The FRW Universe

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Developed by Friedman (22,24), Robertson (36) and Walker (36). I'm loosely following Steven Weinberg's Cosmology and Eli Waxman's notes.

1. Assumptions

We assume that the Universe is isotropic and homogeneous, meaning that we can choose x^μ such that the subspaces $t=\text{const.}$ are homogeneous and isotropic. We can choose a time such that all free falling observers can agree on, $t(S)$, where S is some scalar, e.g. $t(T_{\rm CMB})$.

2. An example - a 2D space

Let's discuss a 2D isotropic and homogeneous space, embedded in a 3D Euclidean space: $ds^2=dx^2+dy^2+dz^2$. The simplest case is a flat 2D space, e.g. $ds^2=dx^2+dy^2$ with some $z=\text{const.}$. Another possibility is a 2D sphere (S^2) with a radius R: $x^2+y^2+z^2=R^2$. We define

$$x=r\cos\theta,\ \ y=r\sin\theta,$$

such that

$$\begin{aligned}
dx &= dr\cos\theta-r\sin\theta d\theta,\ \ dy=dr\sin\theta+r\cos\theta d\theta,\\
dx^2+dy^2 &= dr^2+r^2d\theta^2,\\
z &= \sqrt{R^2-r^2}\Rightarrow dz=-\frac{rdr}{\sqrt{R^2-r^2}}\Rightarrow dz^2=\frac{r^2dr^2}{R^2-r^2}\\
\Rightarrow &\qquad ds^2=\frac{R^2}{R^2-r^2}dr^2+r^2d\theta^2=R^2\left(\frac{d\tilde{r}^2}{1-\tilde{r}^2}+\tilde{r}^2d\theta^2\right),
\end{aligned}$$

where $\tilde{r}$=r/R.

How do you know you're on a sphere? You can take a string with a length l and measure circle circumstance. The string is placed from the coordinate $r=0$ to some coordinate r with a fixed θ. The circumstance is given by $\int_0^{2\pi}rd\theta=2\pi r$. The length for a flat space is given by $l=\int_0^r dr'=r$, such that $2\pi r=2\pi l$. For S^2 we get

$$l = \int_0^r\frac{dr'}{\sqrt{1-r'^2/R^2}}=R\sin^{-1}\left(\frac{r}{R}\right)=r\left[1+\frac{1}{6}\left(\frac{r}{R}\right)^2+O\left(\frac{r^4}{R^4}\right)\right]\Rightarrow 2\pi r<2\pi l.$$

S^2 is isotropic and homogeneous. It is also unbounded but finite.

The last possibility for a 2D isotropic and homogeneous space is (constant negative curvature)

$$ds^2 = R^2\left(\frac{d\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2 d\theta^2\right).$$

Expansion or contraction of these geometries is changing the scale factor R, but leaving $\tilde{r}, \theta$ fixed $\Rightarrow \tilde{r}, \theta$ are comoving coordinates. Distances between comoving coordinates scale with R. For flat space R is not a radius, but just scale the physical distance between comoving points.

3. An extension to 3D space + time

The metric in this case is

$$-c^2 d\tau^2 = g_{00} c^2 dt^2 + 2 g_{i0} c dt dx^i + g_{ij} dx^i dx^j.$$

$g_{i0} = 0$ since otherwise there is a preferred direction. From homogeneity $g_{00}(t)$, so we can scale time:

$$\begin{aligned} -c^2 d\tau^2 &= -c^2 dt^2 + g_{ij} dx^i dx^j, \\ &\Rightarrow \; c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right), \end{aligned}$$

where $k = 0$ is for Euclidean, $k = +1$ is for spherical (sometimes called 'close'), and $k = -1$ is for hyper-spherical (sometimes called 'open'). I'm not proving that these are the only 3 options for the spatial part. r is dimensionless. The 3D spatial curvature is $R_{3D} = k/R^2$. The components of the metric are:

$$\begin{aligned} g_{00} &= -1, g_{0i} = g_{i0} = 0, \; g_{ij} = R^2(t)\tilde{g}_{ij}, \, \tilde{g}_{rr} = \frac{1}{1-kr^2}, \, \tilde{g}_{\theta\theta} = r^2, \tilde{g}_{\phi\phi} = r^2 \sin^2\theta, \\ g^{00} &= -1, \, g^{0i} = g^{i0} = 0, \, g^{ij} = \frac{1}{R^2(t)}\tilde{g}^{ij}. \end{aligned}$$

Note units: $[x^0] = [ct] = \mathrm{cm}$, and x^i is dimensionless, such that g_{00}, g^{00}, $\tilde{g}_{ij}$ and $\tilde{g}^{ij}$ are dimensionless, $[g_{ij}] = \mathrm{cm}^2$ and $[g^{ij}] = \mathrm{cm}^{-2}$.

In these coordinates the metric is diagonal. Sometimes it is more convenient to work with different coordinates:

$$r^{(1)} = r\sin\theta\cos\phi, \; r^{(2)} = r\sin\theta\sin\phi, \; r^{(3)} = r\cos\theta,$$

such that

$$\begin{aligned}
dr^{(1)} &= dr\sin\theta\cos\phi + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi, \\
dr^{(2)} &= dr\sin\theta\sin\phi + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi, \\
dr^{(3)} &= dr\cos\theta - r\sin\theta d\theta, \\
&\Rightarrow d\vec{r}^2 + \frac{k\left(\vec{r}\cdot d\vec{r}\right)^2}{1-kr^2} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + \frac{k}{1-kr^2}\left(rdr\right)^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega, \\
&\Rightarrow \frac{dr^2}{1-kr^2} + r^2 d\Omega = \delta_{ij}dr^i dr^j + \frac{k}{1-kr^2}\left(\delta_{ij}r^j dr^i\right)^2, \\
&\Rightarrow \tilde{g}_{ij} = \delta_{ij} + \frac{kr^i r^j}{1-kr^2}.
\end{aligned}$$

The inverse metric is $\tilde{g}^{ij} = \delta^{ij} - kr^i r^j$. Let's verify:

$$\begin{aligned}
\tilde{g}_{ij}\tilde{g}^{jk} &= \left(\delta_{ij} + \frac{kr^i r^j}{1-kr^2}\right)\left(\delta^{jk} - kr^j r^k\right) \\
&= \delta_{ij}\delta^{jk} - \delta_{ij}kr^j r^k + \frac{kr^i r^j}{1-kr^2}\delta^{jk} - \frac{kr^i r^j}{1-kr^2}kr^j r^k \\
&= \delta_i^k - kr^i r^k + \frac{kr^i r^k}{1-kr^2} - \frac{k^2 r^i r^k r^2}{1-kr^2} \\
&= \delta_i^k + kr^i r^k\left(-1 + \frac{1}{1-kr^2} - \frac{kr^2}{1-kr^2}\right) = \delta_i^k.
\end{aligned}$$

The spatial metric is invariant under quasi translations

$$\vec{r}' = \vec{r} + \vec{r}_0\left[\sqrt{1-kr^2} - \left(1 - \sqrt{1-kr_0^2}\right)\frac{\vec{r}\cdot\vec{r}_0}{r_0^2}\right],$$

which translates the origin to $\vec{r}_0$. To verify, we need to show that

$$g'^{ij} = g^{kl}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}.$$

Using

$$\begin{aligned}
\frac{\partial x'^i}{\partial x^j} &= \delta_{ij} + r_0^i\left(-\frac{2kr}{2\sqrt{1-kr^2}}\frac{2r^j}{2r} - A_0\frac{r_0^j}{r_0^2}\right), \\
&= \delta_{ij} - r_0^i\left(\frac{kr^j}{\sqrt{1-kr^2}} + A_0\frac{r_0^j}{r_0^2}\right),
\end{aligned}$$

where $A_0 = 1 - \sqrt{1-kr_0^2}$ (note that $A_0^2 = 1 - 2\sqrt{1-kr_0^2} + 1 - kr_0^2 = 2A_0 - kr_0^2$), we get

$$\begin{aligned}
g'^{ij} &= \left(\delta^{kl} - kr^k r^l\right)\left[\delta_{ik} - r_0^i\left(\frac{kr^k}{\sqrt{1-kr^2}} + A_0\frac{r_0^k}{r_0^2}\right)\right]\left[\delta_{jl} - r_0^j\left(\frac{kr^l}{\sqrt{1-kr^2}} + A_0\frac{r_0^l}{r_0^2}\right)\right] \\
&= \left[\delta_i^l - r_0^i\left(\frac{kr^l}{\sqrt{1-kr^2}} + A_0\frac{r_0^l}{r_0^2}\right) - kr^i r^l + kr_0^i\left(\frac{kr^2 r^l}{\sqrt{1-kr^2}} + A_0\frac{r^l\vec{r}\cdot\vec{r}_0}{r_0^2}\right)\right] \\
&\times \left[\delta_{jl} - r_0^j\left(\frac{kr^l}{\sqrt{1-kr^2}} + A_0\frac{r_0^l}{r_0^2}\right)\right] \\
&= \delta^{ij} - r_0^j\left(\frac{kr^i}{\sqrt{1-kr^2}} + A_0\frac{r_0^i}{r_0^2}\right) - r_0^i\left(\frac{kr^j}{\sqrt{1-kr^2}} + A_0\frac{r_0^j}{r_0^2}\right) \\
&+ r_0^i r_0^j\left(\frac{k^2r^2}{1-kr^2} + 2\frac{kA_0\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} + \frac{A_0^2}{r_0^2}\right) - kr^i r^j \\
&+ kr_0^j r^i\left(\frac{kr^2}{\sqrt{1-kr^2}} + A_0\frac{\vec{r}\cdot\vec{r}_0}{r_0^2}\right) + kr_0^i\left(\frac{kr^2r^j}{\sqrt{1-kr^2}} + A_0\frac{r^j\vec{r}\cdot\vec{r}_0}{r_0^2}\right) \\
&- kr_0^i r_0^j\left(\frac{k^2r^4}{1-kr^2} + 2A_0\frac{kr^2\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} + A_0^2\frac{(\vec{r}\cdot\vec{r}_0)^2}{r_0^4}\right) \\
&= \delta^{ij} - kr^i r^j + \left(r^i r_0^j + r_0^i r^j\right)\left(-\frac{k}{\sqrt{1-kr^2}} + \frac{k^2r^2}{\sqrt{1-kr^2}} + \frac{kA_0\vec{r}\cdot\vec{r}_0}{r_0^2}\right) \\
&+ r_0^i r_0^j\left(\frac{k^2r^2}{1-kr^2} + 2kA_0\frac{\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} + \frac{A_0^2}{r_0^2} - \frac{k^3r^4}{1-kr^2} - 2k^2A_0\frac{r^2\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} - kA_0^2\frac{(\vec{r}\cdot\vec{r}_0)^2}{r_0^4} - 2\frac{A_0}{r_0^2}\right).
\end{aligned}$$

The lhs is

$$g'^{ij} = \delta^{ij} - k\left[r^i + r_0^i\left(\sqrt{1-kr^2} - A_0\frac{\vec{r}\cdot\vec{r}_0}{r_0^2}\right)\right]\left[r^j + r_0^j\left(\sqrt{1-kr^2} - A_0\frac{\vec{r}\cdot\vec{r}_0}{r_0^2}\right)\right],$$

which we can compare term by term. The only nontrivial term is the $r_0^i r_0^j$ term, for which we need to verify that

$$\begin{aligned}
&-k\left[1 - kr^2 - 2A_0\sqrt{1-kr^2}\frac{\vec{r}\cdot\vec{r}_0}{r_0^2} + A_0^2\frac{(\vec{r}\cdot\vec{r}_0)^2}{r_0^4}\right] \\
&= \frac{k^2r^2}{1-kr^2} + 2kA_0\frac{\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} + \frac{A_0^2}{r_0^2} - \frac{k^3r^4}{1-kr^2} - 2k^2A_0\frac{r^2\vec{r}\cdot\vec{r}_0}{r_0^2\sqrt{1-kr^2}} - kA_0^2\frac{(\vec{r}\cdot\vec{r}_0)^2}{r_0^4} - 2\frac{A_0}{r_0^2}.
\end{aligned}$$

The 6th term on the rhs is the 4th term on the lhs, the 2nd and the 5th terms on the rhs give the 4th term on the lhs, the 3rd and the 7th terms on the rhs give the 1st term on the lhs and the 1st and the 4th terms on the rhs give the 2nd term on the lhs.

4. Free falling particle (geodesic)

The equation of motion is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\kappa}\frac{dx^\nu}{d\tau}\frac{dx^\kappa}{d\tau} = 0,$$

where

$$\Gamma^\mu_{\nu\kappa} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\nu}}{\partial x^\kappa} + \frac{\partial g_{\lambda\kappa}}{\partial x^\nu} - \frac{\partial g_{\nu\kappa}}{\partial x^\lambda}\right).$$

Let's look on a comoving particle with $\vec{r} = \text{const.}$. It follows $dx^i/d\tau = 0$, and therefore

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0.$$

Since

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda 0}}{\partial x^0} + \frac{\partial g_{\lambda 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right) = 0$$

(as $g_{\lambda 0}$ is either 0 or -1), we get $d^2x^i/d\tau^2 = 0$, so $\vec{r} = \text{const.}$ is a geodesic. Proper time interval for this particle:

$$c^2d\tau^2 = c^2dt^2 - g_{ij}dr^idr^j = c^2dt^2 \Rightarrow dt = d\tau,$$

so t is the time measured in the rest frame of a comoving clock.

5. The rest of the affine connections

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0\lambda}\left(\frac{\partial g_{\lambda i}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\lambda}\right) = -\frac{1}{2}\left(\frac{\partial g_{0i}}{\partial x^j} + \frac{\partial g_{0j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^0}\right) = \frac{R\dot{R}}{c}\tilde{g}_{ij},\\
\Gamma^0_{0i} &= \frac{1}{2}g^{0\lambda}\left(\frac{\partial g_{\lambda 0}}{\partial x^i} + \frac{\partial g_{\lambda i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\lambda}\right) = -\frac{1}{2}\left(\frac{\partial g_{00}}{\partial x^i} + \frac{\partial g_{0i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^0}\right) = 0 = \Gamma^0_{i0},\\
\Gamma^i_{0j} &= \frac{1}{2}g^{i\lambda}\left(\frac{\partial g_{\lambda 0}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^0} - \frac{\partial g_{0j}}{\partial x^\lambda}\right) = \frac{1}{2}g^{ik}\frac{\partial g_{kj}}{\partial x^0} = \frac{\dot{R}}{cR}\delta^i_j = \Gamma^i_{j0},\\
\Gamma^i_{jk} &= \frac{1}{2}g^{i\lambda}\left(\frac{\partial g_{\lambda j}}{\partial x^k} + \frac{\partial g_{\lambda k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\lambda}\right) = \frac{1}{2}\tilde{g}^{il}\left(\frac{\partial \tilde{g}_{lj}}{\partial x^k} + \frac{\partial \tilde{g}_{lk}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right) \equiv \tilde{\Gamma}^i_{jk}.
\end{aligned}$$

Note units: $[\Gamma^0_{ij}] = \text{cm}$, $[\Gamma^i_{0j}] = \text{cm}^{-1}$ and Γ^i_{jk}, $\tilde{\Gamma}^i_{jk}$ dimensionless.

6. Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}.$$

Since Γ vanishes for two or three time indices, we get:

$$\begin{aligned} R_{ij} &= \frac{\partial \Gamma^{k}_{ki}}{\partial x^{j}} - \left(\frac{\partial \Gamma^{k}_{ij}}{\partial x^{k}} + \frac{\partial \Gamma^{0}_{ij}}{\partial x^{0}}\right) + \left(\Gamma^{0}_{ik}\Gamma^{k}_{j0} + \Gamma^{k}_{i0}\Gamma^{0}_{jk} + \Gamma^{l}_{ik}\Gamma^{k}_{jl}\right) - \left(\Gamma^{k}_{ij}\Gamma^{l}_{kl} + \Gamma^{0}_{ij}\Gamma^{l}_{0l}\right) \\ &\equiv \tilde{R}_{ij} - \frac{\partial \Gamma^{0}_{ij}}{\partial x^{0}} + \Gamma^{0}_{ik}\Gamma^{k}_{j0} + \Gamma^{k}_{i0}\Gamma^{0}_{jk} - \Gamma^{0}_{ij}\Gamma^{l}_{0l}, \end{aligned}$$

$$R_{00} = \frac{\partial \Gamma^{i}_{i0}}{\partial x^{0}} + \Gamma^{i}_{0j}\Gamma^{j}_{0i},$$

and $R_{0i} = R_{i0} = 0$ because of isotropy.

We need to calculate the following terms:

$$\begin{aligned} \frac{\partial \Gamma^{0}_{ij}}{\partial x^{0}} &= \frac{1}{c^2}\tilde{g}_{ij}\frac{d}{dt}\left(R\dot{R}\right), \\ \Gamma^{0}_{ik}\Gamma^{k}_{j0} &= \frac{R\dot{R}}{c}\tilde{g}_{ij}\frac{\dot{R}}{cR} = \frac{1}{c^2}\tilde{g}_{ij}\dot{R}^2, \\ \Gamma^{k}_{i0}\Gamma^{0}_{jk} &= \frac{\dot{R}}{cR}\delta^{k}_{i}\frac{R\dot{R}}{c}\tilde{g}_{jk} = \frac{1}{c^2}\tilde{g}_{ij}\dot{R}^2, \\ \Gamma^{0}_{ij}\Gamma^{l}_{0l} &= \frac{R\dot{R}}{c}\tilde{g}_{ij}\frac{3\dot{R}}{cR} = \frac{3\dot{R}^2}{c^2}\tilde{g}_{ij}, \\ \frac{\partial \Gamma^{i}_{i0}}{\partial x^{0}} &= \frac{3}{c^2}\frac{d}{dt}\left(\frac{\dot{R}}{R}\right), \\ \Gamma^{i}_{0j}\Gamma^{j}_{0i} &= \frac{\dot{R}}{cR}\delta^{i}_{j}\frac{\dot{R}}{cR}\delta^{j}_{i} = \frac{3\dot{R}^2}{c^2R^2}. \end{aligned}$$

We get:

$$\begin{aligned} R_{ij} &= \tilde{R}_{ij} + \frac{1}{c^2}\tilde{g}_{ij}\left(-\dot{R}^2 - R\ddot{R} + \dot{R}^2 + \dot{R}^2 - 3\dot{R}^2\right) = \tilde{R}_{ij} + \frac{1}{c^2}\tilde{g}_{ij}\left(-R\ddot{R} - 2\dot{R}^2\right), \\ R_{00} &= \frac{3}{c^2}\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} + \frac{\dot{R}^2}{R^2}\right) = \frac{3}{c^2}\frac{\ddot{R}}{R}. \end{aligned}$$

Note units: $[R_{00}] = \mathrm{cm}^{-2}$, R_{ij} and $\tilde{R}_{ij}$ are dimensionless.

To calculate $\tilde{R}_{ij}$, note that we need $\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}$, calculated with $\tilde{g}_{ij}$ and $\tilde{g}^{ij}$. We have

$$\begin{aligned}
\frac{\partial \tilde{g}_{ij}}{\partial x^k} &= \frac{k}{1-kr^2}\left(\delta_{jk}r^i + \delta_{ik}r^j\right) + \frac{k^2 r^i r^j}{(1-kr^2)^2} 2r \frac{2r^k}{2r} \\
&= \frac{k}{1-kr^2}\left(\delta_{jk}r^i + \delta_{ik}r^j\right) + \frac{2k^2}{(1-kr^2)^2} r^i r^j r^k, \\
\frac{\partial \tilde{g}^{ij}}{\partial x^k} &= -k\left(\delta^j_k r^i + \delta^i_k r^j\right).
\end{aligned}$$

Such that

$$\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \frac{1}{2}\tilde{g}^{il}\left(\frac{\partial \tilde{g}_{lj}}{\partial x^k} + \frac{\partial \tilde{g}_{lk}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right) \\
&= \frac{1}{2}\left(\delta^{il} - kr^i r^l\right)\frac{k}{1-kr^2}\left(\delta_{jk}r^l + \delta_{lk}r^j + \delta_{kj}r^l + \delta_{lj}r^k - \delta_{kl}r^j - \delta_{jl}r^k + \frac{2k}{1-kr^2}r^l r^j r^k\right) \\
&= \left(\delta^{il} - kr^i r^l\right)\frac{k}{1-kr^2}\left(\delta_{jk}r^l + \frac{k}{1-kr^2}r^l r^j r^k\right) \\
&= \frac{k}{1-kr^2}\left(\delta_{jk}r^i + \frac{k}{1-kr^2}r^i r^j r^k - k\delta_{jk}r^i r^2 - \frac{k^2}{1-kr^2}r^i r^j r^k r^2\right) \\
&= \frac{kr^i}{1-kr^2}\left[\delta_{jk}\left(1-kr^2\right) + \frac{kr^j r^k}{1-kr^2}\left(1-kr^2\right)\right] \\
&= kr^i\left(\delta_{jk} + \frac{kr^j r^k}{1-kr^2}\right) = kr^i \tilde{g}_{jk}.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial \tilde{\Gamma}^i_{jk}}{\partial x^l} &= k\delta^i_l \tilde{g}_{jk} + kr^i \frac{\partial \tilde{g}_{jk}}{\partial x^l} \\
&= k\delta^i_l \tilde{g}_{jk} + kr^i\left[\frac{k}{1-kr^2}\left(\delta_{kl}r^j + \delta_{jl}r^k\right) + \frac{2k^2}{\left(1-kr^2\right)^2}r^j r^k r^l\right],
\end{aligned}$$

such that

$$\begin{aligned}
\tilde{R}_{ij} &= \frac{\partial \tilde{\Gamma}^k_{ki}}{\partial x^j} - \frac{\partial \tilde{\Gamma}^k_{ij}}{\partial x^k} + \tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl} - \tilde{\Gamma}^k_{ij}\tilde{\Gamma}^l_{kl} \\
&= k\left\{\delta^k_j \tilde{g}_{ki} + r^k\left[\frac{k}{1-kr^2}\left(\delta_{ij}r^k + \delta_{kj}r^i\right) + \frac{2k^2}{\left(1-kr^2\right)^2}r^k r^i r^j\right]\right\} \\
&- k\left\{\delta^k_k \tilde{g}_{ij} + r^k\left[\frac{k}{1-kr^2}\left(\delta_{jk}r^i + \delta_{ik}r^j\right) + \frac{2k^2}{\left(1-kr^2\right)^2}r^i r^j r^k\right]\right\} \\
&+ k^2 r^l r^k \tilde{g}_{ik}\tilde{g}_{jl} - k^2 r^k r^l \tilde{g}_{ij}\tilde{g}_{kl} \\
&= k\left(\tilde{g}_{ji} + \frac{kr^2}{1-kr^2}\delta_{ij} + \frac{kr^i r^j}{1-kr^2} + \frac{2k^2 r^2 r^i r^j}{\left(1-kr^2\right)^2} - 3\tilde{g}_{ij} - \frac{kr^i r^j}{1-kr^2} - \frac{kr^i r^j}{1-kr^2} - \frac{2k^2 r^i r^j r^2}{\left(1-kr^2\right)^2}\right) \\
&+ k^2 r^l r^k\left(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{ij}\tilde{g}_{kl}\right) \\
&= -2k\tilde{g}_{ij} + \frac{k^2}{1-kr^2}\left(r^2\delta_{ij} - r^i r^j\right) + k^2 r^l r^k\left(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{ij}\tilde{g}_{kl}\right).
\end{aligned}$$

The second term is

$$k\left(\frac{kr^2\delta_{ij}}{1-kr^2}-\frac{kr^ir^j}{1-kr^2}\right)=k\left(\frac{kr^2\delta_{ij}}{1-kr^2}+\delta_{ij}-\tilde{g}_{ij}\right)=k\left(\delta_{ij}\frac{1}{1-kr^2}-\tilde{g}_{ij}\right).$$

For the third term we need

$$\begin{aligned}
\tilde{g}_{ij}\tilde{g}_{kl} &= \left(\delta_{ij}+\frac{k}{1-kr^2}r^ir^j\right)\left(\delta_{kl}+\frac{k}{1-kr^2}r^kr^l\right),\\
\tilde{g}_{ik}\tilde{g}_{jl} &= \left(\delta_{ik}+\frac{k}{1-kr^2}r^ir^k\right)\left(\delta_{jl}+\frac{k}{1-kr^2}r^jr^l\right),\\
&\Rightarrow\ \tilde{g}_{ik}\tilde{g}_{jl}-\tilde{g}_{ij}\tilde{g}_{kl}=\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl}+\frac{k}{1-kr^2}\left(\delta_{ik}r^jr^l-\delta_{ij}r^kr^l\right)+\frac{k}{1-kr^2}\left(\delta_{jl}r^ir^k-\delta_{kl}r^ir^j\right)\\
&\Rightarrow\ k^2r^lr^k\left(\tilde{g}_{ik}\tilde{g}_{jl}-\tilde{g}_{ij}\tilde{g}_{kl}\right)=k^2\left[r^ir^j-\delta_{ij}r^2+\frac{k}{1-kr^2}\left(r^2r^ir^j-\delta_{ij}r^4\right)+\frac{k}{1-kr^2}\left(r^ir^jr^2-r^ir^jr^2\right)\right]\\
&= k^2\left(r^ir^j-\delta_{ij}r^2\right)\left(1+\frac{kr^2}{1-kr^2}\right)=\frac{k^2}{1-kr^2}\left(r^ir^j-\delta_{ij}r^2\right).
\end{aligned}$$

Putting these together, we get:

$$\begin{aligned}
\tilde{R}_{ij} &= -2k\tilde{g}_{ij}+k\left(\delta_{ij}\frac{1}{1-kr^2}-\tilde{g}_{ij}+\frac{k}{1-kr^2}r^ir^j-\frac{k}{1-kr^2}\delta_{ij}r^2\right)\\
&= -2k\tilde{g}_{ij}+k\left(\delta_{ij}+\frac{k}{1-kr^2}r^ir^j-\tilde{g}_{ij}\right)=-2k\tilde{g}_{ij}.
\end{aligned}$$

We finally get

$$R_{ij} = \tilde{R}_{ij}+\frac{1}{c^2}\tilde{g}_{ij}\left(-R\ddot{R}-2\dot{R}^2\right)=-\left(2k+\frac{R\ddot{R}}{c^2}+\frac{2\dot{R}^2}{c^2}\right)\tilde{g}_{ij}.$$

7. The energy-momentum tensor

T^{00} is the energy density e, T^{0i} is the the energy flux divided by c, and T^{ij} is the flux in the j-th direction of the i-th momentum component. We assume that the Universe is full with ideal fluid in LTE: the relaxation time is $\ll$ than the expansion time and the diffusion length scale $\sqrt{D\cdot t_{\rm flow}}$ is $\ll$ than typical length scales L of the problem. Under these assumptions entropy is conserved $\delta s=0$. In the rest frame of the fluid $T^{\mu\nu}=\mathrm{diag}(e,p,p,p)$, where p is the pressure. In Special relativity we can make a tensor out of this with $T^{\mu\nu}=(e+p)u^\mu u^\nu/c^2+\eta^{\mu\nu}p$ This is a tensor, since e and p are defined by their values in locally comoving system, so they are scalars, and u^μ is defined to transform as 4-vector and $u^0=c$, $u^i=0$ in a locally comoving system. This velocity vector is normalised such that $g_{\mu\nu}u^\mu u^\nu=-c^2$. The generalisation to general relativity would be $T^{\mu\nu}=(e+p)u^\mu u^\nu/c^2+g^{\mu\nu}p$.

For locally comoving systems:

$$\begin{aligned} T^{\mu\nu} &= (e+p)\delta^{\mu 0}\delta^{\nu 0} + g^{\mu\nu}p, \\ T^{\mu}_{\nu} &= T^{\mu\lambda}g_{\lambda\nu} = -(e+p)\delta^{\mu 0}\delta_{\nu 0} + \delta^{\mu}_{\nu}p \Rightarrow T^{\mu}_{\mu} = 3p - e, \\ T_{\mu\nu} &= T^{\lambda}_{\mu}g_{\lambda\nu} = (e+p)\delta_{\mu 0}\delta_{\nu 0} + g_{\mu\nu}p. \end{aligned}$$

Note units: $[T^{00}] = \mathrm{erg\,cm^{-3}}$, $[T^{ij}] = \mathrm{erg\,cm^{-5}}$, $[T^{0}_{0}] = [T^{i}_{j}] = \mathrm{erg\,cm^{-3}}$, $[T_{00}] = \mathrm{erg\,cm^{-3}}$, and $[T_{ij}] = \mathrm{erg\,cm^{-1}}$.

The conservation laws are:

$$T^{\mu\nu}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\kappa\nu}T^{\kappa\nu} + \Gamma^{\nu}_{\kappa\nu}T^{\mu\kappa} = 0.$$

For energy conservations:

$$\begin{aligned} 0 &= T^{0\mu}_{;\mu} = \frac{\partial T^{0\mu}}{\partial x^{\mu}} + \Gamma^{0}_{\mu\nu}T^{\mu\nu} + \Gamma^{\mu}_{\nu\mu}T^{0\nu} \\ &= \frac{1}{c}\frac{\partial T^{00}}{\partial t} + \Gamma^{0}_{ij}T^{ij} + \Gamma^{i}_{0i}T^{00} = \frac{1}{c}\dot{e} + \frac{R\dot{R}}{c}\tilde{g}_{ij}\frac{1}{R^2}\tilde{g}^{ij}p + \frac{3\dot{R}}{cR}e \\ &= \frac{1}{c}\dot{e} + \frac{3\dot{R}}{cR}(e+p) \Rightarrow \dot{e} + \frac{3\dot{R}}{R}(p+e) = 0. \end{aligned} \tag{1}$$

This can be written as:

$$\begin{aligned} \frac{de}{dR} &= -\frac{3}{R}(e+p) \Rightarrow \frac{de}{dR} + \frac{3e}{R} = \frac{1}{R^3}\frac{d}{dR}\left(R^3 e\right) = -\frac{3p}{R} \\ &\Rightarrow \frac{d}{dR}\left(\frac{4\pi}{3}R^3 e\right) = -4\pi p R^2, \end{aligned}$$

This is simply $dE = -PdV$, since $dS = 0$ for a diagonal $T^{\mu\nu}$. For momentum conservations:

$$\begin{aligned} 0 &= T^{i\mu}_{;\mu} = \frac{\partial T^{i\mu}}{\partial x^{\mu}} + \Gamma^{i}_{\mu\nu}T^{\mu\nu} + \Gamma^{\mu}_{\nu\mu}T^{i\nu} \\ &= \frac{\partial T^{ij}}{\partial x^{j}} + \Gamma^{i}_{jk}T^{jk} + \Gamma^{k}_{jk}T^{ij} \\ &= \frac{p}{R^2}\frac{\partial \tilde{g}^{ij}}{\partial x^{j}} + kr^{i}\tilde{g}_{jk}\frac{p}{R^2}\tilde{g}^{jk} + kr^{k}\tilde{g}_{jk}\frac{p}{R^2}\tilde{g}^{ij} \\ &= \frac{p}{R^2}\left[-k\left(\delta^{j}_{j}r^{i} + \delta^{i}_{j}r^{j}\right) + 3kr^{i} + kr^{k}\delta^{i}_{k}\right] = \frac{pk}{R^2}\left(-\delta^{i}_{j}r^{j} + \delta^{i}_{k}r^{k}\right) = 0, \end{aligned}$$

so no useful information here.

Equation (1) can be solved for $p = we$ (w is a constants):

$$\frac{de}{dt} + \frac{3\dot{R}}{R}(w+1)e = 0 \Rightarrow \frac{de}{dR} = -\frac{3(w+1)}{R}e \Rightarrow e \propto R^{-3-3w},$$

1. cold matter: $p = 0 \Rightarrow e \propto R^{-3}$

2. radiation: $p = e/3 \Rightarrow e \propto R^{-4}$

3. vacuum energy: $p = -e \Rightarrow e$ const.

As long as there is no interchange of energy between different components, these results hold separately for each of them.

8. The Einstein field equations

The Einstein field equations are

$$R_{\mu\nu} + \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} S_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}\right).$$

Note units: $[\lambda] = \mathrm{cm}^{-2}$. So we need $S_{\mu\nu}$:

$$\begin{aligned} S_{\mu\nu} &= (e+p)\delta_{\mu 0}\delta_{\nu 0} + g_{\mu\nu} p - \frac{1}{2} g_{\mu\nu}(3p - e) \\ &= (e+p)\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{2} g_{\mu\nu}(e - p), \end{aligned}$$

We find

$$\begin{aligned} S_{00} &= e + p - \frac{1}{2}(e-p) = \frac{1}{2}(e+3p), \\ S_{0i} &= S_{i0} = 0, \\ S_{ij} &= \frac{1}{2}(e-p) g_{ij} = \frac{1}{2}(e-p) R^2 \tilde{g}_{ij}. \end{aligned}$$

The 00 term of Einstein field equations is

$$\begin{aligned} \frac{3}{c^2}\frac{\ddot{R}}{R} - \lambda &= -\frac{8\pi G}{c^4}\frac{1}{2}(e+3p) \\ &\Rightarrow 3\frac{\ddot{R}}{R} - \lambda c^2 = -\frac{4\pi G}{c^2}(e+3p). \end{aligned}$$

The ij terms of Einstein field equations are

$$\begin{aligned} -\left(2k + \frac{R\ddot{R}}{c^2} + 2\frac{\dot{R}^2}{c^2}\right)\tilde{g}_{ij} + \lambda R^2 \tilde{g}_{ij} &= -\frac{8\pi G}{c^4}\frac{1}{2}(e-p) R^2 \tilde{g}_{ij} \\ &\Rightarrow 2kc^2 + R\ddot{R} + 2\dot{R}^2 - \lambda c^2 R^2 = \frac{4\pi G}{c^2}(e-p) R^2. \end{aligned}$$

We can interpret the λ term as having some energy density e_λ and pressure p_λ if

$$\begin{aligned} \frac{4\pi G}{c^2}(e_\lambda - p_\lambda) &= \lambda c^2, \\ \frac{4\pi G}{c^2}(e_\lambda + 3p_\lambda) &= -\lambda c^2. \end{aligned}$$

Multiplying by 3 the first equation and adding both equations:

$$\frac{4\pi G}{c^4} 4e_\lambda = 2\lambda \Rightarrow e_\lambda = \frac{\lambda c^4}{8\pi G}.$$

First equation minus the second equation:

$$\frac{4\pi G}{c^4}(-4p_\lambda) = 2\lambda \Rightarrow p_\lambda = -\frac{\lambda c^4}{8\pi G} = -e_\lambda.$$

Taking $\ddot{R}$ from the 00 component and substituting into the ij component:

$$\begin{aligned} 2kc^2 &+ R^2\left[\frac{\lambda c^2}{3} - \frac{4\pi G\,(e+3p)}{3c^2}\right] + 2\dot{R}^2 - \lambda c^2 R^2 = \frac{4\pi G}{c^2}(e-p)R^2 \\ &\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = -\frac{kc^2}{R^2} + \frac{1}{3}\lambda c^2 + \frac{2\pi G}{c^2}\frac{4e}{3} \\ &\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3c^2}\left(e + \frac{\lambda c^4}{8\pi G}\right) - \frac{kc^2}{R^2}. \end{aligned}$$

This is the Friedmann equation. It can be written as

$$H^2 = H_0^2\left(\frac{e}{\rho_c c^2} + \frac{\rho_\lambda}{\rho_c} - \frac{kc^2}{H_0^2 R^2}\right),$$

with

$$H = \frac{\dot{R}}{R},\ H_0 = \frac{\dot{R}_0}{R_0},\ \rho_\lambda = \frac{\lambda c^2}{8\pi G},$$

and

$$\rho_c = \frac{3H_0^2}{8\pi G} \approx 1.878\times 10^{-29} h^2 \frac{\mathrm{g}}{\mathrm{cm}^3},\ h = \frac{H_0}{100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}$$

is the critical density. We can compare this with the Newtonian approximation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\rho}{\rho_c} + \frac{\rho_\lambda}{\rho_c} + \left(1 - \frac{\rho_0}{\rho_c} - \frac{\rho_\lambda}{\rho_c}\right)a^{-2}\right].$$

We see that

1. all energy density contributes, $\rho \rightarrow e/c^2$.
2. a is R up to a scale factor, $a = R(t)/R(t_0)$, $H_0 = (\dot{R}/R)|_{t=t_0}$
3. $k/R_0^2 = -H_0^2(1 - \rho_0/\rho_c - \rho_\lambda/\rho_c)/c^2$, such that the current curvature radius is of the order c/H_0 or larger (unless $|\rho_0 + \rho_\lambda| \gg \rho_c$).

So the 16 Einstein field equations reduce to 2 equation for the FRW Universe:

$$\begin{aligned} H^2 &= H_0^2\left(\frac{e}{\rho_c c^2}+\frac{\rho_\lambda}{\rho_c}-\frac{kc^2}{H_0^2R^2}\right), \\ \dot{e}+3H(p+e) &= 0, \end{aligned}$$

which are the Friedmann equation and energy conservation, respectively. We have 2 equations for 3 variables e, p, R, so we need one more equation, which is the equation of state (EOS), $p(e, s)$. Since s = const., we have $p(e, s_0)$, where s_0 is a property of the Universe. We'll see later that $s_0 \sim \eta = n_\gamma/n_b$.

We define for the vacuum, matter, radiation, and curvature, respectively:

$$\begin{aligned} \Omega_\Lambda &= \Lambda = \frac{\rho_\lambda}{\rho_c} = \frac{8\pi G}{3H_0^2}\rho_\lambda = \frac{\lambda c^2}{3H_0^2}, \\ \Omega_M &= \frac{\rho_{M,0}}{\rho_c} = \frac{8\pi G}{3H_0^2}\rho_{M,0}, \\ \Omega_R &= \frac{\rho_{R,0}}{\rho_c} = \frac{8\pi G}{3H_0^2}\rho_{R,0}, \\ \Omega_K &= -\frac{kc^2}{H_0^2R_0^2}, \end{aligned}$$

to write the Friedmann equation as

$$H^2 = H_0^2\left[\Omega_M\left(\frac{R}{R_0}\right)^{-3}+\Omega_R\left(\frac{R}{R_0}\right)^{-4}+\Omega_\Lambda+\Omega_K\left(\frac{R}{R_0}\right)^{-2}\right],$$

and $\Omega_M+\Omega_R+\Omega_\Lambda+\Omega_K = 1$. We can find solutions for $k = 0$:

1. Matter dominated ($\Omega_M = 1$): $\dot{R} = H_0R(R/R_0)^{-3/2} \propto R^{-1/2} \Rightarrow R \propto t^{2/3}$.
2. Radiation dominated ($\Omega_R = 1$): $\dot{R} = H_0R(R/R_0)^{-2} \propto R^{-1} \Rightarrow R \propto t^{1/2}$.
3. Vacuum dominated ($\Omega_\Lambda = 1$): $\dot{R} = H_0R \Rightarrow R \propto \exp(H_0t) = \exp(Ht)$.

9. Currents and numbers

From isotropy, the mean value of any 3-vector v^i must vanish. From homogeneity, the mean value of any 3-scalar (i.e. invariant under spatial translations) can be only a function of time. Therefore, we have for currents

$$J^i = 0,\ J^0 = n(t),$$

where $n(t)$ is the number per proper volume in a comoving frame. From conservation:

$$0 = J^\mu_{;\mu} = \frac{\partial J^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\nu}J^\nu = \frac{\partial n}{\partial(ct)} + \Gamma^i_{i0}n = \frac{\dot{n}}{c} + \frac{3\dot{R}}{cR}n \Rightarrow \frac{\dot{n}}{n} = -\frac{3\dot{R}}{R} \Rightarrow n \propto R^{-3}. \quad (2)$$

10. Distances and redshift

10.1. Proper distance

The distance at time t from the origin to a co-moving object at a radial coordinate r:

$$d_{\rm prop}(r,t) = \int_0^r dr' \sqrt{g_{rr}(r',t)} = R(t)\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = R(t)rf(r),$$

where $f(r) = \sin^{-1}(r)/r$ for $k=1$, $f(r)=1$ for $k=0$, and $f(r)=\sinh^{-1}(r)/r$ for $k=-1$. We can never measure the proper distance. Also,

$$\dot{d}_{\rm prop} = \dot{R}(t)rf(r) = R(t)rf(r)\frac{\dot{R}}{R} = \left(\frac{\dot{R}}{R}\right)d_{\rm prop},$$

which can be larger than c.

10.2. Redshift

Light travels along $cd\tau = 0$. For a photon emitted at r in the $\hat{r}$ direction and received at $r=0$ at time t:

$$c^2d\tau^2 = c^2dt^2 - g_{rr}dr^2 \Rightarrow cdt = -R\frac{dr}{\sqrt{1-kr^2}},$$

where the minus sign is because r decreases as time increases (the photon is coming to us). The time $t_i(r,t)$ at which the photon was emitted is given by:

$$\int_{t_i(r,t)}^t \frac{cdt'}{R(t')} = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = rf(r). \tag{3}$$

Consider now 2 signals, emitted at t_i and than at $t_i+\delta t_i$ (both from r, which is time independent) and received at r at t, $t+\delta t$. Differentiate the last equation wrt t:

$$\frac{c}{R(t)} - \frac{c}{R(t_i)}\frac{\partial t_i}{\partial t} = 0 \Rightarrow \frac{\delta t}{\delta t_i} = \frac{R(t)}{R(t_i(r,t))} \equiv 1+z(r,t).$$

For frequencies: $\nu/\nu_i = R(t_i)/R(t)$, such that the wavelength is redshifted $\lambda \to \lambda(1+z)$. The analogy with Doppler ($v=cz$) only holds for $z \ll 1$. In particular, we care about the increase of $R(t)$ from emission to absorption, and not only on the rate of change of $R(t)$ at the time of emission or absorption. Also, $p = h\nu/c$ for a photon, such that $p \propto R^{-1}$.

Going back to Friedmann equation, we can write:

$$dt = \frac{dR}{H_0R\sqrt{\Omega_M\left(\frac{R}{R_0}\right)^{-3} + \Omega_R\left(\frac{R}{R_0}\right)^{-4} + \Omega_\Lambda + \Omega_K\left(\frac{R}{R_0}\right)^{-2}}}$$

$$dt = \frac{-dz}{H_0(1+z)\sqrt{\Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_\Lambda + \Omega_K(1+z)^2}},$$

where

$$\begin{aligned} x &\equiv a = \frac{R}{R_0} = \frac{1}{1+z}, \\ \frac{dR}{R} &= \frac{dx}{x} = -\frac{(1+z)dz}{(1+z)^2} = -\frac{dz}{1+z}. \end{aligned}$$

If we define the zero of time at infinite redshift, then the time at which light was emitted that reached us with a redshift z, is given by:

$$t(z) = \frac{1}{H_0} \int_0^{1/(1+z)} \frac{dx}{x\sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}.$$

For $z = 0$, the age of the Universe is

$$t_0 = \frac{1}{H_0} \int_0^{1} \frac{dx}{x\sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}.$$

10.3. Angular distance

Consider a sphere of a diameter D lying at r and 2 photons emitted from the sphere's edges reaching to us. They move on a fixed $\hat{\Omega}$, such that the size of the object is (assuming $D \ll rR$)

$$\begin{aligned} D &= rR[t_i(r,t)]d\theta \\ &\Rightarrow d_A(r) = R[t_i(r,t)]r = (1+z)^{-1}R(t)r. \end{aligned}$$

10.4. Luminosity distance

Consider a source with a luminosity L lying at r. Our detector of a diameter D occupies a solid angle of $\pi d\theta^2/4 = \pi[D/(2rR(t))]^2$ as seen by the source. Because the arrival time of photons is larger by the emission by $(1+z)$ and the energy of the photons is decreased by another $(1+z)$, the observed flux is:

$$f = \frac{\pi d\theta^2}{4 \times 4\pi} \frac{L}{\pi D^2/4} \frac{1}{(1+z)^2} = \frac{L}{4\pi r^2 R^2(t)(1+z)^2}.$$

Since

$$f = \frac{L}{4\pi d_L^2} \Rightarrow d_L = (1+z)R(t)r.$$

Note that

$$d_L = (1+z)d_{\rm prop}/f(r) = (1+z)^2 d_A.$$

From Equation (3) we get:

$$r = S\left[\int_{t_i(r,t)}^{t} \frac{cdt'}{R(t')}\right] = S\left[\int_{1/(1+z)}^{1} \frac{cdx}{R_0 H_0 x^2 \sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}\right],$$

where $S[y] = \sin y$ for $k = 1$, $S[y] = y$ for $k = 0$, and $S[y] = \sinh y$ for $k = -1$. This can be written as

$$R_0 r(z) = \frac{c}{H_0 \Omega_K^{1/2}} \sinh\left[\Omega_K^{1/2} \int_{1/(1+z)}^{1} \frac{dx}{x^2 \sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}\right],$$

for all possible k values, by noting that $\sinh(ix) = i \sin x$ and that $\sinh x \to x$ for $x \to 0$. We finally get

$$d_L(z) = (1+z) R_0 r(z) = \frac{(1+z)c}{H_0 \Omega_K^{1/2}} \sinh\left[\Omega_K^{1/2} \int_{1/(1+z)}^{1} \frac{dx}{x^2 \sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}\right].$$

It is instructive to expand $d_L(z)$ for small z. For small z we can expand around R_0:

$$\begin{aligned} \dot{R}(R) &= \dot{R}_0 + \frac{\ddot{R}_0}{\dot{R}_0} \Delta R + O\left(\Delta R^2\right) = R_0 H_0 \left[1 + \frac{R_0 \ddot{R}_0}{\dot{R}_0^2} \frac{\Delta R}{R_0} + O\left(\Delta R^2\right)\right] \\ &= R_0 H_0 \left[1 - q_0 \Delta a + O\left(\Delta a^2\right)\right], \end{aligned}$$

where

$$a = \frac{R}{R_0} = \frac{1}{1+z},\ q_0 \equiv -\frac{R_0 \ddot{R}_0}{\dot{R}_0^2},\ \Delta a = a - 1.$$

Note that $\dot{a} = H_0[1 - q_0 \Delta a + O(\Delta a^2)]$. We now expand both sides of equation (3):

$$\begin{aligned} r f(r) &= r + \frac{1}{6} k r^3 + O(r^5), \\ \int_{t_i(r,t)}^{t} \frac{cdt'}{R(t')} &= \frac{1}{R_0} \int_{a(r)}^{1} \frac{c}{a} \frac{da}{\dot{a}} = \frac{c}{R_0} \int_{\Delta a(r)}^{0} \frac{dx}{(1+x) H_0 \left[1 - q_0 x + O\left(x^2\right)\right]}, \end{aligned}$$

where $x = a - 1 = \Delta a$. We can further evaluate the last expression as

$$\begin{aligned} &\frac{c}{R_0 H_0} \int_{\Delta a(r)}^{0} \frac{dx}{\left[1 + (1 - q_0) x + O\left(x^2\right)\right]} = -\frac{c}{R_0 H_0} \int_{0}^{\Delta a(r)} \left[1 + (q_0 - 1) x + O(x^2)\right] dx \\ &= -\frac{c}{R_0 H_0} \left[\Delta a + \frac{1}{2}(q_0 - 1) \Delta a^2 + O\left(\Delta a^3\right)\right]. \end{aligned}$$

So up to second order in Δa we have

$$\begin{aligned} r &= \frac{c}{R_0 H_0} \left[-\Delta a - \frac{1}{2}(q_0 - 1) \Delta a^2 + O\left(\Delta a^3\right)\right] \\ &\Rightarrow d_L(z) = (1+z) R_0 r = \frac{c(1+z)}{H_0} \left[-\Delta a - \frac{1}{2}(q_0 - 1) \Delta a^2 + O\left(\Delta a^3\right)\right] \\ &\Rightarrow d_L(z) = \frac{c}{H_0} \left[z + \frac{1}{2}(1 - q_0) z^2 + O\left(z^3\right)\right]. \end{aligned} \quad (4)$$

11. The deceleration parameter q_0

Writing the 00 component of the Einstein equation for t_0 we get

$$\begin{aligned} & \frac{\ddot{R}_0}{\dot{R}_0^2} R_0 \frac{\dot{R}_0^2}{R_0^2} = \frac{\lambda c^2}{3} - \frac{4\pi G}{3c^2}\left(e_0 + 3p_0\right) \\ \Rightarrow \quad & q_0 = -\frac{\lambda c^2}{3H_0^2} + \frac{4\pi G}{3H_0^2 c^2}\left(e_0 + 3p_0\right) = \frac{4\pi G}{3H_0^2 c^2}\left(e_0 + 3p_0 + 2p_\lambda\right) \\ = \quad & \frac{e_0 + 3p_0 + 2p_\lambda}{2\rho_c c^2} = \frac{1}{2}\left(\Omega_M + 2\Omega_R - 2\Omega_\Lambda\right). \end{aligned}$$

We see that if matter (radiation, vacuum) dominates today than k is determined by whether q_0 is larger or smaller than 1/2 (1, -1). For the justified approximation $\Omega_R = 0$ we can write

$$q_0 = \frac{3}{4}\left(\Omega_M - \Omega_\Lambda\right) - \frac{1}{4}\left(\Omega_M + \Omega_\Lambda\right).$$

Some observation are more sensitive to $\Omega_M + \Omega_\Lambda$ (like CMB) and some are more sensitive to $\Omega_M - \Omega_\Lambda$ (like type Ia SNe).

12. The fate of the Universe

The expansion can stop is there is a real root to the cubic equation (it is safe to ignore radiation here):

$$\Omega_\Lambda u^3 + \Omega_K u + \Omega_M = 0, \; u = \frac{R}{R_0} > 1.$$

for $u = 1$, we have $\Omega_\Lambda + \Omega_K + \Omega_M = 1$. For $\Omega_\Lambda < 0$, the expression will become negative for large u, meaning that the expansion will stop for some u. Even for $\Omega_\Lambda \geq 0$ it is possible to stop, if $\Omega_K = 1 - \Omega_\Lambda - \Omega_M$ is sufficiently negative (which requires $k = 1$).

13. Massive particles

A massive particle leaves $r = 0$ at t with v. In the local inertial frame around $\{t, r = 0\}$ there is no acceleration ($\Gamma = 0$), such that the particle reaches Δx at $\Delta t = \Delta x / v$ with a constant v (to a first order in Δt). A comoving observer at $\{\Delta t, \Delta x\}$ has a velocity $\Delta x \dot{R}(t)/R(t) = H\Delta x$ (to first order in Δx) and therefore measures the particle velocity:

$$v' = \frac{v - H\Delta x}{1 - v\frac{H\Delta x}{c^2}} \Rightarrow \Delta v = v' - v = \frac{\left(\frac{v^2}{c^2} - 1\right) H\Delta x}{1 - v\frac{H\Delta x}{c^2}} = -\frac{\gamma^{-2} H\Delta x}{1 - v\frac{H\Delta x}{c^2}}.$$

The particle momentum $p = \gamma\beta mc$ is changed by

$$\begin{aligned}
\Delta p &= mc\Delta(\gamma\beta) = mc\Delta\left(\sqrt{\gamma^2 - 1}\right) = mc\beta^{-1}\Delta\gamma = mc\gamma^3\Delta\beta \\
&\Rightarrow \frac{\Delta p}{p} = \gamma^2\frac{\Delta\beta}{\beta} = -\frac{H\Delta t}{1 - \beta^2 H\Delta t} \\
&\Rightarrow \frac{\Delta p}{p} \approx -\frac{\Delta R}{R} \Rightarrow p \propto R^{-1}. \qquad (5)
\end{aligned}$$

For relativistic particles $E = pc \Rightarrow E \propto R^{-1}$. For non-relativistic particles, the kinetic energy is $E = p^2/2m \Rightarrow E \propto R^{-2}$.

Let's derive this result in a different way. Consider the expression

$$p = m_0\sqrt{g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}, \qquad (6)$$

where $c^2 d\tau^2 = c^2 dt^2 - g_{ij}dx^i dx^j$. In a local inertial Cartesian coordinate system $g_{ij} = \delta_{ij}$, such that

$$\begin{aligned}
cd\tau &= cdt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2} \Rightarrow d\tau = dt\sqrt{1-\beta^2} \\
&\Rightarrow p = m_0\frac{d\vec{x}}{d\tau} = m_0\vec{v}\frac{1}{\sqrt{1-\beta^2}} = \gamma\beta m_0 c,
\end{aligned}$$

which is the momentum. Since Equation (6) is invariant under changes in spatial coordinates we can evaluate it for comoving FRW coordinates. Let's look on particle position near the origin $x^i = 0$ with $\tilde{g}_{ij} = \delta ij + O(\vec{x}^2)$. In this case we can ignore the spatial Γ^i_{jk} so the geodesic equation is

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\frac{2\dot{R}}{cR}\frac{dx^i}{d\tau}\frac{d(ct)}{d\tau}.$$

Multiplying by $d\tau/dt$ we get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{dx^i}{d\tau}\right) &= -\frac{2\dot{R}}{R}\frac{dx^i}{d\tau} \Rightarrow \frac{dx^i}{d\tau} \propto \frac{1}{R^2} \\
\Rightarrow\ p &\propto \sqrt{R^2(t)\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} \propto \sqrt{R^2\frac{1}{R^4}} \propto R^{-1}.
\end{aligned}$$

This hold for any nonzero mass, so also in the limit of massless particle, $m_0 \to 0$, $d\tau \to 0$, we still have $p \propto R^{-1}$.

14. The current status

From CMB (mostly Planck 2015) $\Omega_K = -0.005^{+0.016}_{-0.017}$. Adding BAO improves this to $\Omega_K = 0.000 \pm 0.005$. Assuming a flat Universe, CMB constrains $\Omega_M = 0.315 \pm 0.013$, $\Omega_\Lambda = 0.685 \pm 0.013$,

$\Omega_b h^2 = 0.02222 \pm 0.00023$, $H_0 = 67.31 \pm 0.96\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $t_0 = 13.813 \pm 0.038\,\mathrm{Gyr}$, $z_{eq} = 3393 \pm 49$. For $p_\lambda = w e_\lambda$, CMB constrains $w = -1.54^{+0.62}_{-0.50}$, and adding BAO and Type Ia SNe the constraint improves $w = -1.019^{+0.075}_{-0.080}$. These values indicate that $q_0 \approx -0.55 \Rightarrow \ddot{R} > 0$ such that the expansion of the Universe is accalerating.

Note that the value of Ω_Λ is extremely small. From zero point energy fluctuations of some field of mass m up to some cutoff energy $\Lambda \gg m$, the vacuum energy density is $\sim \Lambda^4/\hbar^3 c^5$. For $\Lambda = 300\,\mathrm{GeV}$ we get $\sim 10^{27}\,\mathrm{g/cm^3}$ which is some 56 orders of magnitudes larger than $\rho_\lambda \sim \rho_c \sim 10^{-29}\,\mathrm{g/cm^3}$. For the Planck energy scale, $\Lambda \sim \sqrt{\hbar c^5/G} \sim 10^{19}\,\mathrm{GeV}$, the situation is much worse.

Distances and the age of the Universe as a function of z for $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$ and $h = 0.67$ are plotted in Figure 1.

15. The age of the Universe for different Cosmologies

Before the exact measurements of the CMB, it was reasonable to consider the following 3 cosmologies:

1. $\Omega_M = 0.3$, $\Omega_K = 0.7$: This roughly corresponds to the directly observed mass density in the Universe, ($\Omega_M \lesssim 0.3$), with nothing else.
2. $\Omega_M = 1$: This could be the case if we are missing some mass from direct observations, and we have a strong prior for a flat Universe.
3. $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$: Here we trust the directly observed mass estimation and we are having a strong prior for a flat Universe.

The age of the Universe can be calculated for the above possibilities (and others possibilities). For $\Omega_K = 0$ we get

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x\sqrt{\Omega_M x^{-3} + \Omega_\Lambda}} = \frac{2}{3H_0}\frac{\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}}\right)}{\sqrt{\Omega_\Lambda}}.$$

In the limit $\Omega_M \to 1$ we get $t_0 H_0 = 2/3$ and in the limit $\Omega_M \to 0$ we get $t_0 \to \infty$. For $\Omega_M = 1$ we get $t_0 \approx 9.68\,\mathrm{Gyr}$, which is too short compared with stellar ages of $\sim 13\,\mathrm{Gyr}$, although the discrepancy is not huge and maybe can be resolved with systematic errors in H_0 or in the stellar ages estimation. For $\Omega_M = 0.3$ we find $t_0 \approx 14\,\mathrm{Gyr}$, which agrees with stellar ages. For $\Omega_M + \Omega_K = 1$ we find

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x\sqrt{\Omega_M x^{-3} + \Omega_K x^{-2}}} = \frac{1}{H_0}\left[\frac{1}{\Omega_K} - \frac{\Omega_M \log\left(\frac{2+2\sqrt{\Omega_K}-\Omega_M}{\Omega_M}\right)}{2\Omega_K^{3/2}}\right].$$

In the limit $\Omega_M \to 1$ we get $t_0 H_0 = 2/3$ and in the limit $\Omega_M \to 0$ we get $t_0 H_0 = 1$. For $\Omega_M = 0$ we get $t_0 \approx 14.5\,\mathrm{Gyr}$, which agrees with stellar ages. For $\Omega_M = 0.3$ we find $t_0 \approx 0.81/H_0 \approx 12\,\mathrm{Gyr}$,

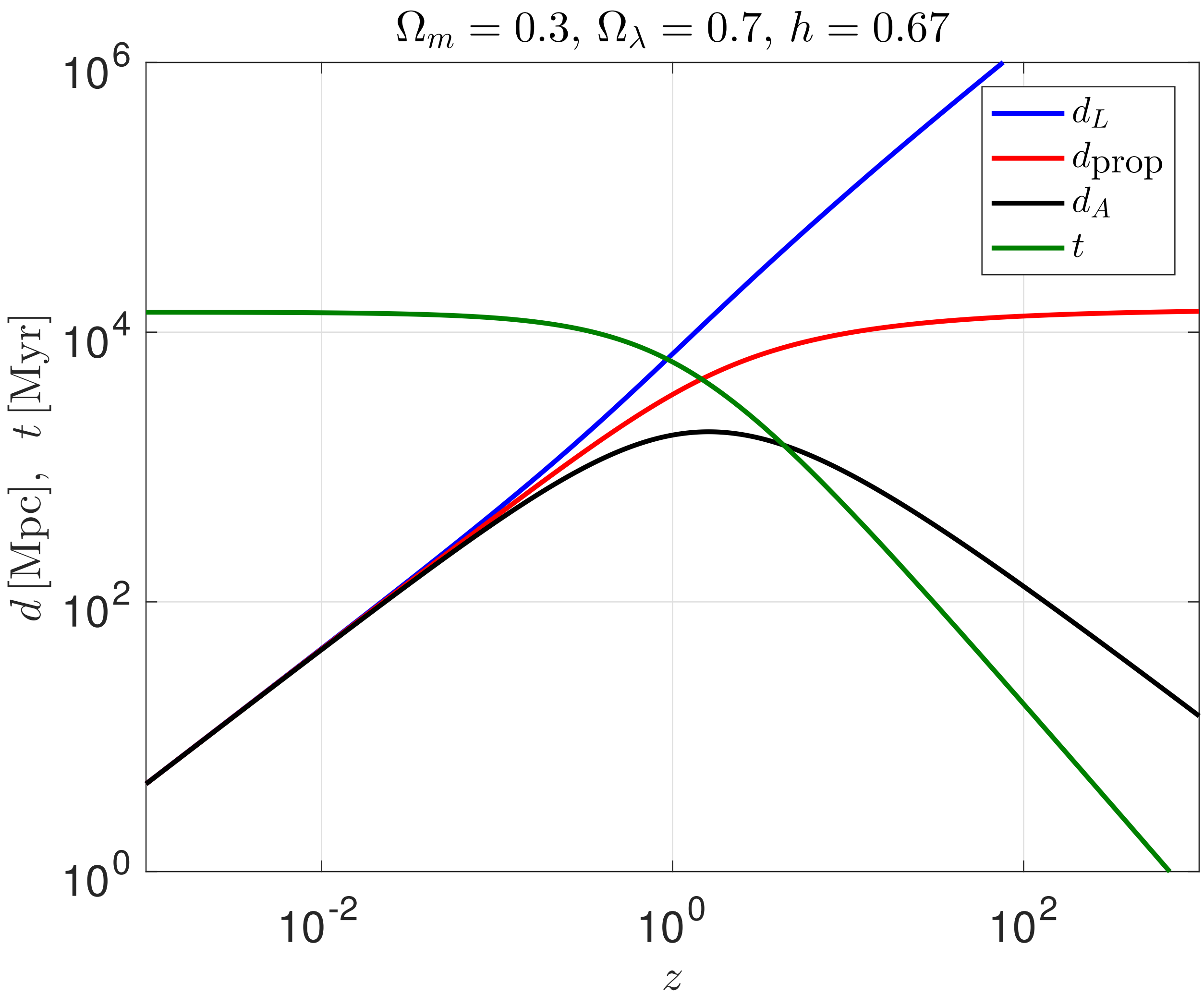

Fig. 1.— d_L (blue), $d_{\rm prop}$ (red), d_A (black) and time (green) as a function of the redshift for $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$ and $h = 0.67$.

which also roughly agrees with stellar ages. This means that $\Omega_\Lambda = 0$ with a small $\Omega_M \gtrsim 0$ is not contradicted by stellar ages.

For $\Omega_K = 0$ we have the general expression

$$\begin{aligned} t(z) &= \frac{1}{H_0}\int_0^{1/(1+z)=a} \frac{dx}{x\sqrt{\Omega_M x^{-3} + \Omega_\Lambda}} = \frac{2}{3H_0}\frac{\sinh^{-1}\left(\sqrt{\frac{a^3\Omega_\Lambda}{\Omega_M}}\right)}{\sqrt{\Omega_\Lambda}}. \\ &\Rightarrow\ a = \left[\sqrt{\frac{\Omega_M}{\Omega_\Lambda}}\sinh\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}t\right)\right]^{2/3}. \end{aligned}$$

16. Luminosity distance for different Cosmologies

For $\Omega_M = 1$ we get

$$d_L(z) = \frac{(1+z)c}{H_0}\int_{1/(1+z)}^1 \frac{dx}{\sqrt{x}} = \frac{(1+z)c}{H_0}2\left[1-(1+z)^{-1/2}\right] = \frac{2c}{H_0}\left(1+z-\sqrt{1+z}\right).$$

For $\Omega_\Lambda = 1$ we get

$$d_L(z) = \frac{(1+z)c}{H_0}\int_{1/(1+z)}^1 \frac{dx}{x^2} = \frac{c}{H_0}\left[(1+z)^2 - 1 - z\right] = \frac{c}{H_0}\left(z^2+z\right),$$

for which the second order approximation, Equation (4), is accurate. For $\Omega_\Lambda = 0$ ($k = -1$) we have

$$\begin{aligned} d_L(z) &= \frac{(1+z)c}{H_0\Omega_K^{1/2}}\sinh\left(\Omega_K^{1/2}\int_{1/(1+z)}^1 \frac{dx}{x^2\sqrt{\Omega_M x^{-3} + \Omega_K x^{-2}}}\right) \\ &= \frac{(1+z)c}{H_0\Omega_K^{1/2}}\sinh\left[2\sinh^{-1}\left(\sqrt{\frac{\Omega_K}{\Omega_M}}\right) - 2\sinh^{-1}\left(\sqrt{\frac{1}{1+z}\frac{\Omega_K}{\Omega_M}}\right)\right]. \end{aligned}$$

For $\Omega_K = 1$ ($k = -1$) we have

$$\begin{aligned} d_L(z) &= \frac{(1+z)c}{H_0}\sinh\left(\int_{1/(1+z)}^1 \frac{dx}{x}\right) \\ &= \frac{(1+z)c}{H_0}\sinh\left[\log(1+z)\right] = \frac{c}{2H_0}\left(z^2+2z\right), \end{aligned}$$

for which the second order approximation, Equation (4), is accurate. The case $\Omega_M + \Omega_\Lambda = 1$ has to be calculated numerically in general. These models are compared in Figure 2.

At $z = 1$ the 3 cosmologies from Section 15 give $d_L(z = 1) = c/H_0(1.37, 1.17, 1.54)$, respectively. So there is a $\sim 10\%$ effect at $z = 1$. The second order approximation, Equation (4), gives $d_L(z = 1) \approx c/H_0(1.42, 1.25, 1.77)$, so it is not accurate enough for $z = 1$. The second order approximation is useful up to $z \sim 0.5$ (for the cases in which Equation (4) is not exact).

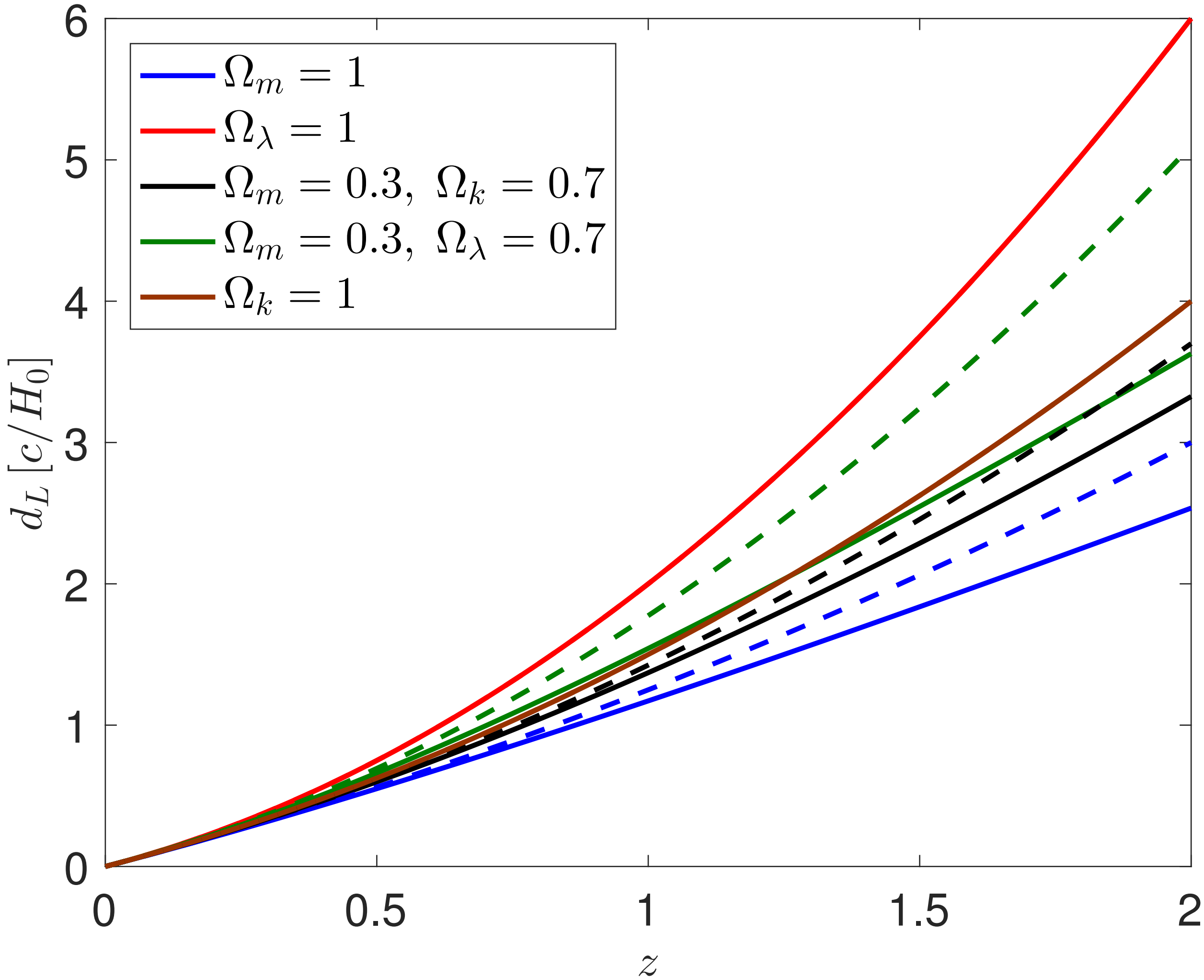

Fig. 2.— $d_L(z)$ for different cosmologies. Solid lines are the exact integrals, and dashed lines are the second order approximation, Equation (4).

The $\sim$ 10% difference at $z = 1$ can be measured with Type Ia SNe, since their intrinsic luminosity can be calibrated to this accuracy by using the Phillips relation. Note that without the Phillips relation the scatter in the peak luminosity is a factor of a few, meaning that type Ia SNe are not standard candles (so it is hard to imagine that all are explosions of the same star, as in the popular Chandrasekhar model). The measurement with Type Ia SNe established that $\Omega_\Lambda > 0$ and that the expansion of the Universe is accelerating.

17. Horizons

17.1. Particle horizon

Particle horizon is the limit on distances at which past events can be observed. Observer at a time t is able to receive signals only from $r < r_{\rm max}(t)$:

$$\int_0^t \frac{cdt'}{R(t')} = \int_0^{r_{\rm max}(t)} \frac{dr'}{\sqrt{1-kr'^2}} = r_{\rm max} f(r_{\rm max}).$$

There is an horizon if the lhs converges. For radiation dominated Universe we get $R(t) \propto t^{1/2}$ and the lhs converges. The proper distance of the horizon is:

$$d_{\rm max}(t) = R(t) r_{\rm max} f(r_{\rm max}) = R(t) \int_0^t \frac{cdt'}{R(t')}.$$

For radiation dominated Universe we have $R(t) = At^{1/2} \Rightarrow d_{\rm max}(t) = ct^{1/2} 2t^{1/2} = 2ct = c/H$, since $H = \dot{R}/R = 1/(2t)$. For matter dominated Universe, $R \propto t^{2/3}$, we have $d_{\rm max}(t) = ct^{2/3} 3t^{1/3} = 3ct = 2c/H$, since $H = 2/(3t)$. In general, the particle horizon today is given by

$$d_{\rm max}(t) = \frac{c}{H_0} \int_0^1 \frac{dx}{x^2\sqrt{\Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_\Lambda + \Omega_K x^{-2}}}.$$

If Ω_Λ dominates in the early Universe then $d_{\rm max}(t_0) \to \infty$.

17.2. Event horizon

Event horizon is the limit on distances at which it will ever be possible to observe future events. Observer at a time t will be able to observe events only for $r < r_{\rm max}(t)$:

$$\int_t^\infty \frac{cdt'}{R(t')} = \int_0^{r_{\rm max}(t)} \frac{dr'}{\sqrt{1-kr'^2}} = r_{\rm max} f(r_{\rm max}).$$

The proper distance to the event horizon is given by:

$$d_{\rm max}(t) = R(t) r_{\rm max} f(r_{\rm max}) = R(t) \int_t^\infty \frac{cdt'}{R(t')}.$$

If $\Omega_\Lambda = 0$ then $R(t) \propto t^{2/3}$ and the integral diverges, so there is no event horizon. With $\Omega_\Lambda > 0$, eventually $H = H_0\Omega_\Lambda^{1/2} \Rightarrow R(t) \propto \exp(Ht) \Rightarrow d_{\rm max}(t) = c\exp(Ht)\int_t^\infty \exp(-Ht) = c/H$, which is ≈ 5.2 Gpc for the current Universe. Every object which is further from this will become unobservable in the future (since signals from there are redshifted indefinitely, we never 'see' they are lost). This is also the largest distance we will ever be able to travel.