The thermal history of the early Universe

Doron Kushnir

I'm following Steven Weinberg's Cosmology Yossi Nir's notes.

1. Age-temperature relation

From Einstein equations

$$\begin{aligned}\frac{\dot{R}^2}{R^2} &= \frac{8\pi Ge}{3c^2} \\ &\Rightarrow\ dt = \frac{dR}{R\sqrt{\frac{8\pi Ge}{3c^2}}} \\ &\Rightarrow\ \frac{dt}{dT} = \frac{1}{R}\frac{dR}{dT}\left(\frac{8\pi Ge}{3c^2}\right)^{-1/2}.\end{aligned}$$

We showed that sR^3 is conserved for $|\mu| \ll T$, so

$$\begin{aligned}&\frac{ds}{dT}R^3 + 3sR^2\frac{dR}{dT} = 0 \\ \Rightarrow\ &\frac{1}{R}\frac{dR}{dT} = -\frac{1}{3s}\frac{ds}{dT} \\ \Rightarrow\ &\frac{dt}{dT} = -\frac{s'}{3s}\left(\frac{8\pi Ge}{3c^2}\right)^{-1/2} \\ \Rightarrow\ &t = -\int\frac{s'dT}{s\sqrt{\frac{24\pi Ge}{c^2}}} + \text{const.}\end{aligned} \tag{1}$$

for UR particles, all with $T_i = T$ we have

$$s = \frac{4}{3}g\frac{\bar{a}_B}{2}T^3, \quad e = g\frac{\bar{a}_B}{2}T^4,$$

so

$$\begin{aligned}t &= -\int\frac{3dT}{T}\left(\frac{12\pi Gg\bar{a}_BT^4}{c^2}\right)^{-1/2} + \text{const.} \\ &= -3\sqrt{\frac{c^2}{12\pi Gg\bar{a}_B}}\int\frac{dT}{T^3} + \text{const.} \\ &= \frac{3}{2}\sqrt{\frac{c^2}{12\pi Gg\bar{a}_B}}\frac{1}{T'^2}\Bigg|_{T^*}^{T} + \text{const} \\ &= \sqrt{\frac{3c^2}{16\pi Gg\bar{a}_B}}\left(\frac{1}{T^2} - \frac{1}{T^{*2}}\right) + t(T^*).\end{aligned} \tag{2}$$

T^* is chosen such that g is a constant (does not depends on temperature, i.e. same species in the Universe) for $T < T^*$. Equation (2) holds as long as g does not change. We can usually use Equation (2) for $T \ll T^*$, and then

$$t \approx \sqrt{\frac{3c^2}{16\pi G g \bar{a}_B} \frac{1}{T^2}}.$$

Note that this is just the dynamical time $\sim 1/\sqrt{G\rho}$, where $\rho c^2 = \bar{a}_B T^4$.

2. $T \lesssim 10^{11}$ K

We consider $1.2 \times 10^{12}\,\mathrm{K} \approx m_\mu c^2 \gg T \gg m_e c^2 \approx 6 \times 10^9\,\mathrm{K}$. This temperature range is too cold for reaction like $\nu_\mu + e \leftrightarrow \mu + \nu_e$ and $\nu_\tau + e \leftrightarrow \tau + \nu_e$, but ν_τ, $\bar{\nu}_\tau$, ν_μ, $\bar{\nu}_\mu$ are in TE through neutral current reactions like $e^- + e^+ \leftrightarrow \nu + \bar{\nu}$. So we have γ ($g_\gamma = 2$), 3ν ($g_\nu = 1$, since they are only left-handed), $3\bar{\nu}$ ($g_{\bar{\nu}} = 1$, since they are only right-handed), e^- ($g_{e^-} = 2$) and e^+ ($g_{e^+} = 2$) (baryons are neglected here). All are in TE and UR: $g = 2 + 7(6+4)/8 = 43/4$, so

$$t \approx 0.994 \left(\frac{T}{10^{10}\,\mathrm{K}}\right)^{-2} \mathrm{s}. \tag{3}$$

The weak interaction corss-section for $\nu - e$ scattering is

$$\sigma_{wk} \approx \left(c\hbar G_{wk} T\right)^2,$$

where $G_{wk} \approx 1.16637 \times 10^{-5}\,\mathrm{GeV}^{-2}$, and since $n_e \approx (T/\hbar c)^3$, the collision rate of ν with e^- or with e^+ is

$$\Gamma_\nu = n_e \sigma_{wk} c \approx \frac{T^3}{\hbar^3 c^3} c \left(c\hbar G_{wk} T\right)^2 = \frac{G_{wk}^2 T^5}{\hbar}.$$

This can be compared to $H \approx \sqrt{GT^4/\hbar^3 c^5}$:

$$\frac{\Gamma_\nu}{H} \approx \frac{G_{wk}^2 T^5}{\hbar \sqrt{G} T^2} \hbar^{3/2} c^{5/2} = G_{wk}^2 \sqrt{\frac{\hbar c^5}{G}} T^3 \approx \left(\frac{T}{10^{10}\,\mathrm{K}}\right)^3 \frac{10^{-9} \times 10^{19}}{10^{10}},$$

where we have used the Planck mass $M_{pl} c^2 = \sqrt{\hbar c^5/G} \approx 10^{19}\,\mathrm{GeV}$ and that $10^{10}\,\mathrm{K} \approx \mathrm{MeV}$. We see that $\Gamma_\nu \approx H$ for $T \approx 10^{10}\,\mathrm{K}$, which is just a little bit greater than $m_e c^2$, so for lower T the $e^\pm$ pairs disappear from equilibrium, and $\Gamma_\nu / H \ll 1$ (also because this ratio $\propto T^3$). Then the neutrinos begin a free expansion with $T_\nu \propto R^{-1}$.

At lower T we must consider the finite mass of $e^\pm$, so the temperature of $e^\pm$, γ (which are still in TE) no longer falls as $1/R$. The neutrinos, however, preserve a Fermi-Dirac distribution with $T_\nu \propto R^{-1}$, so there is a difference between T and T_ν (actually the decoupling of the neutrinos is not instantaneous, so the Fermi-Dirac distribution is slightly modified, parameterised as an increase of the effective number of ν species, from 3 to ≈ 3.04). The entropy density of $e^{\pm}$, γ is given by

$$\begin{aligned} s(T) &= \frac{4}{3}\bar{a}_B T^3 + \frac{g_{e^-} + g_{e^+}}{T}\int_0^\infty \frac{4\pi}{h^3}\frac{p^2 dp}{\exp(\beta\varepsilon)+1}\left(\sqrt{c^2p^2+m_e^2c^4} + \frac{c^2p^2}{3\sqrt{c^2p^2+m_e^2c^4}}\right) \\ &= \frac{4}{3}\bar{a}_B T^3 \mathcal{S}\left(\frac{m_e c^2}{T}\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{S}(x) &= 1 + \frac{12\pi}{h^3T^4\bar{a}_B}\int_0^\infty \frac{p^2 dp}{\exp\left(\sqrt{\frac{c^2p^2}{T^2}+\frac{m_e^2c^4}{T^2}}\right)+1}\left(T\sqrt{\frac{c^2p^2}{T^2}+\frac{m_e^2c^4}{T^2}} + \frac{T^2\frac{c^2p^2}{T^2}}{3T\sqrt{\frac{c^2p^2}{T^2}+\frac{m_e^2c^4}{T^2}}}\right) \\ &= 1 + \frac{12\pi}{c^3h^3\bar{a}_B}\int_0^\infty \frac{y^2 dy}{\exp\left(\sqrt{y^2+x^2}\right)+1}\left(\sqrt{y^2+x^2} + \frac{y^2}{3\sqrt{y^2+x^2}}\right) \\ &= 1 + \frac{45}{2\pi^4}\int_0^\infty \frac{y^2 dy}{\exp\left(\sqrt{y^2+x^2}\right)+1}\left(\sqrt{y^2+x^2} + \frac{y^2}{3\sqrt{y^2+x^2}}\right), \end{aligned}$$

where we changed the variable of integration to $y = cp/T$. From entropy conservation $R^3T^3\mathcal{S}(m_ec^2/T) =$ const., and since $T_\nu \propto 1/R$, we have $T_\nu \propto T\mathcal{S}^{1/3}(m_ec^2/T)$. We have $T = T_\nu$ for $T \gg m_ec^2$, and $\mathcal{S}(0) = 1 + 2 \times 7/8 = 11/4$ so

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T\mathcal{S}^{1/3}\left(\frac{m_ec^2}{T}\right).$$

We get that $T/T_\nu = 1$ for $T > 10^{11}\,\mathrm{K}$ and then increases as T decreases. Since $\mathcal{S}(\infty) = 1$, we find for $T \ll m_ec^2$ that $T/T_\nu \to (11/4)^{1/3} \approx 1.401$, so today there is a relic neutrino background with $T_\nu = (4/11)^{1/3}T_{\mathrm{CMB}} \approx 1.945\,\mathrm{K}$ (so far not detected). The ratio T/T_ν is shown in Figure 1. Note that the factor 11/4 is simply the ratio of g for $e^{\pm}$, γ (11/2) to g for just γ (2).

The total energy density in this period (from $\bar{\nu}$, ν, $e^{\pm}$, γ) is given by:

$$\begin{aligned} e(T) &= 6\frac{7}{8}\frac{\bar{a}_B T_\nu^4}{2} + \bar{a}_B T^4 + \frac{16\pi}{h^3}\int_0^\infty \frac{\varepsilon p^2 dp}{\exp(\beta\varepsilon)+1} \\ &= \bar{a}_B T^4 \mathcal{E}\left(\frac{m_ec^2}{T}\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}(x) &= 1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\mathcal{S}^{4/3}(x) + \frac{16\pi}{h^3c^3\bar{a}_B}\int_0^\infty \frac{y^2 dy}{\exp\left(\sqrt{y^2+x^2}\right)+1}\sqrt{y^2+x^2} \\ &= 1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\mathcal{S}^{4/3}(x) + \frac{30}{\pi^4}\int_0^\infty \frac{y^2 dy}{\exp\left(\sqrt{y^2+x^2}\right)+1}\sqrt{y^2+x^2}. \end{aligned}$$

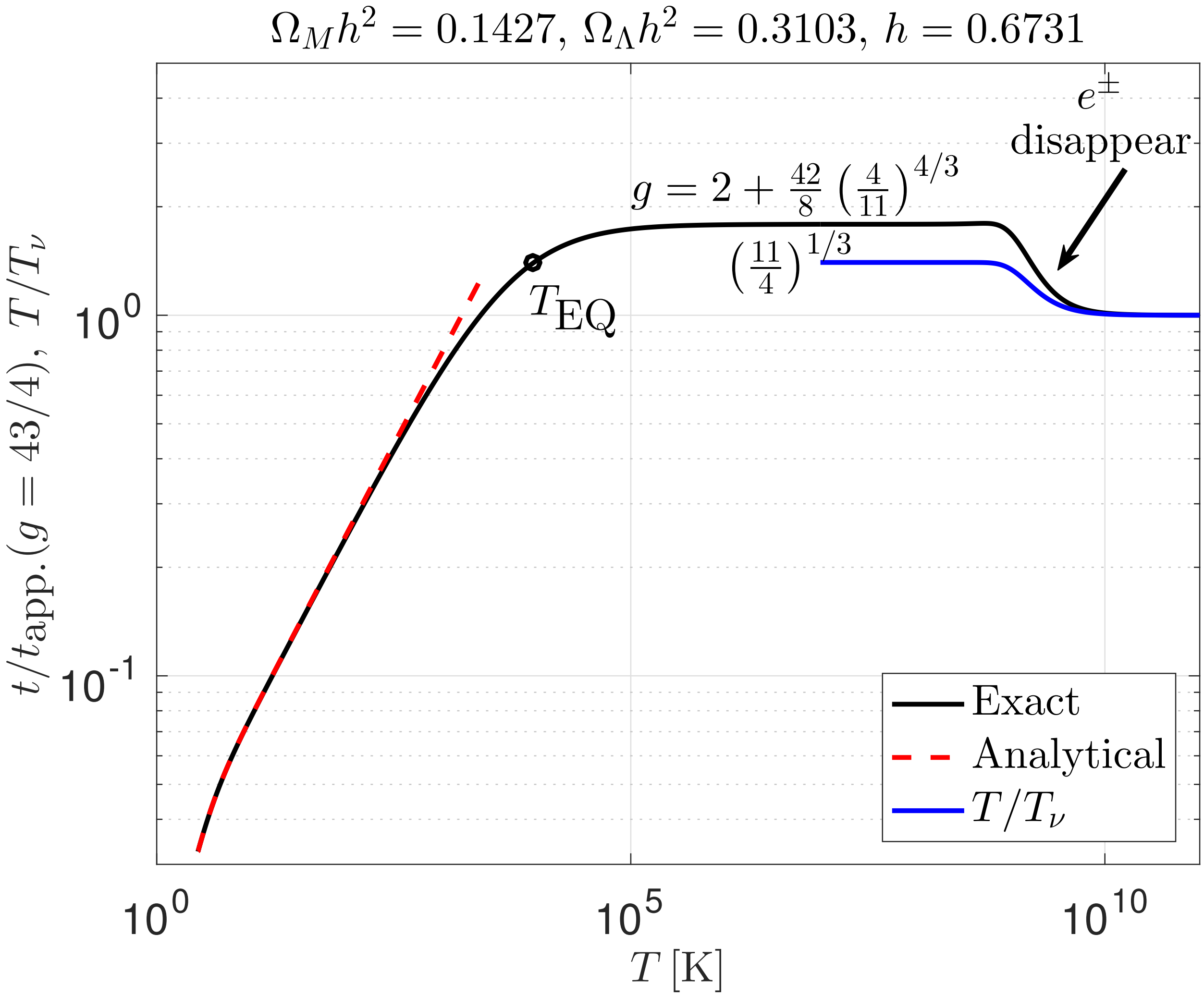

Fig. 1.— T/T_ν (blue), and the time (black) normalized to Equation (3) (which provides the time for $g = 43/4$, relevant for $T \lesssim 10^{11}$ K with $g = 43/3$). The time was calculated for $\Omega_M h^2 = 0.1427$, $\Omega_\Lambda h^2 = 0.3103$, $h = 0.6731$. Dashed red is the closed analytical formula for the time when the radiation is ignored.

Using the expression for $s(T)$ and $e(T)$ in Equation (1) we find (note that $s'(T) = 3s/T - (m_ec^2/T^2)s\mathcal{S}'/\mathcal{S}$):

$$\begin{aligned} t &= \int \left(\frac{\mathcal{S}'}{\mathcal{S}}\frac{m_ec^2}{T} - 3\right) \frac{dT}{T\sqrt{\frac{24\pi G\bar{a}_B T^4 \mathcal{E}}{c^2}}} + \text{const.} \\ &= \int \left(3 - \frac{\mathcal{S}'x}{\mathcal{S}}\right) \frac{xdx}{m_e^2c^4\sqrt{\frac{24\pi G\bar{a}_B \mathcal{E}}{c^2}}} + \text{const.} \\ &= t_e \int \left(3 - \frac{\mathcal{S}'x}{\mathcal{S}}\right) \mathcal{E}^{-1/2} xdx + \text{const.}, \end{aligned}$$

where we changed the variable of integration to $x = m_ec^2/T$ and $t_e = (24\pi G\bar{a}_B m_e^4 c^6)^{-1/2} \approx 4.3694\,\text{s}$. The result of this integration is shown in Figure 1.

Following $e^{\pm}$ annihilation, $e(T)$ is dominated by $\bar{\nu}$, ν, γ, all UR, so $s(T) \propto T^3$, and

$$e(T) = \bar{a}_B T^4 + \frac{7}{8}3\bar{a}_B T_\nu^4 = \bar{a}_B T^4 \left[1 + \frac{7}{8}3\left(\frac{4}{11}\right)^{4/3}\right] \approx \frac{3.363}{2}\bar{a}_B T^4,$$

so $g \approx 3.363$ and

$$t \approx 1.78 \left(\frac{T}{10^{10}\,\text{K}}\right)^{-2} \text{s} + \text{const.} \tag{4}$$

This holds until $T \sim 10^6\,\text{K}$, where we need to take Ω_M into account (Figure 1).

3. BBN

First calculated by Alpher, Gamow and Herman in the late 40s, assuming initially pure neutron composition. Hayashi (50') pointed out that the initial composition is equal fractions of neutrons and protons. Modern calculation of the neutron fraction by Alpher, Follin, Herman (53'). Modern theory by Peebles (66') and Zel'dovich (65') (unknown in the west).

3.1. $n - p$ conversion

In the thermal history at $10^4\,\text{K} \lesssim T \lesssim 10^{10}\,\text{K}$ we have ignored the presence of a small number of nucleaons and a small excess of electrons over positrons. Now we will consider them. Weak interactions allow $n - p$ conversion through 6 processes, which are outlined below together with their energies (note that we consider $T \ll m_nc^2$, so nucleons are at rest):

$$\begin{aligned} n + \nu &\leftrightarrow p + e^-;\;\; E_e - E_\nu = Q, \\ n + e^+ &\leftrightarrow p + \bar{\nu};\;\; E_\nu - E_e = Q, \\ n &\leftrightarrow p + e^- + \bar{\nu};\;\; E_\nu + E_e = Q, \end{aligned}$$

where $Q = (m_n - m_p)c^2 \approx 1.293\,\mathrm{MeV}$. The conversion rates are:

$$\lambda(n \to p) = A \int \sqrt{1 - \frac{m_e^2 c^4}{(Q+q)^2}} \frac{(Q+q)^2 q^2 dq}{\left[1 + \exp\left(\frac{q}{T_\nu}\right)\right]\left[1 + \exp\left(-\frac{Q+q}{T}\right)\right]},$$

$$\lambda(p \to n) = A \int \sqrt{1 - \frac{m_e^2 c^4}{(Q+q)^2}} \frac{(Q+q)^2 q^2 dq}{\left[1 + \exp\left(-\frac{q}{T_\nu}\right)\right]\left[1 + \exp\left(\frac{Q+q}{T}\right)\right]},$$

where

$$A = \frac{G_{wk}^2 \left(1 + 3g_A^2\right) \cos^2 \theta_c}{\pi^2 h} \approx 1.72 \times 10^{27}\,\mathrm{erg}^{-5}\,\mathrm{s}^{-1},$$

$g_A \approx 1.257$ is the axial vector coupling beta decay, $\cos^2 \theta_c \approx 0.9745$ where θ_c is the Cabibbo angle, and the integrals go over all values of q for which the integrand is real: $-\infty < q < -Q - m_e c^2$; $-Q + m_e c^2 < q < \infty$. With the rates known, we can calculate:

$$\frac{dX_n}{dt} = -\lambda(n \to p) X_n + \lambda(p \to n)\left(1 - X_n\right), \tag{5}$$

where $X_i = n_i/n$ is the number fraction of of specie i to all nucleons (in this case $X_n + X_p = 1$). For $T = T_\nu$, the integrand of $\lambda(p \to n)$ will have in the denominator $[1 + \exp(-q/T)][1 + \exp(Q/T + q/T)] = \exp(Q/T)[1 + \exp(q/T)][1 + \exp(-Q/T - q/T)]$, which is the denominator of $\lambda(n \to p)$ times $\exp(Q/T)$. Since the rest of the integrands are the same in both expression, we get $\lambda(p \to n)/\lambda(n \to p) = \exp(-Q/T)$, so the time independent solution of Equation (5) is

$$\frac{X_n}{X_p} = \frac{X_n}{1 - X_n} = \exp\left(-\frac{Q}{T}\right),$$

as expected in equilibrium. So it is the inequality of T and T_ν, as well as the time dependence of these temperatures that drive X_n/X_p away from its equilibrium value.

For $T \gg Q$ we get $X_n = X_p$ and $\lambda(n \to p) = \lambda(p \to n)$. The integrals get contribution only for $q \sim T$, so we can evaluate them by taking $T = T_\nu$, $Q = m_e c^2 = 0$ to obtain:

$$\begin{aligned} \lambda(n \to p) &= \lambda(p \to n) = A \int_{-\infty}^{\infty} \frac{q^4 dq}{\left[1 + \exp\left(\frac{q}{T}\right)\right]\left[1 + \exp\left(-\frac{q}{T}\right)\right]} \\ &= AT^5 \int_{-\infty}^{\infty} \frac{x^4 dx}{[1 + \exp(x)]\,[1 + \exp(-x)]} \\ &= A\frac{7\pi^4}{15} T^5 \approx 0.39 \left(\frac{T}{10^{10}\,\mathrm{K}}\right)^5 \mathrm{s}^{-1}, \end{aligned}$$

where we changed the variable of integration to $x = q/T$. For times in which we can use Equation (3), and by using $H = 1/2t$ for radiation dominated Universe, we get

$$\frac{\lambda}{H} = 2\lambda t \approx 0.8 \left(\frac{T}{10^{10}\,\mathrm{K}}\right)^3,$$

so $\lambda/H > 1$ for $T \gtrsim 1.1 \times 10^{10}$ K. However, this derivation is not accurate since $Q \approx 1.5 \times 10^{10}$ K, so we could not take $Q = 0$ in the derivation of λ. Nevertheless, accurate integration shows that equilibrium is obtained for $T \gtrsim 3 \times 10^{10}$ K:

$$X_n = \frac{\lambda(p \to n)}{\lambda(p \to n) + \lambda(n \to p)} \approx \frac{1}{1 + \exp\left(\frac{Q}{T}\right)}.$$

This result only relies on $\mu = 0$ for leptons. We can now integrate Equation (5) to obtain $X_n(t)$, $X_p(t)$. The integration is shown in Figure 2 (note that the figure presents the mass fractions $\bar{X}_i$ and not the number fractions X_i, see discussion below). We get that $\lambda \ll H$ after $T < 10^{10}$ K, so $n \to p$ continues only through n decay with a lifetime $\tau_n = 885.7 \pm 0.9$ s and $X_n(t) \propto \exp(-t/\tau_n)$. From the exact integration we find that at late times

$$X_n(t) \approx 0.164 \exp(-t/\tau_n). \tag{6}$$

The conversion of $n \to p$ is stopped by the formation of heavy nuclei (not included in the calculation so far), in which neutrons are stable.

3.2. formation of heavy nuclei

For a nuclei of type i, with a mass number A_i and a charge Z_i, that is is equilibrium with neutrons and protons, $Z_i p + (A_i - Z_i)n \leftrightarrow i$, we have $Z_i\mu_p + (A_i - Z_i)\mu_n = \mu_i$. Using the expression of n_i for NR

$$\mu_i = m_i c^2 + T \ln\left[\frac{n_i}{g_i}\left(\frac{2\pi m_i T}{h^2}\right)^{-3/2}\right],$$

we get

$$\begin{aligned} & \ln\left[\frac{n_i}{g_i}\left(\frac{2\pi m_i T}{h^2}\right)^{-3/2}\right] + \frac{m_i c^2}{T} \\ = \; & Z_i \ln\left[\frac{n_p}{2}\left(\frac{2\pi m_p T}{h^2}\right)^{-3/2}\right] + \frac{Z_i m_p c^2}{T} + (A_i - Z_i)\ln\left[\frac{n_n}{2}\left(\frac{2\pi m_n T}{h^2}\right)^{-3/2}\right] + \frac{(A_i - Z_i) m_n c^2}{T}, \\ \Rightarrow \; & \frac{n_i}{g_i}\left(\frac{n_p}{2}\right)^{-Z_i}\left(\frac{n_n}{2}\right)^{Z_i - A_i}\left(\frac{2\pi m_n T}{h^2}\right)^{-3/2 + 3Z_i/2 + 3(A_i - Z_i)/2} A_i^{-3/2} = \exp\left(\frac{B_i}{T}\right), \end{aligned}$$

where we used $g_p = g_n = 2$, $B_i = [(A_i - Z_i)m_n + Z_i m_p - m_i]c^2$ and we approximated $m_n \approx m_p$ outside the exponent. From here it follows

$$\begin{aligned} \frac{n_i}{n_p^{Z_i} n_n^{A_i - Z_i}} &= \frac{g_i}{2^{A_i}} A_i^{3/2}\left(\frac{2\pi m_n T}{h^2}\right)^{3(1 - A_i)/2} \exp\left(\frac{B_i}{T}\right), \\ \Rightarrow \; X_i &= \frac{g_i}{2} A_i^{3/2} X_p^{Z_i} X_n^{A_i - Z_i} \epsilon^{A_i - 1} \exp\left(\frac{B_i}{T}\right), \end{aligned} \tag{7}$$

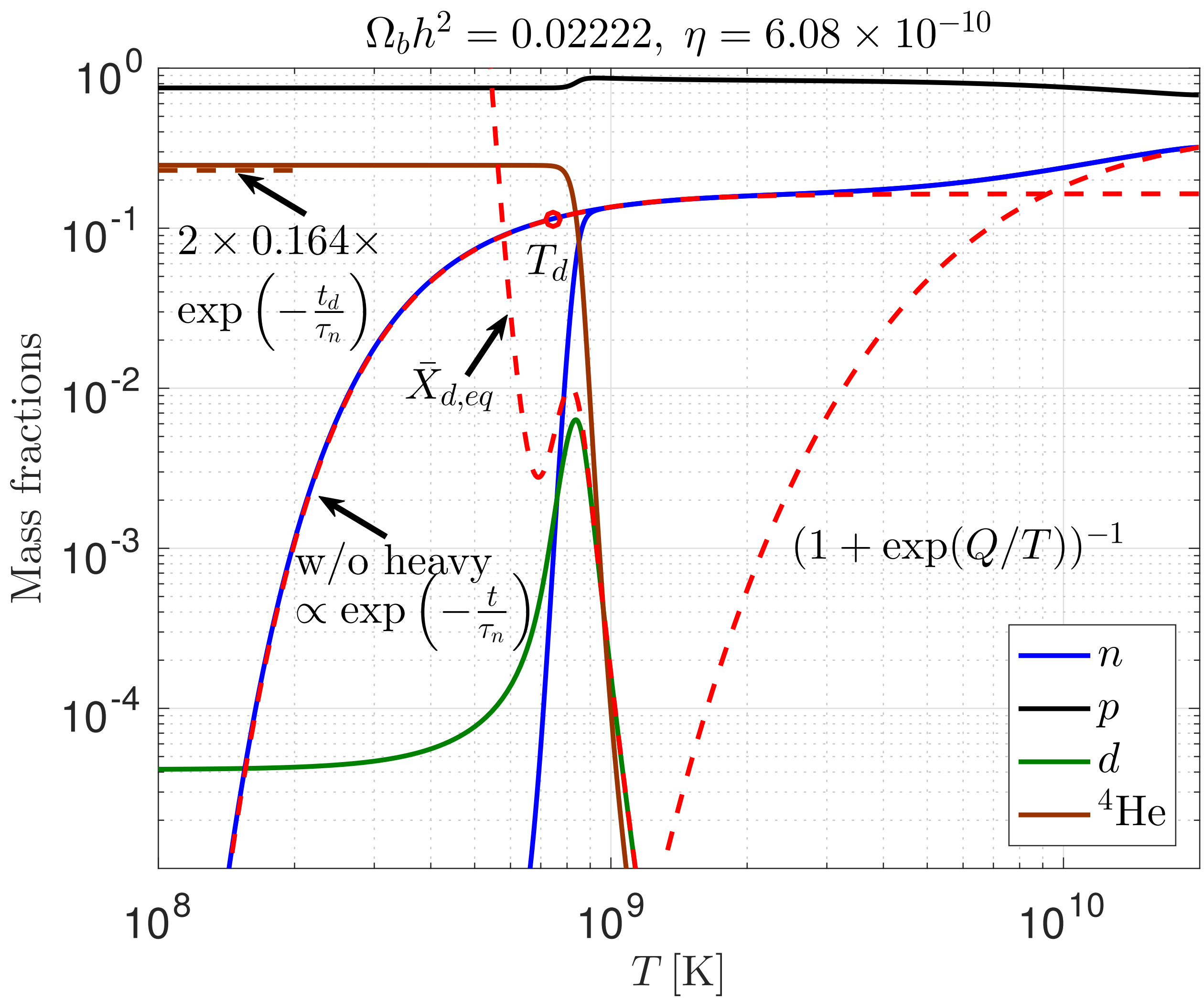

Fig. 2.— Mass fractions of different nuclei as a function of the temperature for $\Omega_b h^2 = 0.02222$ ($\eta = 6.08 \times 10^{-10}$). The two blue lines are $\bar{X}_n$ for a calculation of only n and p and for a calculation that includes heavy nuclei. At high temperatures both $\bar{X}_n$ are the same, where the transition from the equilibrium value, $[1+\exp(Q/T)]^{-1}$, to the free neutron decay, $\approx 0.164\exp(-t/\tau_n)$, is obtained. The decay of free neutrons is stopped by the formation of heavy nuclei, in which neutrons are stable, slightly below 10^9 K. The rest of the solid lines (p, black, d, green, ^{4}He, brown) are for a calculation with the heavy nuclei. The process $p + n \to d + \gamma$ is very efficient at high temperature, so d is in equilibrium (compare to the red dashed line, which uses Equation (8) with X_n and X_p from the full calculation). d are very rare until $T \sim T_d \approx 7.5 \times 10^8$ K, which prohibits ^{4}He production. When finally $T \sim T_d$, all free neutrons make ^{4}He. Further synthesise is blocked because the triple-α reaction is too slow. The estimate $Y_p \approx 2 \times 0.164 \times \exp(-t_d/\tau_n)$ is shown in dashed brown. The time of d burning is actually slightly earlier than t_d, which leads to higher Y_p compared with the estimation from above (because of less neutron decay).

where

$$\begin{aligned}\epsilon &= \frac{n}{2}\left(\frac{2\pi m_n T}{h^2}\right)^{-3/2} = \frac{1}{2}\eta 16\pi\zeta(3)\left(\frac{T}{hc}\right)^3\left(\frac{2\pi m_n T}{h^2}\right)^{-3/2} \\ &= \frac{8\pi\zeta(3)}{2\pi\sqrt{2\pi}}\eta\left(\frac{T}{m_n c^2}\right)^{3/2} = \frac{4\zeta(3)}{\sqrt{2\pi}}\eta\left(\frac{T}{m_n c^2}\right)^{3/2} \approx 1.46\times 10^{-12}\left(\frac{T}{10^{10}\,\mathrm{K}}\right)^{3/2}\Omega_b h^2,\end{aligned}$$

where we used $\eta = n/n_\gamma \approx 2.74\times 10^{-8}\Omega_b h^2$ (note that we also used $T \propto 1/R$, which holds after $e^\pm$ annihilate). Because $\epsilon_i \ll 1$ the temperature needs to drop significantly below B_i in order to get a significant fraction X_i. We can estimate that species of type i are absent until

$$\begin{aligned}1 &\approx \epsilon^{A_i-1}\exp\left(\frac{B_i}{T}\right) \Rightarrow (1-A_i)\ln\epsilon = \frac{B_i}{T} \\ &\Rightarrow T = \frac{B_i}{(1-A_i)\ln\epsilon} = \frac{B_i}{(A_i-1)\,|\ln\epsilon|}.\end{aligned}$$

In Table 1 we calculate this temperature, T_i, for different nuclei, where we used for ϵ a fixed value at $T = 10^9$ K and we used $\Omega_b h^2 = 0.02$. We also provide the value of g_i (only for the ground state, ignoring contribution from excited states). Note that $B_i/(A_i-1)$ of ^{4}He is larger than the values of ^{12}C and ^{56}Ni, but this is just because we divide by $A_i - 1$ and not by A_i (The binding energy per baryon of ^{12}C and ^{56}Ni is larger than the binding energy per baryon of ^{4}He). Looking at the table, we would expect that at high temperatures we would have ^{4}He and heavier nuclei, and as T drops we will have ^{3}He and t followed by d.

This is not what actually happens, since heavy nuclei are built from lighter nuclei through the following reactions:

$$\begin{aligned}\text{Step I}: &\quad p + n \to d + \gamma \\ \text{Step II}: &\quad d + d \to t + p;\;\; d + d \to^3 \mathrm{He} + n \\ \text{Step III}: &\quad d + t \to^4 \mathrm{He} + n;\;\; d +^3 \mathrm{He} \to^4 \mathrm{He} + p\end{aligned}$$

(there are more slower processes that involve photons, like $p + d \to^3 \mathrm{He} + \gamma$ and $n + d \to^3 t + \gamma$).

Table 1: Some parameters for different nuclei

nucleus	B_i [MeV]	$B_i/(A_i-1)$ [MeV]	T_i [K]	g_i
^{2}H=d	2.22	2.22	7.46×10^8	3
^{3}H=t	8.48	4.24	1.42×10^9	2
^{3}He	7.72	3.86	1.29×10^9	2
^{4}He	28.3	9.43	3.16×10^9	1
^{12}C	92.16	8.38	2.80×10^9	1
^{56}Ni	484.0	8.80	2.95×10^9	1

There is no problem with step I, since the rate of d production per free neutron is:

$$\begin{aligned}\lambda_d &\gtrsim \left(3\times 10^4\,\mathrm{cm^3\,s^{-1}\,mol^{-1}}\right) n_p = \frac{1}{N_A} 3\times 10^4 X_p \eta 16\pi\zeta(3)\left(\frac{T}{hc}\right)^3 \\ &\approx 2.8\times 10^4 \left(\frac{T}{10^{10}\,\mathrm{K}}\right)^3 X_p \Omega_b h^2\,\mathrm{s^{-1}} \\ &\Rightarrow \lambda_d t \approx 4.9\times 10^4 \left(\frac{T}{10^{10}\,\mathrm{K}}\right) X_p \Omega_b h^2 \gg 1,\end{aligned}$$

where we have used Equation (4) for the time in the last line. So even for $T < 10^9\,\mathrm{K}$ we can still have d in equilibrium. We get from Equation (7) that

$$X_d = \frac{3}{2} 2^{3/2} X_p X_n \epsilon \exp\left(\frac{B_d}{T}\right) = 3\sqrt{2} X_p X_n \epsilon \exp\left(\frac{B_d}{T}\right). \qquad (8)$$

The problem is that $T_d \approx 7.5\times 10^8\,\mathrm{K}$, so d are very rare well after ^{4}He should have been formed. The rate of $d + d \to t + p$ and of $d + d \to^3$ He $+ n$ are small per d (although $p + d \to^3$ He $+ \gamma$ and $n + d \to^3 t + \gamma$ are not small per d, they are too slow), which prohibits ^{4}He production. This is called the d bottleneck. When finally $T \sim T_d$, all free neutrons make ^{4}He. Further synthesise is blocked because the triple-α reaction is too slow. This process is demonstrated in Figure 2. If we assume that all neutrons end up in ^{4}He nuclei, then we can relate the fractions of different species at $T > T_d$, $X_i^>$, to the fractions at $T < T_d$, $X_i^<$. Note that the number fractions, X_i, satisfy $X_i = n_i/n$ (such that $\sum_i X_i = 1$) and that the mass fractions, $\bar{X}_i$, satisfy $\bar{X}_i = n_i A_i / \sum_i n_i A_i$ (such that $\sum_i \bar{X}_i = 1$). We get that

$$X_i = \frac{\bar{X}_i}{A_i}\frac{\sum_i n_i A_i}{\sum_i n_i} \Rightarrow X_i = \bar{X}_i \frac{\bar{A}}{A_i},$$

where $\bar{A} = \sum_i n_i A_i / \sum_i n_i = \sum_i X_i A_i$. So we get $\bar{A}^> = 1$, $X_n^> = \bar{X}_n^>$, $X_p^> = \bar{X}_p^>$, and $\bar{A}^< = 4X_{^4\mathrm{He}}^< + X_p^<$, $\bar{X}_p^< = X_p^</\bar{A}^<$, $\bar{X}_{^4\mathrm{He}}^< = 4X_{^4\mathrm{He}}^</\bar{A}^<$. Since all neutrons end up in ^{4}He nuclei, we get $X_n^> = 2X_{^4\mathrm{He}}^<$ and $X_p^> = X_p^< + 2X_{^4\mathrm{He}}^<$, so

$$Y_p \equiv \bar{X}_{^4\mathrm{He}}^< = \frac{4X_{^4\mathrm{He}}^<}{4X_{^4\mathrm{He}}^< + X_p^<} = \frac{2X_n^>}{2X_n^> + (1 - X_n^>) - X_n^>} = 2X_n^>.$$

From Equation (4) we can find $t_d \equiv t(T_d) \approx 1.78(T_d/10^{10}\,\mathrm{K})^{-2}\,\mathrm{s} \approx 320\,\mathrm{s}$, so from Equation (6) $Y_p \approx 2\times 0.164\times\exp(-320/886) \approx 0.23$. This estimate is shown in Figure 2. In a more exact calculation the time of d burning is slightly earlier, which leads to higher Y_p (because of less neutron decay). Also some residual d is left after burning, see Figure 2.

In general, larger $\Omega_b h^2$ will result earlier burning of d so larger Y_p and less residual d. The numerical result for Y_p and $(\mathrm{D/H})_p \equiv X_d/X_p$ as a function of η are shown in Figure 3. The results can be fitted by $Y_p \approx 0.245 + 0.01\ln(\eta/5\times 10^{-10})$ and $(\mathrm{D/H})_p \approx 3.6\times 10^{-5}(\eta/5\times 10^{-10})^{-1.6}$. We also provide the results for $(^3\mathrm{He/H})_p$ (that can be approximated by $\approx 1.2\times 10^{-5}(\eta/5\times 10^{-10})^{-0.63}$, note that t decays to ^{3}He with a half-life of $\approx 12.3\,\mathrm{yr}$, so it also contributes to the primordial ^{3}He) and the results for $(^7\mathrm{Li/H})_p$ (that can be approximated by $\approx 1.2\times10^{-11}[(\eta/5\times10^{-10})^{-2.38}+21.7(\eta/5\times10^{-10})^{2.38}]$, note that ^{7}Be decays to ^{7}Li with a half-life of $\approx 53.2\,\mathrm{day}$, so it also contributes to the primordial ^{7}Li).

Y_p can be inferred from low metallicity HII regions (mainly blue compact galaxies without many stars). The best measured value is $Y_p = 0.2477\pm0.0029$ which results $\eta = (5.8\pm1.8)\times10^{-10}$. The uncertainty in η is large because of the weak dependance of Y_p on η. $(\mathrm{D/H})_p$ is inferred from absorption lines of high redshift QSO. The best value is $(\mathrm{D/H})_p = 2.78^{+0.44}_{-0.38}\times10^{-5}$, which translates to $\eta = (5.9\pm0.5)\times10^{-10}$ or $\Omega_b h^2 = 0.0214\pm0.0020$. This was the first evidence that $\Omega_b \ll \Omega_M$. There are also estimates for $(^3\mathrm{He/H})_p$ and $(^7\mathrm{Li/H})_p$ but they are much less certain. We also mention that the inferred value of $\Omega_b h^2$ is much larger than the mass in stars and in the interstellar medium, which is known as the missing baryons problem.

4. $T \gtrsim 10^{11}\,\mathrm{K}$

We showed that a decoupled particle i since T_D will have $T_i \propto R^{-1} \propto T[g(T)/g(T_D)]^{1/3}$, such that $T_i/T = [g(T)/g(T_D)]^{1/3}$. Now we will calculate $g(T)$ for $T \lesssim 4\times10^{15}\,\mathrm{K}$. We will need the masses and the degeneracies of particles in the standard model, given in Table 2. Several key events in the thermal history of the Universe and the value of g are given in Table 3. $(T_D)_\nu$ is the decoupling temperature of the neutrinos, T^{qh} is the quark-hadron phase transition (note that it happens during s annihilation) and T^{EWPT} is the electroweak phase transition.

5. Cold dark matter

We showed that $\Omega_b \ll \Omega_M$, so which particle contributes to Ω_M? We'll see later that 'cold' means that the particle had to be NR when $t_1 \sim 1\,\mathrm{yr}$ (when the particle horizon is ~ 1 galaxy). If this particle L was in a thermal equilibrium and it decoupled at $T_D \gg m_L c^2$, then this condition is $T_L(t_1) = T_L(t_D)[R(t_D)/R(t_1)] \approx T(t_1) \approx 0.2\,\mathrm{keV} \ll m_L c^2$. If it decoupled NR, then $T_L(t_1) = T_L(t_D)[R(t_D)/R(t_1)]^2 \ll m_L c^2$. One possibility for this particle is "Weakly Interacting Massive Particles" (WIMPs), which are massive, neutral, weakly interacting, stable particles that were in a thermal equilibrium and decoupled at some point. They survive until today if $\mu_L/T \neq 0$ or if they could not annihilate.

Let us calculate their abundance today assuming $\mu_L = 0$. The annihilation rate per particle is $n\langle\sigma v\rangle$ (where $[\langle\sigma v\rangle] = \mathrm{cm^3\,s^{-1}}$), so

$$\frac{d\left(nR^3\right)}{dt} = -n^2R^3\langle\sigma v\rangle + \text{creation from the thermal background.}$$

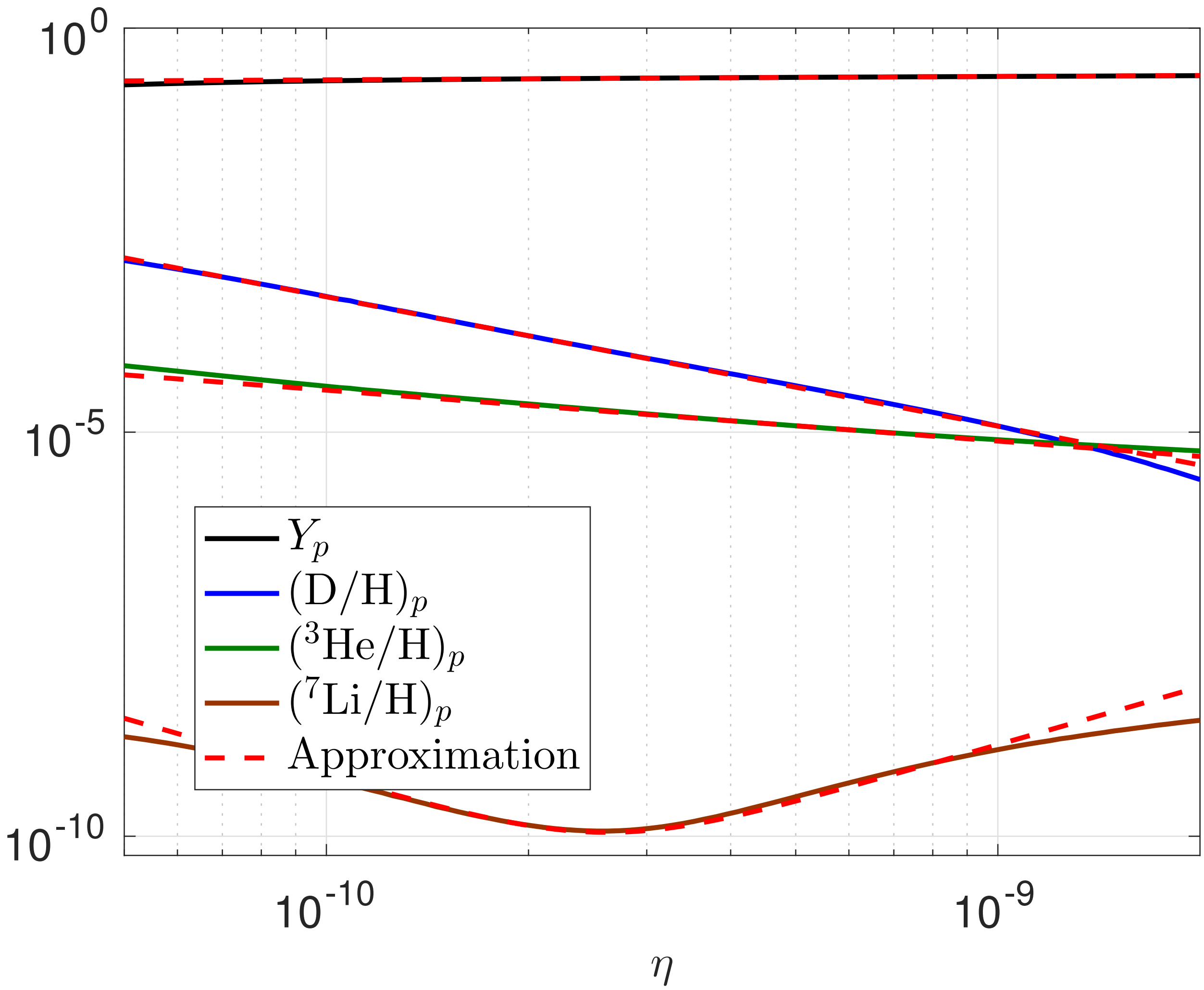

Fig. 3.— Y_p (black), $(\mathrm{D/H})_p$ (blue), $(^3\mathrm{He/H})_p$ (green) and $(^7\mathrm{Li/H})_p$ (brown) as a function of η. The red dashed lines are fits to the results, given in the text.

Table 2: The particles of the standard model

		Mass		Spin	g
Quarks	t	$172.44\,\mathrm{GeV}/c^2$	$\bar{t}$	$\frac{1}{2}$	6
	b	$4.18\,\mathrm{GeV}/c^2$	$\bar{b}$	3 colors	
	c	$1.275\,\mathrm{GeV}/c^2$	$\bar{c}$		
	s	$95\,\mathrm{MeV}/c^2$	$\bar{s}$		
	d	$4.8\,\mathrm{MeV}/c^2$	$\bar{d}$		
	u	$2.4\,\mathrm{MeV}/c^2$	$\bar{u}$		
Gluons		8 massless bosons		1	2
Leptons	τ^-	$1.7768\,\mathrm{GeV}/c^2$	τ^+	$\frac{1}{2}$	2
	μ^-	$105.67\,\mathrm{MeV}/c^2$	μ^+		
	e^-	$0.511\,\mathrm{MeV}/c^2$	e^+		
	ν_τ		$\bar{\nu}_\tau$	$\frac{1}{2}$	1
	ν_μ		$\bar{\nu}_\mu$		
	ν_e		$\bar{\nu}_e$		
Electroweak gauge bosons	W^+	$80.39\,\mathrm{GeV}/c^2$	W^-	1	3
	Z^0	$91.19\,\mathrm{GeV}/c^2$			
	γ			1	2
Higgs boson	H^0	$125.09\,\mathrm{GeV}/c^2$		0	1

Table 3: Several key events in the thermal history of the Universe and the value of g

T/c^2	Threshold [GeV]	Threshold [K]	particle content	g
$< m_e$	5×10^{-4}	6×10^9	γ (+3 decoupled ν)	2
$m_e - (T_D)_\nu$	10^{-3}	10^{10}	add $e^\pm$	$2 + 4 \times \frac{7}{8} = \frac{11}{2}$
$(T_D)_\nu - m_\mu$	0.1	10^{12}	ν's interact	$\frac{11}{2} + 6 \times \frac{7}{8} = \frac{43}{4}$
$m_\mu - m_\pi$	0.135	1.5×10^{12}	add $\mu^\pm$	$\frac{43}{4} + 4 \times \frac{7}{8} = \frac{57}{4}$
$m_\pi - \frac{T^{qh}}{c^2}$	0.15	1.7×10^{12}	add $\pi^\pm$, π^0	$\frac{57}{4} + 3 = \frac{69}{4}$
$\frac{T^{qh}}{c^2} - m_s$	0.2	2×10^{12}	γ, 3ν, $e^\pm$, $\mu^\pm$, u, $\bar{u}$, d, $\bar{d}$, $8g$	$\frac{57}{4} + 4 \times 6 \times \frac{7}{8} + 8 \times 2 = \frac{205}{4}$
$m_s - m_c$	1.3	1.5×10^{13}	add s, $\bar{s}$	$\frac{205}{4} + 2 \times 6 \times \frac{7}{8} = \frac{247}{4}$
$m_c - m_\tau$	1.8	2×10^{13}	add c, $\bar{c}$	$\frac{247}{4} + 2 \times 6 \times \frac{7}{8} = \frac{289}{4}$
$m_\tau - m_b$	4.2	5×10^{13}	add $\tau^\pm$	$\frac{289}{4} + 2 \times 2 \times \frac{7}{8} = \frac{303}{4}$
$m_b - m_W$	80	9×10^{14}	add b, $\bar{b}$	$\frac{303}{4} + 2 \times 6 \times \frac{7}{8} = \frac{345}{4}$
$m_W - m_H$	125	1.5×10^{15}	add $W^\pm$, Z^0	$\frac{345}{4} + 3 \times 3 = \frac{381}{4}$
$m_H - m_t$	170	2×10^{15}	add H^0	$\frac{381}{4} + 1 = \frac{385}{4}$
$m_t - \frac{T^{\mathrm{EWPT}}}{c^2}$	300	4×10^{15}	add t, $\bar{t}$	$\frac{385}{4} + 2 \times 6 \times \frac{7}{8} = \frac{427}{4}$

Since in equilibrium the creation balances the annihilation, we get

$$\frac{d\left(nR^3\right)}{dt} = -\left(n^2 - n_{eq}^2\right) R^3 \langle\sigma v\rangle. \tag{9}$$

For $T \gg m_L c^2$, we have $n_{eq} \propto T^3$ and $T \propto R^{-1}$ so $n = n_{eq}$ is a solution. For lower $T \lesssim m_L c^2$ the creation is negligible, and

$$\begin{aligned}
\frac{d\left(nR^3\right)}{dt} &= -n^2 R^3 \langle\sigma v\rangle \\
&\Rightarrow \frac{dy}{dt} = -n\langle\sigma v\rangle y \\
&\Rightarrow \frac{dy}{y} = -ndt\langle\sigma v\rangle = -\frac{y}{R^3}\langle\sigma v\rangle dt \\
&\Rightarrow \frac{1}{y} = \int \frac{\langle\sigma v\rangle}{R^3} dt + \text{const.} \\
&\Rightarrow n(t)R^3(t) = \frac{n(t_1)R^3(t_1)}{1 + n(t_1)R^3(t_1)\int_{t_1}^{t} \frac{\langle\sigma v\rangle}{R^3(t')} dt'},
\end{aligned}$$

where we changed variable to $y = nR^3$, and t_1 is chosen such that the creation can be ignored. The integral in the nominator is converging for $t \to \infty$, since $R^3(t) \propto t^{3/2}$ for radiation dominated Universe (and even faster later), and $\langle\sigma v\rangle \to$ const. for low T (as there are less excited states of the particle). So the residual abundance of the L particles is

$$n(t)R^3(t) \to \frac{n(t_1)R^3(t_1)}{1 + n(t_1)R^3(t_1)\int_{t_1}^{\infty} \frac{\langle\sigma v\rangle}{R^3(t')} dt'}.$$

At the time of WIMPs annihilation the Universe is radiation dominated, so $R \propto 1/T$ and

$$\begin{aligned}
dt &= -2\sqrt{\frac{3c^2}{16\pi G g \bar{a}_B}} \frac{dT}{T^3} = -2\sqrt{\frac{45h^3c^5}{8 \times 16\pi^6 G g}} \frac{dT}{T^3} \\
&= -\sqrt{\frac{45h^3c^5}{32\pi^6 G g}} \frac{dx}{x^3} \frac{1}{\left(m_L c^2\right)^2} = -\sqrt{\frac{45h^3}{32\pi^6 c^3 G g}} \frac{dx}{x^3} \frac{1}{m_L^2},
\end{aligned}$$

where we changed the variable to $x = T/m_L c^2$. By defining $u = n(\hbar c)^3/T^3$, we can write Equation (9) as

$$\begin{aligned}
\frac{du}{dt} &= -\left(u^2 - u_{eq}^2\right)\langle\sigma v\rangle \left(\frac{T}{\hbar c}\right)^3 \\
&\Rightarrow \frac{du}{dx} = \sqrt{\frac{45h^3}{32c^3\pi^6 G g}} \frac{1}{x^3}\left(u^2 - u_{eq}^2\right)\langle\sigma v\rangle \left(\frac{T}{\hbar c}\right)^3 \frac{1}{m_L^2} = \sqrt{\frac{45h^3}{32c^3\pi^6 G g}} \frac{\left(m_L c^2\right)^3}{(\hbar c)^3}\left(u^2 - u_{eq}^2\right)\langle\sigma v\rangle \frac{1}{m_L^2} \\
&\Rightarrow \frac{du}{dx} = B\left(u^2 - u_{eq}^2\right),
\end{aligned} \tag{10}$$

where

$$\begin{aligned} B &= \sqrt{\frac{45h^3}{32c^3\pi^6 Gg}\frac{m_L(2\pi)^3c^3}{h^3}\langle\sigma v\rangle} = \sqrt{\frac{45}{32Gg}8m_L\left(\frac{c}{h}\right)^{3/2}\langle\sigma v\rangle} \\ &= \sqrt{\frac{90}{g}}\sqrt{\frac{\hbar c^5}{G}}m_L\left(\frac{c}{h}\right)^{3/2}\frac{1}{\sqrt{\hbar c^5}}\langle\sigma v\rangle = \sqrt{\frac{90}{g}}M_{pl}c^2 m_L c^2\frac{\sqrt{2\pi}}{c^3h^2}\langle\sigma v\rangle \\ &= \sqrt{\frac{180\pi}{g}}M_{pl}c^2 m_L c^2\frac{\langle\sigma v\rangle}{c^3h^2}. \end{aligned}$$

This number is typically going to be $\gg 1$. The leftover value of $n(\hbar c)^3/T^3$ depends only on B and on g_L through

$$\begin{aligned} u_{eq}(x) &= \left(\frac{\hbar c}{T}\right)^3\frac{4\pi g_L}{h^3}\int_0^\infty\frac{p^2dp}{\exp(\beta\varepsilon)\pm 1} \\ &= \frac{g_L}{2\pi^2}\int_0^\infty\frac{y^2dy}{\exp\left(\sqrt{y^2+x^{-2}}\right)\pm 1}, \end{aligned}$$

where we changed the variable of integration to $y = cp/T$. Equation (10) can be solved numerically, and the result for $g_L = 2$ and FD distribution is shown in Figure 4. We can fit the values of $u(0)$ as $u(0) \approx 6\times B^{-0.95}$, and in what follows we will take the index of B to be -1.

We get for the current mass density of particle L (assuming it annihilate before ν decoupling)

$$\rho_{L,0} = m_L u(0)\left(\frac{T_{\rm CMB}}{\hbar c}\right)^3\left(\frac{4}{11}\right) \approx 6m_L\left(\frac{T_{\rm CMB}}{\hbar c}\right)^3\left(\frac{4}{11}\right)B^{-1},$$

so it is independent of m_L and scales as $1/\langle\sigma v\rangle$. We find

$$\Omega_L = \frac{8\pi G\rho_L}{3H_0^2} = \frac{8\pi G}{3H_0^2}6m_L\left(\frac{T_{\rm CMB}}{\hbar c}\right)^3\left(\frac{4}{11}\right)B^{-1}. \qquad (11)$$

If we demand $\Omega_L = \Omega_M$, then

$$\begin{aligned} \Omega_M &= \frac{8\pi G}{3H_0^2}6m_L\left(\frac{T_{\rm CMB}}{\hbar c}\right)^3\left(\frac{4}{11}\right)\sqrt{\frac{g}{180\pi}}\left(M_{pl}c^2\right)^{-1}\left(m_Lc^2\right)^{-1}\frac{c^3h^2}{\langle\sigma v\rangle} \\ &= \frac{8\pi G}{3H_0^2}6\left(\frac{T_{\rm CMB}}{hc}\right)^3 8\pi^3\left(\frac{4}{11}\right)\sqrt{\frac{g}{180\pi}}\frac{ch^2}{M_{pl}c^2\langle\sigma v\rangle} \\ \Rightarrow\ \langle\sigma v\rangle &= \frac{\Omega_\gamma}{\Omega_M}\frac{c^2}{T_{\rm CMB}}\frac{15}{8\pi^5}8\pi^3\left(\frac{4}{11}\right)6\sqrt{\frac{g}{180\pi}}\frac{ch^2}{M_{pl}c^2} \\ &= \frac{\Omega_\gamma}{\Omega_M}\left(\frac{4}{11}\right)\frac{90}{\pi^2}\sqrt{\frac{g}{180\pi}}\frac{c^3h^2}{T_{\rm CMB}M_{pl}c^2} \approx \left(\frac{g}{100}\right)^{1/2}4\times 10^{-26}\,{\rm cm^3\,s^{-1}}. \end{aligned} \qquad (12)$$

In natural units this is written as

$$\langle\sigma v\rangle \sim \frac{\Omega_\gamma}{\Omega_M}\frac{1}{M_{pl}T_{\rm CMB}} \sim 10^{-4}\frac{1}{10^{19}\,{\rm GeV}\,10^{-13}\,{\rm GeV}} \sim \frac{10^{-4}}{{\rm TeV}^2}.$$

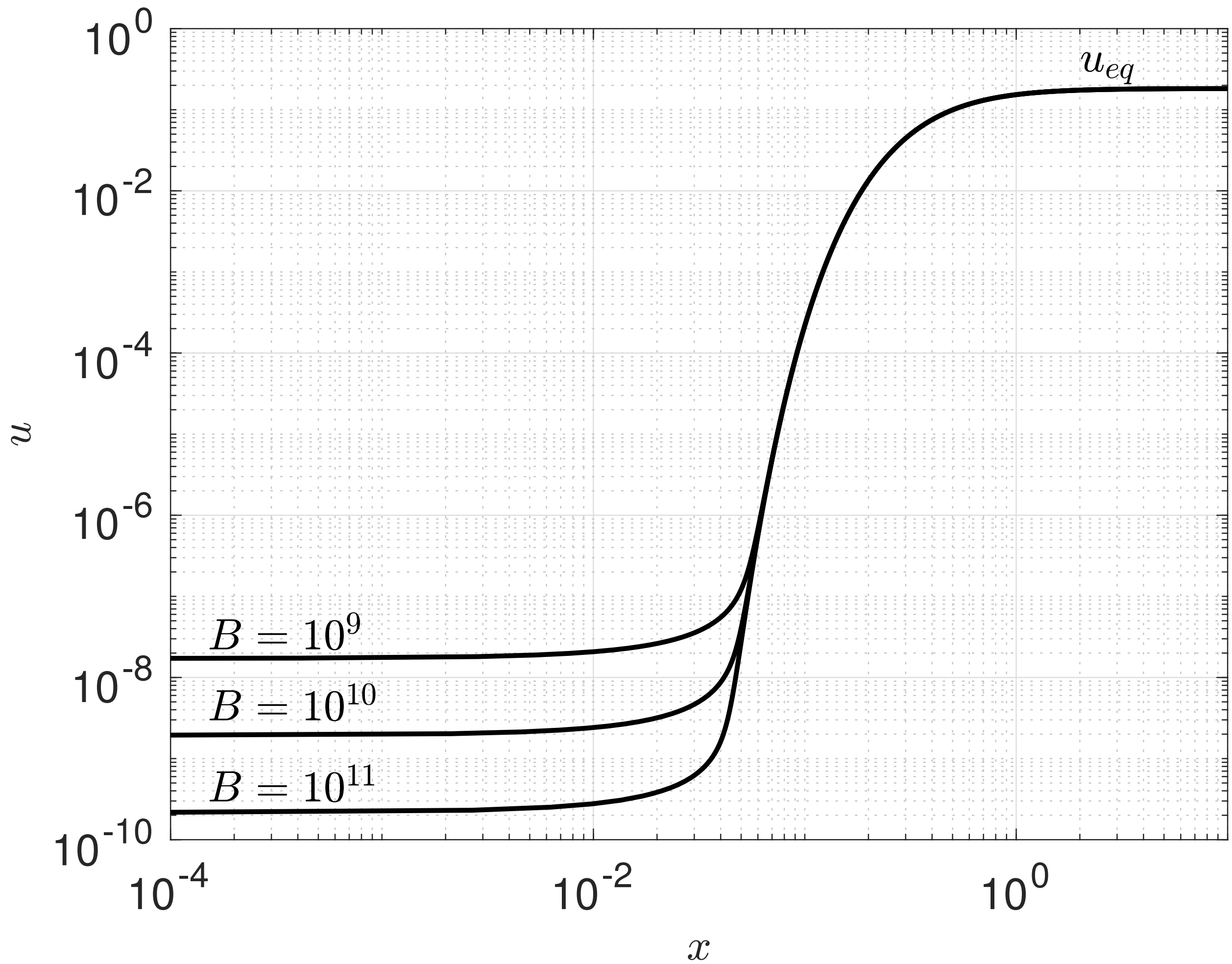

Fig. 4.— u as a function of x for $g_L = 2$ and FD distribution.

For the weak scale, $\langle\sigma v\rangle \sim G_{wk}^2 m_L^2 c^4 c(c\hbar)^2 \Rightarrow m_L c^2 \sim \sqrt{\langle\sigma v\rangle / G_{wk}^2 c^3 \hbar^2} \sim 5\,\text{Gev}$. This suggests that this particle should have been found already at the LHC, or that the cross-section is not given by the weak scale.

We can use Equation (12) to estimate the residual mass density of baryons under the same assumptions ($\mu = 0$):

$$\Omega_b \approx \Omega_\gamma \left(\frac{4}{11}\right) \frac{90}{\pi^2} \sqrt{\frac{g}{180\pi}} \frac{c^3 h^2}{\langle\sigma v\rangle T_{\text{CMB}} M_{pl} c^2}.$$

For baryons $\langle\sigma v\rangle \sim (hc)^2 c / m_\pi^2 c^4 = h^2 / m_\pi^2 c$, so

$$\Omega_b \approx \Omega_\gamma \left(\frac{4}{11}\right) \frac{90}{\pi^2} \sqrt{\frac{g}{180\pi}} \frac{m_\pi^2 c^4}{T_{\text{CMB}} M_{pl} c^2}.$$

The last factor is $(m_\pi / M_{pl})(m_\pi c^2 / T_{\text{CMB}}) \sim (0.1/10^{19})(0.1/10^{-13}) = 10^{-20} 10^{12} = 10^{-8}$, so we get far too small Ω_b in this case.