the degeneracy, $\beta = 1/T$, $\varepsilon_A(p) = \sqrt{c^2p^2 + m_A^2c^4}$, μ_A is the chemical potential, the plus sign is for fermions and the minus sign is for bosons.

2. Number density, energy density and pressure

For clarity we drop now the subscript A. The number density, the energy density and the pressure are given by

$$\begin{aligned} n &= \frac{4\pi g}{h^3}\int_0^\infty \frac{p^2 dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} = \frac{4\pi g}{(hc)^3}\int_{mc^2}^\infty \frac{\sqrt{\varepsilon^2-m^2c^4}\varepsilon d\varepsilon}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} \\ e &= \frac{4\pi g}{h^3}\int_0^\infty \frac{\varepsilon p^2 dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} = \frac{4\pi g}{(hc)^3}\int_{mc^2}^\infty \frac{\sqrt{\varepsilon^2-m^2c^4}\varepsilon^2 d\varepsilon}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} \\ P &= \frac{4\pi g}{3h^3}\int_0^\infty \frac{p^2 v p dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} = \frac{4\pi g c^2}{3h^3}\int_0^\infty \frac{p^4 dp}{\varepsilon\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} \\ &= \frac{4\pi g}{3(hc)^3}\int_{mc^2}^\infty \frac{(\varepsilon^2-m^2c^4)^{3/2} d\varepsilon}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1} \end{aligned}$$

where we have used for the velocity $v = c^2 p/\varepsilon = d\varepsilon/dp$. Note that in the ultra-relativistic (UR) limit we get $v \approx c$ and in the non-relativistic (NR) limit we get $v \approx p/m$.

For later reference:

$$\begin{aligned} \int_0^\infty x^n e^{-x} dx &= \Gamma(n+1), \text{for } n > -1, \\ \int_0^\infty x^n e^{-x^2} dx &= \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right), \text{for } n > -1, \\ \int_0^\infty \frac{x^n}{e^x-1} dx &= n!\zeta(n+1), \text{for } n > 0, \\ \int_0^\infty \frac{x^n}{e^x+1} dx &= \left(1-\frac{1}{2^n}\right) n!\zeta(n+1), \text{for } n > -1. \end{aligned}$$

3. Thermodynamics

Let's calculate dP/dT:

$$\frac{dP}{dT} = \frac{4\pi g c^2}{3h^3}\int_0^\infty \frac{p^4 dp}{\varepsilon}\frac{\exp\left[\beta(\varepsilon-\mu)\right]}{\left\{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1\right\}^2}\left[(\varepsilon-\mu)\beta^2 + \beta\frac{\partial\mu}{\partial T}\right].$$

Since

$$\begin{aligned} \frac{d}{dp}\left\{\frac{1}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\right\} &= -\frac{\exp\left[\beta(\varepsilon-\mu)\right]}{\left\{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1\right\}^2}\beta\frac{\partial\varepsilon}{\partial p} \\ = \quad & -\frac{c^2\exp\left[\beta(\varepsilon-\mu)\right]}{\left\{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1\right\}^2}\beta\frac{p}{\varepsilon}, \end{aligned}$$

we get

$$
\begin{aligned}
\frac{dP}{dT} &= -\frac{4\pi g}{3h^3}\int_0^\infty \frac{p^3 dp}{\beta}\frac{d}{dp}\left\{\frac{1}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\right\}\left[\varepsilon\beta^2+\frac{\partial}{\partial T}\left(\beta\mu\right)\right] \\
&= -\frac{4\pi g}{3h^3}\frac{p^3}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\left[\varepsilon\beta+T\frac{\partial}{\partial T}\left(\beta\mu\right)\right]\Bigg|_0^\infty \\
&+ \frac{4\pi g}{3h^3}\int_0^\infty \frac{dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\left\{3p^2\left[\varepsilon\beta+T\frac{\partial}{\partial T}\left(\beta\mu\right)\right]+\frac{c^2p^4\beta}{\varepsilon}\right\} \\
&= \frac{4\pi g}{3h^3}\left\{\beta\int_0^\infty \frac{dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\left(3p^2\varepsilon+\frac{c^2p^4}{\varepsilon}\right)+T\frac{\partial}{\partial T}\left(\beta\mu\right)\int_0^\infty \frac{3p^2 dp}{\exp\left[\beta(\varepsilon-\mu)\right]\pm 1}\right\} \\
&= \beta(e+P)+nT\frac{\partial}{\partial T}\left(\beta\mu\right),
\end{aligned}
$$

where we integrated by parts and the term on the second line is zero. For instance, we have for photons $P = e/3$ and $\mu = 0$, such that we get $4e = Tde/dT \Rightarrow e \propto T^4$.

We already know $dE = -PdV$ for the expanding Universe:

$$
\begin{aligned}
d\left(eR^3\right) &= -Pd\left(R^3\right) \Rightarrow \frac{d}{dT}\left[(e+P)R^3\right] = R^3\frac{dP}{dT} \\
&\Rightarrow \frac{d}{dT}\left[(e+P)R^3\right] = \frac{R^3(e+P)}{T}+nR^3T\frac{\partial}{\partial T}\left(\beta\mu\right).
\end{aligned}
$$

We define $s = (e + P - n\mu)/T$, for which we get

$$
\begin{aligned}
d\left(sR^3\right) &= d\left[\frac{R^3(e+P-n\mu)}{T}\right] \\
&= \frac{1}{T}d\left[R^3(e+P)\right]-\frac{R^3(e+P)}{T^2}dT-nR^3d\left(\frac{\mu}{T}\right)-\frac{\mu}{T}d\left(nR^3\right) \\
&= -\frac{\mu}{T}d\left(nR^3\right).
\end{aligned}
$$

So the quantity sR^3 is approximately conserved if nR^3 is approximately conserved and/or $|\mu| \ll T$.

In the case $|\mu| \ll T$ we have $s \approx (e + P)/T$, so

$$
\begin{aligned}
Td\left(sR^3\right) &= Td\left(R^3\frac{e+P}{T}\right) = d\left[R^3(e+P)\right]-(e+P)R^3\frac{dT}{T} \\
&\approx d\left[R^3(e+P)\right]-R^3dP = d\left(R^3e\right)+Pd\left(R^3\right).
\end{aligned}
$$

Since $dE = -PdV + TdS$, we see that in thus case s is the entropy density and $s \propto R^{-3}$ (it is clear that the μ part will give μdN in general).

4. UR with $|\mu| \ll T$

Here $\varepsilon = cp$ and

$$e = \frac{4\pi g}{h^3}\int_0^\infty \frac{cp^3 dp}{\exp\left(\frac{cp}{T}\right) \pm 1} = \frac{4\pi g}{(hc)^3} T^4 \int_0^\infty \frac{x^3 dx}{\exp(x) \pm 1} = \begin{cases} \frac{7}{8}\frac{g}{2}\bar{a}_B T^4 & \text{fermions} \\ \frac{g}{2}\bar{a}_B T^4 & \text{bosons} \end{cases},$$

where we have used $\zeta(4) = \pi^4/90$ and $\bar{a}_B = 8\pi^5/15(hc)^3 \approx 2.0822\times10^{49}\,\mathrm{cm}^{-3}\,\mathrm{erg}^{-3}$ is the radiation constants written for temperature is energy units. For temperature in units of Kelvin, we have $a_B = \bar{a}_B k_B^4 \approx 7.56577 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$. For the number density

$$\begin{aligned} n &= \frac{4\pi g}{h^3}\int_0^\infty \frac{p^2 dp}{\exp\left(\frac{cp}{T}\right) \pm 1} = \frac{4\pi g}{(hc)^3} T^3 \int_0^\infty \frac{x^2 dx}{\exp(x) \pm 1} \\ &= \begin{cases} \frac{3}{4}\frac{16\pi}{(hc)^3}\zeta(3)\frac{g}{2}T^3 = \frac{3}{4}\frac{30}{\pi^4}\zeta(3)\frac{g}{2}\bar{a}_B T^3 \approx 0.2777\frac{g}{2}\bar{a}_B T^3 & \text{fermions} \\ \frac{16\pi}{(hc)^3}\zeta(3)\frac{g}{2}T^3 = \frac{30}{\pi^4}\zeta(3)\frac{g}{2}\bar{a}_B T^3 \approx 0.3702\frac{g}{2}\bar{a}_B T^3 & \text{bosons} \end{cases}, \end{aligned}$$

where $\zeta(3) \approx 1.202$. Finally, $P = e/3$, $s = 4e/3T$ and the average energy is

$$\langle\varepsilon\rangle = \frac{e}{n} = \begin{cases} \frac{7}{6}\frac{\pi^4}{30\zeta(3)}T \approx 3.15T & \text{fermions} \\ \frac{\pi^4}{30\zeta(3)}T \approx 2.70T & \text{bosons} \end{cases}.$$

We can write the total energy density for many species as:

$$e_{tot} = \bar{a}_B\left(\sum_{\text{bosons}} \frac{g_i}{2}T_i^4 + \frac{7}{8}\sum_{\text{fermions}}\frac{g_i}{2}T_i^4\right) \equiv \frac{g}{2}\bar{a}_B T^4,$$

where

$$g = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^4.$$

If all species have $T_i = T$, then

$$g = \sum_{\text{bosons}} g_i + \frac{7}{8}\sum_{\text{fermions}} g_i \equiv g_B + g_F.$$

Similarly for the total entropy density

$$s_{tot} = \frac{4}{3}\bar{a}_B\left(\sum_{\text{bosons}} \frac{g_i}{2}T_i^3 + \frac{7}{8}\sum_{\text{fermions}}\frac{g_i}{2}T_i^3\right) \equiv \frac{4}{3}\frac{q}{2}\bar{a}_B T^3,$$

where

$$q = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^3.$$

If all species have $T_i = T$, then $q = g$. Note that qT^3R^3 is conserved during expansion. Finally for the number density

$$n_{tot} = \frac{30}{\pi^4}\zeta(3)\bar{a}_B \left(\sum_{\text{bosons}} \frac{g_i}{2} T_i^3 + \frac{3}{4} \sum_{\text{fermions}} \frac{g_i}{2} T_i^3 \right) \equiv \frac{30}{\pi^4}\zeta(3)\frac{r}{2}\bar{a}_B T^3,$$

where

$$r = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{3}{4} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3.$$

If all species have $T_i = T$, then $s \propto n$. Also, since for photons $g_\gamma = 2$, then

$$n_\gamma = \frac{30}{\pi^4}\zeta(3)\bar{a}_B T^3 \Rightarrow s = \frac{4}{3}\frac{q}{2} n_\gamma \frac{\pi^4}{30\zeta(3)} \approx 1.801 q n_\gamma.$$

5. NR

Here $\varepsilon \approx mc^2 + p^2/2m \gg T$ and we further assume $\varepsilon - \mu \gg T$. In this case

$$\frac{1}{\exp\left[\beta\left(\varepsilon - \mu\right)\right] \pm 1} \approx \exp\left[-\beta(\varepsilon - \mu)\right],$$

regardless of the nature of the particle (a boson or a fermion). We get for the number density (note that $\Gamma(3/2) = \sqrt{\pi}/2$)

$$\begin{aligned} n &= \frac{4\pi g}{h^3} \int_0^\infty p^2 \exp\left[-\beta\left(mc^2 - \mu\right)\right] \exp\left(-\frac{\beta p^2}{2m}\right) dp \\ &= \frac{4\pi g}{h^3} \exp\left[-\beta\left(mc^2 - \mu\right)\right] \left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty x^2 \exp\left(-x^2\right) dx \\ &= g\left(\frac{2\pi m T}{h^2}\right)^{3/2} \exp\left[-\beta\left(mc^2 - \mu\right)\right], \end{aligned}$$

and for the energy density $e \approx mc^2 n$. We see that in a radiation dominated Universe, the number (and energy) density of non-relativistic particles with $|\mu| \ll T$ are exponentially suppressed by $\exp(-mc^2/T)$ compared with relativistic particles.

For the pressure we get (note that $\Gamma(5/2) = 3\sqrt{\pi}/4$)

$$\begin{aligned} P &\approx \frac{4\pi g}{3mh^3} \exp\left[-\beta\left(mc^2 - \mu\right)\right] \left(\frac{2m}{\beta}\right)^{5/2} \int_0^\infty x^4 \exp\left(-x^2\right) dx \\ &= g\left(\frac{2\pi m T}{h^2}\right)^{3/2} \exp\left[-\beta\left(mc^2 - \mu\right)\right] T = nT \ll e. \end{aligned}$$

6. $t > t_D$

After decouple, the distribution is given by Equation (1).

6.1. $T_D \gg mc^2$

For particles that decouple while they are relativistic, the shape of the distribution remains the same if $T(t) = T_D[R(t_D)/R(t)]$. In free expansion these particles are not in TE, but we can still define a temperature, which follows 'T'$\propto R^{-1}$. The entropy of these A particles, $S_A = s_A R^3$, is separately conserved. For particles still in TE the conserved entropy is qT^3R^3 so $T \propto q^{-1/3}R^{-1}$, which falls more slowly then R^{-1} (since $q(t)$ is decreasing with decreasing temperature). The number density of the decoupled particles is

$$n(t > t_D) \quad = \quad \begin{cases} \frac{3}{4}\frac{30}{\pi^4}\zeta(3)\frac{g}{2}\bar{a}_B T_D^3 \left[\frac{R(t_D)}{R(t)}\right]^3 & \text{fermions} \\ \frac{30}{\pi^4}\zeta(3)\frac{g}{2}\bar{a}_B T_D^3 \left[\frac{R(t_D)}{R(t)}\right]^3 & \text{bosons} \end{cases},$$

which is comparable to n_γ at any given time. Specifically, relic background of decoupled particles is present today with $n \sim n_\gamma$.

The distribution is of UR particles, i.e. at decouple $\varepsilon(t_D) \sim T_D \sim cp(t_D)$, but the momentum is redshifted as the Universe expands, and it is possible to get to the point where $T(t) = T_D R(t_D)/R(t) \sim mc^2 \equiv T_{NR}$. Then the energy of each particle becomes $\varepsilon(t) = \sqrt{c^2p^2(t) + m^2c^4} \sim mc^2$, so $e \sim nmc^2$ where n is UR.

6.2. $T_D \ll mc^2$

Here for $|\mu| \ll T$ the shape of the distribution remains the same if $T(t) = T_D[R(t_D)/R(t)]^2$. The number density is given in this case by

$$n(t > t_D) = g\left(\frac{2\pi m T_D}{h^2}\right)^{3/2}\left(\frac{R(t_D)}{R(t)}\right)^3 \exp\left(-\frac{mc^2}{T_D}\right),$$

so $n \propto R^{-3}$ and $e \sim mc^2 n$.

7. Excess of fermion over their antiparticles

Assume some fermions are able to annihilate through $f + \bar{f} \leftrightarrow \gamma' s$, then in TE $\mu_f + \mu_{\bar{f}} = 0$. The number density excess of f over $\bar{f}$ is given by

$$n_f - n_{\bar{f}} = \frac{4\pi g}{h^3}\int_0^\infty p^2 dp \left\{\frac{1}{\exp\left[\beta\left(\varepsilon - \mu\right)\right] + 1} - \frac{1}{\exp\left[\beta\left(\varepsilon + \mu\right)\right] + 1}\right\},$$

and the total energy density in these particles is given by

$$e_f + e_{\bar{f}} = \frac{4\pi g}{h^3} \int_0^\infty \varepsilon p^2 dp \left\{ \frac{1}{\exp\left[\beta\left(\varepsilon - \mu\right)\right] + 1} + \frac{1}{\exp\left[\beta\left(\varepsilon + \mu\right)\right] + 1} \right\}.$$

7.1. UR

Here $\varepsilon = cp$ and

$$\begin{aligned} n_f - n_{\bar{f}} &= \frac{4\pi g}{h^3} \int_0^\infty p^2 dp \left\{ \frac{1}{\exp\left[\beta\left(cp - \mu\right)\right] + 1} - \frac{1}{\exp\left[\beta\left(cp + \mu\right)\right] + 1} \right\} \\ &= 4\pi g \left(\frac{T}{ch}\right)^3 \int_0^\infty x^2 dx \left[\frac{1}{\exp\left(x - \beta\mu\right) + 1} - \frac{1}{\exp\left(x + \beta\mu\right) + 1} \right] \\ &= 4\pi g \left(\frac{T}{ch}\right)^3 \mathcal{M}\left(\frac{\mu}{T}\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}(x) &= \int_0^\infty y^2 dy \left[\frac{1}{\exp\left(y - x\right) + 1} - \frac{1}{\exp\left(y + x\right) + 1} \right] \\ &= \frac{x}{3}\left(\pi^2 + x^2\right), \end{aligned}$$

such that

$$n_f - n_{\bar{f}} = \frac{4\pi g}{3} \left(\frac{T}{ch}\right)^3 \frac{\mu}{T} \left[\pi^2 + \left(\frac{\mu}{T}\right)^2 \right].$$

The total energy density in these particles is

$$\begin{aligned} e_f + e_{\bar{f}} &= \frac{4\pi g}{h^3} \int_0^\infty cp^3 dp \left\{ \frac{1}{\exp\left[\beta\left(cp - \mu\right)\right] + 1} + \frac{1}{\exp\left[\beta\left(cp + \mu\right)\right] + 1} \right\} \\ &= 4\pi g \left(\frac{T}{ch}\right)^3 T \int_0^\infty x^3 dx \left[\frac{1}{\exp\left(x - \beta\mu\right) + 1} + \frac{1}{\exp\left(x + \beta\mu\right) + 1} \right] \\ &= 4\pi g \left(\frac{T}{ch}\right)^3 T\mathcal{P}\left(\frac{\mu}{T}\right), \end{aligned}$$

where

$$\mathcal{P}(x) = \int_0^\infty y^3 dy \left[\frac{1}{\exp\left(y - x\right) + 1} + \frac{1}{\exp\left(y + x\right) + 1} \right].$$

We have $\mathcal{P}(0) = 7\pi^4/60$, and

$$\begin{aligned} \mathcal{P}'(x) &= \int_0^\infty y^3 dy \left\{ \frac{\exp(y - x)}{\left[\exp\left(y - x\right) + 1\right]^2} - \frac{\exp(y + x)}{\left[\exp\left(y + x\right) + 1\right]^2} \right\} \\ &= \int_0^\infty \frac{y^3 dy}{\left[\exp\left(y - x\right) + 1\right]^2 \left[\exp\left(y + x\right) + 1\right]^2} \left[\exp(-x) + \exp(2y + x) - \exp(x) - \exp(2y - x)\right]. \end{aligned}$$

The rightmost term is $-2\sinh(x) + 2\exp(2y)\sinh(x) = 2\sinh(x)[\exp(2y) - 1]$, which is positive for $x > 0$, so we have $\mathcal{P}'(x) > 0$ for $x > 0$. Since $\mathcal{P}(x)$ is symmetric, we have $\mathcal{P}(x) \geq \mathcal{P}(0)$, and $\mathcal{P}(x) = \mathcal{P}(0)$ only for $x = 0$.

In the limit $x \gg 1$ we have

$$\mathcal{P}(x) \approx \int_0^\infty \frac{y^3}{\exp(y-x)+1}dy \approx \frac{x^4}{4},$$

such that

$$e_f + e_{\bar{f}} \approx \pi g \left(\frac{T}{ch}\right)^3 T \left(\frac{\mu}{T}\right)^4 = \pi g \frac{\mu^4}{(ch)^3} \gg \bar{a}_B T^4.$$

7.2. NR

Here $\varepsilon \sim mc^2 + p^2/2m$ and we further assume $|\mu| \ll \varepsilon$. We have

$$\begin{aligned}
n_f - n_{\bar{f}} &= \frac{4\pi g}{h^3}\int_0^\infty p^2 dp \left\{\exp\left[-\beta\left(\varepsilon-\mu\right)\right] - \exp\left[-\beta\left(\varepsilon+\mu\right)\right]\right\} \\
&= \frac{4\pi g}{h^3}\int_0^\infty p^2 dp \exp\left(-\frac{mc^2}{T}\right)\exp\left(-\frac{p^2}{2mT}\right)\left[\exp\left(\frac{\mu}{T}\right) - \exp\left(-\frac{\mu}{T}\right)\right] \\
&= g\left(\frac{2\pi mT}{h^3}\right)^{3/2}\exp(-\beta mc^2) 2\sinh\left(\frac{\mu}{T}\right).
\end{aligned}$$