Inhomogeneous Universe: Linear Perturbation Theory & the CMB

June 12, 2018

Contents

1 Linear Perturbations

We have so far discussed the evolution of a homogeneous universe. The universe we see today is, however, highly inhomogeneous. We see structures on a wide range of scales, from the solar system, on scale of 1 A.U.$= 1.5 \times 10^{13}$ cm, through galaxies on scales of 10 kpc (1 kpc$= 3.15 \times 10^{18}$ cm) and galaxy clusters on 1 Mpc scale, to super-clusters on 30 Mpc scale. The inhomogeneities on small scales, $\ll 10$ Mpc, are highly non-liner, that is, the fractional density fluctuations on these scales, $\delta \equiv \delta\rho/\rho$ where ρ is the average density, are large, $\delta \gg 1$. $\delta \approx 1$ on scales ~ 10 Mpc and $\delta \ll 1$ on much larger scales.

As we have mentioned in the introductory lecture, the cosmic microwave background (CMB) radiation is highly isotropic, with variations not exceeding 1 part in 10^5. Since the radiation has decoupled from matter at $z = z_{\rm dec.} = 1.1 \times 10^3$, the CMB anisotropy reflects the level of inhomogeneities in that early epoch. The smallness of the CMB fluctuations imply that the universe has been nearly homogeneous at $z = z_{\rm dec.}$. We therefore believe that the structures that we see today are the result of the evolution of these small perturbations, which were generated at an early epoch of the universe evolution.

As we shall see below, perturbations grow due to the effect of gravity. Gravity slows the expansion of the universe. Hence, a perturbed over dense region will suffer stronger than average deceleration, leading to slower than average expansion and hence to increase of over density with time. In this chapter, we will discuss the evolution of perturbations in the linear regime. Since the initial perturbations were small, $\delta \ll 1$, much of their evolution may be described by a linear perturbation analysis. Moreover, the evolution of perturbations on scales > 10 Mpc is well described by linear analysis all the way through the present epoch. In the following chapter, we will use the results of this chapter to derive constraints on the cosmological model from the observed CMB fluctuations and from the observed large scale (galaxies, clusters of galaxies) structure of the local ($z = 0$) universe.

The plan of this chapter is as follows. In the first sections, we will consider the evolution of perturbations in a universe composed of radiation and ordinary (baryonic) matter. In §1.1 we will derive the characteristic scales over which various physical processes are operating. In §1.2 we will discuss the evolution of perturbations on scales where only gravity is important, and in §**??** we will derive the equations describing the evolution of perturbations on arbitrary scale. In §1.4 we will discuss the evolution of perturbations in the presence of dark, non-baryonic component of matter. In §1.5 we will summarize the main conclusions of our analysis by deriving the relation between the present day, $z = 0$, spectrum of perturbations and the initial perturbations at the early universe.

1.1 Characteristic scales

The expansion of the (homogeneous) universe is described by the equation

$$\dot{a}^2/a^2 = H_0^2\left[\frac{e}{\rho_c c^2} + \frac{\rho_\Lambda}{\rho_c} + \left(1 - \frac{\rho_0}{\rho_c} - \frac{\rho_\Lambda}{\rho_c}\right)a^{-2}\right]. \qquad (1)$$

Here, a is the scale factor, for which we will use the normalization $a(t = t_0) = 1$, where t_0 is the present ($z = 0$) age of the universe. H_0 is the Hubble constant, e is the energy density, $\rho_c \equiv 3H_0^2/8\pi G$ is the critical density and $\rho_\Lambda \equiv \lambda c^2/8\pi G$ is the "effective" Λ density.

Since ρ_Λ is time independent, and since ρ decreases with time, at sufficiently early time, $a \ll 1$, we may neglect the curvature and Λ terms, $\dot{a}^2 = H_0^2(e/\rho_c c^2)a^2$. No characteristic time scale appears in this equation. This implies that a is proportional to some power of t, and that the characteristic time scale for expansion, $a/\dot{a}$, is proportional to t. As we shall see in the following section, small perturbations on scales where only gravity is important also grow, at $a \ll 1$, as a power of time, i.e. $\delta \propto a^\alpha$, implying that the characteristic time scale for growth of perturbations, $\delta/\dot{\delta} = \alpha^{-1}a/\dot{a}$, is also proportional to t. This result is to be expected, since under the effects of gravity alone, no characteristic time scale appears in the problem at $a \ll 1$.

Since the universe expansion as well as the evolution of perturbations occur, for $a \ll 1$, on time scale $a/\dot{a}$, the instantaneous horizon scale, defined as $\lambda_H \equiv ca/\dot{a}$, has an important physical significance. It is the largest scale which is causally connected, i.e. across which information can be communicated, during the time over which significant expansion, or perturbation growth, takes place. In what follows, we will express length scales in terms of "comoving units". The proper distance between points which move with the (homogeneous) Hubble flow increases linearly with a: Two points separated by a proper length $L_{\rm prop.}$ at time t are separated at present, $t = t_0$, by $L_{\rm prop.}/a(t)$. A perturbation on scale $\lambda_{\rm prop.}$ at time t therefore corresponds at present to a perturbation on scale $\lambda_{\rm co.} \equiv \lambda_{\rm prop.}/a(t)$. In what follows we drop the subscript "co." and express all length scales in comoving units, unless otherwise stated. Thus, $\lambda_H = a^{-1}ca/\dot{a} = c/\dot{a}$.

Radiation and matter energy densities were equal at $a = a_{\rm eq.}$, where $a_{\rm eq.}^{-1} \equiv \Omega\rho_c/(4\sigma/c)T_{CMB}^4 = 2.3\Omega h_{75}^2 \times 10^4$. Here Ω is the matter energy density (today, measured in units of ρ_c), $H_0 = 75h_{75}{\rm km/Mpc\,s}$ and $T_{CMB} =$

2.73 K is the CMB temperature. At early times during matter domination, $a_{\rm eq.} \ll a \ll 1$, we have $\dot{a}^2 = H_0^2\Omega a^{-1}$, while during radiation domination, $a_{\rm eq.} \gg a$, we have $\dot{a}^2 = H_0^2\Omega a_{\rm eq.} a^{-2}$. Thus,

$$\lambda_H \equiv \frac{c}{\dot{a}} = \lambda_{\rm eq.} \times \begin{cases} (a/a_{\rm eq.})^{1/2}, & \text{for } a \gg a_{\rm eq.}; \\ (a/a_{\rm eq.})^1, & \text{for } a \ll a_{\rm eq.}. \end{cases} \qquad \lambda_{\rm eq.} = 27(\Omega h_{75}^2)^{-1}\text{Mpc}. \tag{2}$$

The baryon mass enclosed in a sphere of radius λ is

$$M_\lambda \equiv \frac{4\pi}{3}\lambda^3\Omega_b\rho_c = 6.6 \times 10^9 \Omega_{b,-2} h_{75}^2 \lambda_{\rm Mpc}^3 M_\odot, \tag{3}$$

where $\Omega_b = 10^{-2}\Omega_{b,-2}$ is the baryon energy density today measured in units of ρ_c, $\lambda = 1\lambda_{\rm Mpc}$ Mpc and $M_\odot$ is solar mass. We use different notations for the total matter density, Ω, and the baryonic density Ω_b, to allow for the presence of non-baryonic matter ($\Omega > \Omega_b$). Thus, present day galaxies, which contain $\sim 10^{10}M_\odot$ of baryons, were formed out of fluctuations on a scale ~ 1 Mpc. Since the fluctuations we see in the present day universe are large on galaxy-cluster scale, the theory of structure formation should allow for the growth of perturbations to $\delta \sim 1$ on galaxy cluster scale.

1.1.1 The Jeans scale

While gravity acts to increase density fluctuations, plasma pressure acts to suppress them. The pressure in a denser region, which expands slower than average, would decrease slower than the average pressure. The over-pressure would act to accelerate the over-dense region expansion. In order to estimate the scales at which pressure effects are important, we recall that (linear) pressure perturbations in an ideal fluid, where dissipative processes may be neglected, propagate at the speed of sound c_s, which is given by the derivative of the pressure with respect to energy density at constant entropy, $c_s^2 \equiv c^2(\partial p/\partial e)_S$ (see Appendix A). Thus, over-pressure will prevent the growth of perturbations on a scale for which the sound crossing time, $\lambda_{\rm prop.}/c_s$ is shorter than the time scale for the growth the of perturbation, $a/\dot{a}$, i.e. for $\lambda_{\rm prop.} < c_s a/\dot{a}$ corresponding to a comoving scales $\lambda < c_s/\dot{a}$. Thus, on scales much larger than the Jeans scale,

$$\lambda_J \equiv \frac{c_s}{\dot{a}} = \frac{c_s}{c}\lambda_H, \tag{4}$$

$\lambda \gg \lambda_J$, pressure effects may be neglected. At small scales, $\lambda \ll \lambda_J$, pressure prevents the growth of perturbations, pressure gradients dominate over gravitational force.For $\lambda \ll \lambda_J$, the perturbations become sound waves (see Appendix A), with oscillation period given by $\lambda_{\rm prop.}/c_s$.

During radiation domination, the speed of sound is $c_s = c/\sqrt{3}$ (see Appendix A). Thus, $\lambda_J \approx \lambda_H$ for $a < a_{\rm eq.}$. For $a > a_{\rm eq.}$ and before decoupling, $a < a_{\rm dec.}$, the plasma pressure is dominated by the photons, $p = (1/3)(4\sigma/c)T^4$. Since the entropy is proportional to the number of photons per baryon, $S \propto n_\gamma/n_b \propto T^3/\rho \propto p^{3/4}/\rho$, we find $(c_s/c)^2 = (4/3)p/\rho = (4/9)(4\sigma/c)T^4/\rho = (4/9)(a/a_{\rm eq.})^{-1}$, i.e. $c_s/c = (2/3)(a/a_{\rm eq.})^{-1/2}$. Since for $a > a_{\rm eq.}$ we have $\dot{a} \propto a^{-1/2}$, we find that for $a_{\rm eq.} < a < a_{\rm dec.}$ the Jeans scale is time independent and given by $\lambda_J = (2/3)\lambda_{\rm eq.}$.

Finally, following decoupling the baryons no longer feel the photons pressure, and the pressure is dominated by the thermal motion of the baryons, i.e. $c_s^2 = (5/3)T/m_p$ where T is the baryon temperature. Since following decoupling $T = 0.25(a/a_{\rm dec.})^{-2}$ eV, we find $c_s/c = 2 \times 10^{-5}(a/a_{\rm dec.})^{-1}$, and hence $\lambda_J = 2 \times 10^{-5}\lambda_{\rm dec.}(a/a_{\rm dec.})^{-1/2}$. Combining the results for the various epochs,

$$\lambda_J = \lambda_{\rm eq.} \times \begin{cases} 3^{-1/2}(a/a_{\rm eq.}), & \text{for } a < a_{\rm eq.}; \\ 2/3, & \text{for } a_{\rm eq.} < a < a_{\rm dec.}; \\ 2 \times 10^{-5}(a_{\rm dec.}/a_{\rm eq.})(a/a_{\rm eq.})^{-1/2}, & \text{for } a > a_{\rm dec.}. \end{cases} \tag{5}$$

1.1.2 Diffusion scale

In the discussion of the previous section, we have treated the baryon-radiation plasma as an ideal fluid. We have neglected, in particular, effects that may arise from the dissipation induced by diffusion of photons. As long as radiation and matter are coupled, the energy density of photons in over-dense regions is higher than average. In this situation, photons will try to diffusively "escape" from over-dense regions, carrying with them energy and momentum flux. During matter domination, the effects of diffusion are small, since the energy density carried by the photons is small. However, during radiation domination, photons dominate the energy density and if they "escape" over-dense regions on a time scale shorter than that required for gravitational growth of perturbations, then they will "smear" out the energy density perturbation, and hence suppress the gravitational instability growth.

Photons are scattered, at $a < a_{\rm dec.}$, by free electrons. This scattering is roughly isotropic, and hence photons random walk in space. The average distance a photon propagates after N collisions is therefore $\approx \sqrt{N}\, l_\gamma$, where l_γ is the mean free path. Since the gravitational growth time is $a/\dot{a}$, photon diffusion during the radiation dominated phase will suppress perturbations on (comoving) scales $\lambda < \lambda_D(a) \approx a^{-1}\sqrt{ca/\dot{a}l_\gamma}\, l_\gamma = \sqrt{cl_\gamma/a\dot{a}} = \sqrt{\lambda_H l_\gamma/a}$. The photon mean free path is given by $l_\gamma = 1/n_e\sigma_T$ and the electron density is $n_e = \Omega_b(\rho_c/m_p)a^{-3}$, so that $\lambda_D(a) \approx \sqrt{\lambda_H/an_e\sigma_T} = a\sqrt{m_p\lambda_H/\sigma_T\Omega_b\rho_c}$. The diffusion scale therefore grows with time, and perturbations with $\lambda < \lambda_D(a = a_{\rm eq.})$ would be suppressed due to diffusion,

$$\lambda_{D,{\rm eq.}} \equiv \lambda_D(a = a_{\rm eq.}) \approx a_{\rm eq.}\sqrt{\frac{m_p\lambda_{\rm eq.}}{\sigma_T\Omega_b\rho_c}} \approx 0.2\left(\frac{\Omega}{5\Omega_b}\right)^{1/2}(\Omega h_{75}^2)^{-2}\,{\rm Mpc}. \quad (6)$$

1.2 Evolution of perturbations with $\lambda \gg \lambda_J$

On scales $\lambda \gg \lambda_J$ we may neglect the effects of pressure and diffusion. The perturbations on such scales are therefore described by the energy density perturbation, $\delta \equiv \delta e/e$, and by the deviation $\delta\mathbf{v}$ of the velocity field from the Hubble flow, $\mathbf{v}$. The pressure perturbation δp does not affect the perturbation evolution.

Since the flow is adiabatic, entropy perturbations do not evolve with time (see also A.2), so that the non-adiabatic perturbation modes do not contribute to the growth of inhomogeneities. The incompressible, $\{\nabla\delta\mathbf{v} = 0, \nabla\times\delta\mathbf{v} \neq 0\}$ (see also A.2) part of the adiabatic perturbations also do not contribute, since they do not lead to modification of the density (in fact, since this component is rotational, $\nabla\times\delta\mathbf{v} \neq 0$, it carries angular momentum and conservation of angular momentum implies that as the universe expands the rotational mode amplitude must decrease $\propto 1/a$). We will therefore focus on the adiabatic compressible modes, $\{\nabla\delta\mathbf{v} \neq 0, \nabla\times\delta\mathbf{v} = 0\}$, for which the velocity may be expressed as the gradient of a potential field.

Since the perturbations in which we are interested are described by two scalar functions, the energy density perturbation and the velocity potential, there are two independent modes of perturbation for each wavelength. In what follows, we will derive the temporal evolution of these two modes.

1.2.1 Growth Rates

Consider a spherical region where the energy density is homogeneously perturbed. Since we are discussing scales $\lambda \gg \lambda_J$, the perturbed region may interact with the rest of the universe only through gravity- the thermal motion of particles leading to energy and momentum transfer (pressure and diffusion effects) may be neglected. According to Birkhoff's theorem, we may ignore the gravitational effects of the universe outside our perturbed sphere, provided that the universe is homogeneous outside the sphere. Although we are discussing the evolution of a non-homogeneous universe, we are considering only small, linear, perturbations. For such perturbations, the gravitational effects of the inhomogeneities outside the sphere on the evolution of the sphere, i.e. the interaction between perturbations, is a second order effect, which may therefore be ignored.

The perturbed sphere would thus evolve as if it were a part of a homogeneous universe, where the energy density is everywhere different than that of the unperturbed homogeneous universe. The evolution of the perturbed region is therefore described by a solution $a(t)$ of the equation (1), with parameters that differ from that of the unperturbed universe. Since we are free to choose the normalization of the expansion factor a, we shall choose a normalization of a for the perturbed solution so that both the perturbed and non perturbed universes have the same density for $a = 1$ (of course, the two solutions may reach $a = 1$ at different times t). With this choice of a normalization, Eq. (1) depends on a single parameter, the curvature term $\alpha_1 = kc^2/R_0^2$. The solution for $a(t)$ depends on one additional parameter, the integration constant of the first order differential equation. It will be useful below to consider time, t to be a function of expansion factor, a, $t(a; \alpha_1, \alpha_2)$, where α_2 is the integration constant,

$$t = \alpha_2 + \int^a \frac{da}{\dot{a}(\alpha_1)}. \tag{7}$$

The energy density has a power law dependence on a, $\rho \propto a^{-m}$ with $m = 3$ for matter domination and $m = 4$ for radiation domination. Since we have normalized a so that the energy density of both (perturbed and unperturbed) solutions is the same for given a , the fractional energy density perturbation related to a perturbation corresponding to modification of the parameters α_i is $\delta \equiv \delta\rho/\rho = -m\delta a/a = -ma^{-1}(\partial a/\partial\alpha_i)\delta\alpha_i$. Note, that the derivative of a with respect to α_i is taken at constant t, since the energy density perturbation is defined as the difference between the density of the two (perturbed and unperturbed) solutions at some fixed time t. The variation of the solution $a(t; \alpha_1, \alpha_2)$ with respect to α_i, $\partial a/\partial\alpha_i$, may be obtained by the following consideration. Taking the partial derivatives with respect to α_i of the *rhs* and *lhs* of Eq. (7) at constant t we find,

$$0 = \frac{\partial t}{\partial\alpha_1} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_1} - \int^a \frac{da}{\dot{a}^2}\frac{\partial\dot{a}}{\partial\alpha_1} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_1} - \int^a \frac{da}{2\dot{a}^3}\,, \tag{8}$$

and

$$0 = \frac{\partial t}{\partial\alpha_2} = \frac{1}{\dot{a}}\frac{\partial a}{\partial\alpha_2} + 1\,. \tag{9}$$

Thus,

$$\frac{\partial a}{\partial \alpha_1} = \dot{a}\int^a \frac{da}{2\dot{a}^3}\,, \quad \frac{\partial a}{\partial \alpha_2} = -\dot{a}, \tag{10}$$

and the growth of the two perturbations modes is given by

$$\delta_1 \propto \frac{\dot{a}}{a}\int^a \frac{da}{2\dot{a}^3}\,, \quad \delta_2 \propto \frac{\dot{a}}{a}. \tag{11}$$

Note, that the evolution of perturbations is independent of the perturbation wavelength, as long as $\lambda \gg \lambda_J$.

At early time, $a \ll 1$, we may approximate $\dot{a}^2 \propto a^{2-m}$, with $m = 3$ for matter domination and $m = 4$ for radiation domination, which implies

$$\delta_1 \propto a^{m-2}\,, \quad \delta_2 \propto a^{-m/2}. \tag{12}$$

Thus, as discussed in §1.1, the perturbation amplitude evolves as a power of a. The growing mode evolves as a^2 during radiation domination, and as a during matter domination.

1.2.2 Amplification Factors

In the next chapter, we will compare the amplitude of perturbations today, inferred from galaxy surveys, to their amplitude at decoupling, inferred from CMB anisotropy. It is therefore useful to derive approximate analytic expressions for the factor by which the perturbations amplitude has grown from decoupling till today, $\delta_1(t_0)/\delta_1(t_{\rm dec.})$. We will consider three different sets of cosmological parameters: flat universe with zero cosmological constant, $\Omega = 1$ and $\Lambda = 0$, low-density open universe with zero cosmological constant, $\Omega \ll 1$ and $\Lambda = 0$, and a flat low-density universe with non-zero cosmological constant, $\Omega \ll 1$ and $\Omega + \Omega_\Lambda = 1$ where $\Omega_\Lambda \equiv (c/H_0)^2\Lambda/3$.

For the $\{\Omega = 1, \Lambda = 0\}$ case we have $\dot{a} \propto a^{-1/2}$ and hence $\delta_1 \propto a$. In this case therefore $\delta_1(t_0)/\delta_1(t_{\rm dec.}) = 1/a_{\rm dec.}$.

For the $\{\Omega \ll 1, \Lambda = 0\}$ case we may approximate Eq. (1) by $\dot{a}^2 = H_0^2$ for $a \gg \Omega$ and by $\dot{a}^2 = H_0^2\Omega a^{-1}$ for $a \ll \Omega$. Assuming $\Omega >> 1/a_{\rm dec.}$, we therefore have

$$\left[\frac{\dot{a}}{a}\int^a \frac{da}{2\dot{a}^3}\right]_{a=a_{\rm dec.}} = \frac{\dot{a}}{a}\int^{a_{\rm dec.}} \frac{da}{(H_0^2\Omega)^{3/2}}a^{3/2} = \frac{2}{5}\frac{a_{\rm dec.}}{H_0^2\Omega}\,, \tag{13}$$

and

$$\left[\frac{\dot{a}}{a}\int^a \frac{da}{2\dot{a}^3}\right]_{a=1} = H_0\int^{\Omega} \frac{da}{(H_0^2\Omega)^{3/2}}a^{3/2} + H_0\int_{\Omega}^{1} da\, H_0^{-3} = H_0^{-2}\left(1 - \frac{3}{5}\Omega\right). \tag{14}$$

For this case we obtain $\delta_1(t_0)/\delta_1(t_{\rm dec.}) = (5/2)a_{\rm dec.}^{-1}\Omega(1-3\Omega/5)$.

For the $\{\Omega \ll 1, \Omega+\Omega_\Lambda = 1\}$ case we may approximate Eq. (1) by $\dot{a}^2 = H_0^2 a^2$ for $a \gg \Omega^{1/3}$ and by $\dot{a}^2 = H_0^2\Omega a^{-1}$ for $a \ll \Omega^{1/3}$. Assuming $\Omega^{1/3} >> 1/a_{\rm dec.}$, we therefore have

$$\left[\frac{\dot{a}}{a}\int^a \frac{da}{2\dot{a}^3}\right]_{a=1} = H_0\int^{\Omega^{1/3}} \frac{da}{(H_0^2\Omega)^{3/2}}a^{3/2} + H_0\int_{\Omega^{1/3}}^1 da\,(aH_0)^{-3} = H_0^{-2}\left(-\frac{1}{2}+\frac{9}{10\Omega^{2/3}}\right). \tag{15}$$

Using Eq. (15) and Eq. (13) we obtain $\delta_1(t_0)/\delta_1(t_{\rm dec.}) = (5/2)a_{\rm dec.}^{-1}\Omega(9\Omega^{-2/3}/10-1/2)$.

We may therefore approximate

$$f_{\rm amp.} \equiv \frac{\delta_1(t_0)}{\delta_1(t_{\rm dec.})} \approx \frac{5}{2}a_{\rm dec.}^{-1}\Omega \times \begin{cases} 1-\frac{3}{5}\Omega, & \text{for } \Lambda = 0; \\ \frac{9}{10}\Omega^{-2/3}-\frac{1}{2}, & \text{for } \Omega \ll 1, \Omega+\Omega_\Lambda = 1. \end{cases} \tag{16}$$

Thus, in a $\{\Omega = 1, \Lambda = 0\}$ universe, perturbations are amplified by gravity by a factor $f_{\rm amp.} = a_{\rm dec.}^{-1} = 1100$ from decoupling till the present. The amplification factor is much smaller for an open low density universe, $\{\Omega \ll 1, \Lambda = 0\}$, for which $f_{\rm amp.} \approx 2.5\Omega a_{\rm dec.}^{-1}$. The dependence of amplification on Ω is weaker for a Λ dominated universe, e.g. for $\{\Omega = 0.1, \Omega_\Lambda = 0.9\}$ we have $f_{\rm amp.} \approx 0.9a_{\rm dec.}^{-1}$, similar to the $\{\Omega = 1, \Lambda = 0\}$ universe. Note, however, that the evolution of perturbations with time, given by Eq. (11), is different for the $\{\Omega = 0.1, \Omega_\Lambda = 0.9\}$ and $\{\Omega = 1, \Omega_\Lambda = 0\}$ universes.

For a universe with $\Omega = \Omega_b \ll 1$, the amplification factor is too small to account for the growth of the perturbations' amplitude from $\delta \sim 10^{-5}$ at decoupling to $\delta \sim 1$ today. This is discussed in detail in the next chapter. The accepted solution to this discrepancy is the presence of "Dark Matter", an unknown particle which does not interact electromagnetically and dominates the matter density, so that $\Omega > \Omega_b$. The effect of the presence of such particles on the growth of perturbations is discussed in § 1.4.

1.3 Pressure effects: oscillations

1.3.1 The evolution at scales $\lambda < \lambda_H$ during matter domination

Let us consider sub-horizon scale, $\lambda \ll \lambda_H$, adiabatic perturbations during matter domination, $a > a_{\rm eq.}$. For this regime, we may use the Newtonian approximation,

$$(\partial_t + \mathbf{v}.\nabla)\rho + \rho\nabla\mathbf{v} = 0, \tag{17}$$

$$(\partial_t + \mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\phi. \tag{18}$$

The unperturbed homogeneous solution is $\mathbf{v}_0 = H(t)\mathbf{r}$ and $\rho = \rho_0(t)$. We are interested in compressible adiabatic perturbations. These are described by two functions, which we will choose to be the perturbations in the velocity divergence and in the density. The equation describing the evolution of the velocity divergence may be obtained by taking the divergence of the second equation above,

$$\partial_t(\partial_i v_i) + v_j\partial_j(\partial_i v_i) + (\partial_i v_j)(\partial_j v_i) = -\partial_i(\rho^{-1}\partial_i p) - \partial_i^2\Phi, \qquad (19)$$

which may be written as

$$(\partial_t + v_j\partial_j)(3H + \partial_i\delta v_i) + (H\delta_{ij} + \partial_i\delta v_j)(H\delta_{ij} + \partial_j\delta v_i) = -\partial_i(\rho^{-1}\partial_i p) - \partial_i^2\Phi. \qquad (20)$$

Taking the first order terms only we have

$$\delta\dot{H} + 2H\delta H = -\frac{1}{3}\rho_0^{-1}\partial_i^2\delta p - \frac{1}{3}\partial_i^2\delta\Phi = -\frac{1}{3}\rho_0^{-1}c_s^2\delta\rho - \frac{4\pi G}{3}\delta\rho. \qquad (21)$$

Here, we have defined $\partial_i\delta v_i = 3\delta H$, the full time derivative stands for derivation along the fluid's trajectory, $d/dt = (\partial_t + v_j\partial_j)$, and $\delta p = (\partial p/\partial_\rho)_s\delta\rho = c_s^2\delta\rho$.

The first order part of the continuity equation is

$$\dot{\delta\rho} + 3H\delta\rho + 3\rho_0\delta H = 0. \qquad (22)$$

Defining the perturbation amplitude $\delta = \delta\rho/\rho_0$, this yields

$$\dot{\delta} = -3\delta H. \qquad (23)$$

Using this result in eq. (21) we finally have

$$\ddot{\delta} + 2H\dot{\delta} = c_s^2\nabla^2\delta + 4\pi G\rho_0\delta. \qquad (24)$$

The first term on the RHS represents the effect of pressure, and the second represents the effect of gravity. In the absence of expansion, $H = 0$, and pressure, perturbations grow exponentially on the free-fall time scale $1/\sqrt{G\rho_0}$. In the presence of expansion, and neglecting pressure, the perturbations evolve differently with time. For $a \ll 1$ we have $H^2 = 8\pi G\rho_0/3$ and $H = 2/3t$, and the perturbations evolve as a power of time, $\delta \propto t^x$, with $x = -1$ (decaying mode, $\delta \propto a^{-3/2}$) or $x = 2/3$ (growing mode, $\delta \propto a$). This is consistent with the result of eq. (12). The two modes differ in the relationship between the velocity and density perturbations, see eq. (23).

The pressure effects are negligible for perturbations on scales $\lambda_{\rm prop.} \gg \lambda_{J,\rm prop.} = c_s/H$, since the pressure term is of order $(c_s/\lambda_{\rm prop.})^2\delta$ while the gravity and expansion terms are of order $H^2\delta$. The pressure term dominates at $\lambda \ll \lambda_J$, where perturbations oscillate at a frequency $c_s/\lambda_{\rm prop.}$.

1.3.2 The evolution at scales $\lambda < \lambda_H$ during radiation domination

See Tutorials & Problem sets.

1.4 Dark Matter

The evidence from gravitationally bound systems for the existence of "Dark Matter", matter component that we can not observe through its electromagnetic interaction (emission/absorption of light) and the existence of which is inferred from its gravitational effects only, was discussed in the introductory lectures. Based on observations of galaxies and of clusters of galaxies, it is inferred that the density of dark matter in the universe exceeds that of luminous matter, that we can see via its electromagnetic interaction, by a factor of ~ 5 to one. As we have discussed, this Dark Matter is not necessarily non-baryonic. It may be ordinary, baryonic, matter in a form that is hard to observe electromagnetically (rarified hot gas which has little emission and absorption, low mass stars that have very low luminosity, etc.). In this section, however, we will assume that the dark matter is non-baryonic, and consider its effect on the evolution of perturbations. We shall assume that the dark matter particles have decoupled from the rest of the universe plasma at $a \ll a_{\rm eq.}$ while still relativistic, that following decoupling their interaction (between themselves and with the rest of the plasma) is only through gravity, and that by the time of matter-radiation equality they are sub-relativistic, i.e. that they become non-relativistic at $a = a_{NR} < a_{\rm eq.}$. As we show below, the dark matter particles must be non-relativistic at matter-radiation equality, $a_{NR} < a_{\rm eq.}$, in order to resolve the problem of the too small perturbation amplification factor obtained for a universe with $\Omega - \Omega_b$ (see § 1.2.2).

Since the indication from large-scale structure is that the dark matter density is much higher than that of ordinary matter, we will assume $\Omega \gg \Omega_b$.

1.4.1 Evolution of Perturbations

During radiation domination, the evolution of perturbations is governed by the radiation. Matter, both dark and not, follows the evolution of perturbations in the radiation energy density, which determine the gravitational potential. At $a > a_{\rm eq.}$ the evolution of perturbations is governed by matter. Our discussion, in the previous section, of the evolution perturbations on scales $\lambda > \lambda_J \simeq \lambda_{\rm eq.}$ holds for both ordinary and dark matter: This evolution is governed by gravity only, and hence it does not differentiate between dark and ordinary matter.

Differences in the evolution of perturbations at $a > a_{\rm eq.}$ due to the presence of dark-matter may arise on small scales $\lambda \leq \lambda_J \simeq \lambda_{\rm eq.}$. The Jeans scale is determined, at $a < a_{\rm dec.}$, by the radiation pressure. This pressure prevents perturbations in the baryonic matter component from growing on scales $\lambda \leq \lambda_J \simeq \lambda_{\rm eq.}$. On such scales the baryon density perturbations oscillate during $a_{\rm eq.} < a < a_{\rm dec.}$. Since dark matter particles are not coupled to the radiation they do not "feel" the pressure. Thus, if the density of dark matter is much higher than that of baryonic matter, $\Omega \gg \Omega_b$, then the baryonic density oscillations will have only a little effect on the dark matter distribution. In this case, perturbations in the dark matter component on all scales, including $\lambda \leq \lambda_{\rm eq.}$, will continue to grow, $\delta \propto a$, during $a_{\rm eq.} < a < a_{\rm dec.}$. After decoupling, the baryon Jeans scale drops by many orders of magnitude, the radiation pressure no longer supports the baryons and they will therefore fall into the gravitational potential wells created by the dark matter. Thus, in the presence of dark matter, perturbations on scales $\lambda \leq \lambda_{\rm eq.}$ are amplified by a factor $a_{\rm dec.}/a_{\rm eq.} = 20\Omega h_{75}^2$ beyond their amplification in the absence of dark matter.

Although dark matter particles do not "feel" the radiation pressure, there are other processes that suppress the growth of dark matter perturbations on small scales. Obviously, such effects may be important only on scales $\lambda \ll \lambda_H$, for which, at $a > a_{\rm eq.}$, we may use the Newtonian approximation. For simplicity, we will neglect the expansion of homogeneous universe in the derivation below of the gravitational evolution of dark matter perturbations. It is straightforward to generalize the derivation to include such expansion of the background solution, and the conclusions are unchanged.

Let $f(\mathbf{x}, \mathbf{v}, t)$ be the distribution function of dark matter particles, so that $f(\mathbf{x}, \mathbf{v}, t)d^3xd^3v$ gives the number of particles in the infinitesimal phase space element $d^6q \equiv d^3xd^3v$ positioned at $\mathbf{q} \equiv \{\mathbf{x}, \mathbf{v}\}$. Conservation of the number of particles implies (continuity equation in $\mathbf{q}$ space)

$$\partial_t f + \nabla_{\mathbf{q}}(\dot{\mathbf{q}} f) = \partial_t f + \nabla_{\mathbf{x}}(\dot{\mathbf{x}} f) + \nabla_{\mathbf{v}}(\dot{\mathbf{v}} f) = 0. \tag{25}$$

Since the particles respond only to gravity, the acceleration $\dot{\mathbf{v}}$ is not a function of the velocity, $\dot{\mathbf{v}} = -\nabla\Phi(\mathbf{x})$, where Φ is the gravitational potential. Thus, eq. (25) may be written as

$$\partial_t f + \mathbf{v}.\nabla f - \nabla\Phi.\nabla_{\mathbf{v}} f = 0. \tag{26}$$

The unperturbed distribution function is homogeneous in space, i.e. independent of $\mathbf{x}$. In the Newtonian approximation, where the system must be infinite in size, the gravitational potential Φ can not be determined. However, as mentioned in the introductory lectures, Birkhoff's theorem for general relativity implies that we may use $\Phi = 0$ for the unperturbed solution. We denote the steady homogeneous distribution by $f_0(v)$. We assume that unperturbed velocity distribution is isotropic, f_0 depends only on $v \equiv |\mathbf{v}|$, since there is no preferred direction in space.

Let us now add a small perturbation to the dark matter distribution, $f = f_0 + f_1$. The evolution of the perturbation is described by

$$\partial_t f_1 + \mathbf{v}.\nabla f_1 - \nabla\Phi.\nabla_{\mathbf{v}} f_0 = 0, \tag{27}$$

with

$$\nabla^2\Phi = 4\pi G\delta\rho\,, \quad \delta\rho \equiv m\int d^3v\, f_1(\mathbf{x},\mathbf{v},t). \tag{28}$$

We have neglected the term $\nabla\Phi.\nabla_{\mathbf{v}} f_1$ in Eq. (27) since it is of second order, and denoted in Eq. (28) the mass of each dark matter particle by m.

Since the equations are linear, we look for solutions of the form $f_1 \propto \exp i(\mathbf{k}.\mathbf{x} - \omega t)$. For such solutions, Eq. (27) gives

$$-i\omega f_1 + i\mathbf{v}.\mathbf{k} f_1 - i\Phi\mathbf{k}.\mathbf{v}\frac{1}{v}\frac{df_0}{dv} = 0\,, \tag{29}$$

where we have used $\nabla_{\mathbf{v}} f_0(v) = v^{-1}\mathbf{v}(df_0/dv)$, and Eq. (28) gives

$$\Phi = -k^{-2}4\pi Gm\delta\rho. \tag{30}$$

replacing Φ in Eq. (29) with the expression given by Eq. (30), and integrating over velocities we find

$$\left[1 - \frac{4\pi Gm}{k^2}\int d^3v\,\frac{\mathbf{k}.\mathbf{v}}{\omega - \mathbf{k}.\mathbf{v}}\frac{1}{v}\frac{df_0}{dv}\right]\delta\rho = 0. \tag{31}$$

Non trivial solutions are obtained therefore only for

$$1 - \frac{4\pi Gm}{k^2}\int d^3v\,\frac{\mathbf{k}.\mathbf{v}}{\omega - \mathbf{k}.\mathbf{v}}\frac{1}{v}\frac{df_0}{dv} = 0. \tag{32}$$

This is a dispersion equation, which determines a dispersion relation $\omega(\mathbf{k})$ for which the equality is satisfied.

Let us first obtain the long wavelength, $k \to 0$, limit of the dispersion relation $\omega(\mathbf{k})$ imposed by Eq. (32). We shall assume that in this limit $\omega/k \to \infty$, so that $\mathbf{k}.\mathbf{v}/\omega \ll 1$ and may be considered a small parameter. This approximation will hold for wave numbers $k \ll \omega/v_0$ where v_0 is the characteristic velocity of the dark matter particle distribution. Keeping terms up to second order in $\mathbf{k}.\mathbf{v}/\omega$, Eq. (32) may be written as

$$1-\frac{4\pi Gm}{k^2\omega^2}\int d^3v\,\omega^2\frac{\mathbf{k}.\mathbf{v}}{\omega}\left(1+\frac{\mathbf{k}.\mathbf{v}}{\omega}\right)\frac{1}{v}\frac{df_0}{dv}=0. \tag{33}$$

The term linear in $\mathbf{k}.\mathbf{v}/\omega$ vanishes upon integration, since $k\ll\omega/v_0$ chances signs under $\mathbf{v}\to-\mathbf{v}$ (while the rest of the integrand is independent of $\mathbf{v}$ direction). It is straightforward to show (using integration by parts) that the integral of the $(\mathbf{k}.\mathbf{v}/\omega)^2$ term, $\int d^3v\,(\mathbf{k}.\mathbf{v})^2v^{-1}(df_0/dv)$ gives $-k^2\int d^3v\,f_0$, leading to the dispersion relation $\omega^2=-4\pi G\rho$. Note, that here ρ is the mass, rather then the energy, density.

In the long wavelength limit, perturbations grow therefore on a time scale $2\pi/i\omega=\sqrt{\pi/G\rho}$, independent of perturbation scale. This result has a straight forward interpretation. Consider a sphere of radius R, mass M and mass density ρ. Since the acceleration of a point at the edge of the sphere is $g=GM/R^2$, the sphere would undergo gravitational collapse on a time scale $\approx\sqrt{2R/g}=\sqrt{2R^3/GM}=\sqrt{3/2\pi G\rho}$. $1/\sqrt{G\rho}$ is usually termed the "free fall time", being the characteristic time scale for collapse under gravitation in the absence of processes (e.g. pressure) that resist the collapse.

The long wavelength result we obtained is consistent with the general results obtained in §1.2. The growth of perturbations in the matter dominated universe at early times is described by $\delta\propto a$, implying that the characteristic time for perturbation growth is $a/\dot{a}$. Using the approximation $\dot{a}^2=H_0^2\Omega a^{-1}$ and recalling that $H_0^2\Omega=H_0^2(\rho a^3/\rho_c)=(8\pi G/3)\rho a^3$, we find $a/\dot{a}=\sqrt{3/8\pi G\rho}$ (here too, ρ is the mass density).

The result $\omega^2=-4\pi G\rho$ holds for long wavelength. Let us estimate the wavelength scale where this gravitational instability is suppressed. We may obtain such estimate by finding the wavelength k_c for which $\omega=0$. Using Eq. (32), we find

$$k_c^2=-4\pi Gm\int d^3v\,\frac{1}{v}\frac{df_0}{dv}=4\pi Gm\int d^3v\,\frac{1}{v^2}f_0=4\pi G\rho\overline{v^{-2}}. \tag{34}$$

Here, the over-bar denotes average over the distribution function f_0. The corresponding wavelength is

$$\lambda_c\equiv\frac{2\pi}{k_c}=\sqrt{\frac{\pi}{G\rho}}\,(\overline{v^{-2}})^{-1/2}. \tag{35}$$

This result is similar to that obtained for normal matter. The Jeans scale is given by the product of the perturbation growth time and the speed of sound of the fluid (which is comparable to the thermal speed of fluid particles). Here, the scale below which perturbation growth is suppressed is given by the product of the perturbation growth time and the characteristic dispersion in the dark matter particle velocities, defined as $(\overline{v^{-2}})^{-1/2}$.

1.4.2 "Free Streaming"

Contrary to the effects of pressure, that lead to oscillations, the velocity dispersion of dark matter particles lead to suppression of perturbations. On scales $\lambda < \lambda_c$, dark matter particles propagate a distance larger than λ on perturbation growth time scale (this follows from the discussion at the end of the previous sub-section). During matter domination, where gravitational perturbation growth is driven by dark matter, this "free streaming" of particles will "smear out" and erase the perturbations, much like the photon diffusion suppression of perturbations on scales $\lambda < \lambda_D$ during radiation domination.

We therefore define the free streaming scale as the distance that dark matter particles propagate on a gravitational perturbation growth time scale, $\lambda_{FS} \equiv a^{-1}(a/\dot{a})v = v/\dot{a}$, where v is the characteristic velocity dispersion of dark matter particles. λ_{FS} evolves with time as $\lambda_{FS} \propto va^{m/2-1}$, with $m = 4$ for radiation domination, and $m = 3$ for matter domination. As long as the dark matter particles are relativistic, $v \approx c$, λ_{FS} increases with time, $\lambda_{FS} \propto a^1$ (recall that we have assumed that the dark matter particles become non-relativistic at $a = a_{NR} < a_{\text{dec.}}$). For $a > a_{NR}$, the redshift of dark matter particle momentum implies $v \propto 1/a$, and hence $\lambda_{FS} \propto a^{m/2-2}$. The free streaming scale is constant at $a_{NR} < a < a_{\text{eq.}}$, and decrease afterwards. Thus, free streaming of dark matter particles will erase perturbations on scales $\lambda < \lambda_{FS}(a = a_{NR}) = c/\dot{a}(a = a_{NR})$.

Dark matter particles become non-relativistic when their temperature, T_{DM}, becomes smaller than the rest mass, m. We will define a_{NR} to be the value of the scale factor when $T_{DM} = mc^2/3$. As long as dark matter particles are coupled to radiation, their temperature is the same as that of the radiation, $T_{DM} = T_\gamma$. After their decoupling, the momentum redshift of dark matter particles implies $T_{DM} \propto a^{-1}$, as long as the particles are relativistic. T_{DM} would therefore follow T_γ for $a < a_{NR}$. Since, however, the entropy of the photon gas may be increased after the decoupling of dark matter, T_γ may be increased above the a^{-1} decline, leading to $T_\gamma > T_{DM}$ (this is similar to the process leading to the neutrino background temperature being lower than T_γ).

In order to determine $T_{DM}(a)$, we may use the following consideration.

In the absence of processes increasing the photon entropy, the ratio of photon and dark matter number densities, n_{DM}/n_γ, does not change after photon and dark matter decoupling, since both densities drop as a^{-3}. Let us assume that the photon temperature is increased after photon-dark matter decoupling, due to entropy increase of the photons (e.g. electron-positron annihilation), by a factor x above the value it would have had in the absence of entropy change. The number density of photons, $n_\gamma \propto T_\gamma^3$, is increased in this case by a factor x^3, and the present ratio of n_{DM}/n_γ, $n_{DM,0}/n_{\gamma,0}$, is smaller than the ratio at photon-dark matter decoupling by a factor x^3. As long as dark matter and radiation are coupled, $T_{DM} = T_\gamma = T$, the number densities of the photons and dark matter particles are similar, both given by $\approx (T/hc)^3$. Thus, the value of the photon temperature increase x is approximately given by $x \approx (n_{DM,0}/n_{\gamma,0})^{-1/3}$. Approximating $n_{DM,0} = \Omega\rho_c/mc^2$, where m is the dark matter particle mass and we have neglected the contribution of baryonic matter to Ω, we find

$$x \approx 4\left(\frac{m_{\rm keV}}{\Omega h_{75}^2}\right)^{1/3}, \tag{36}$$

where $m_{\rm keV}$ is the particle mass in keV units, $mc^2 = 1 m_{\rm keV}$ keV.

The dark matter temperature at times later than photon-dark matter decoupling is smaller by a factor x than the extrapolation of the present CMB temperature to that time, i.e. $T_{DM}(a) = a^{-1}T_{\gamma,0}/x$, where $T_{\gamma,0}$ is the present CMB temperature. Thus,

$$a_{NR} \approx 2 \times 10^{-7} m_{\rm keV}^{-4/3} (\Omega h_{75}^2)^{1/3}\,. \tag{37}$$

Using the approximation $\dot{a}^2 = H_0^2 \Omega a_{\rm eq.} a^{-2}$ we therefore have

$$\lambda_{FS,NR} \equiv \lambda_{FS}(a = a_{NR}) = \frac{c}{\dot{a}(a = a_{NR})} \approx 0.1 m_{\rm keV}^{-4/3} (\Omega h_{75}^2)^{1/3}\,{\rm Mpc}. \tag{38}$$

1.5 The Resulting Perturbation Spectrum

In order to discuss perturbations on different length scales, it is convenient to decompose the perturbation $\delta(\mathbf{x})$ into its Fourier components, $\delta_\mathbf{k}$, where $\delta_\mathbf{k}$ is the amplitude of the component $\exp(i\mathbf{k}.\mathbf{x})$. The power in the $\mathbf{k}$ mode is defined as $P(\mathbf{k}) \equiv |\delta_\mathbf{k}|^2$. Let us assume that at some initial time, when $a = a_i$, small deviations from homogeneity have been introduced by some process. We assume that the process was random, i.e. that the amplitudes $\delta_\mathbf{k}$ were randomly drawn from some given distribution. Since we assume

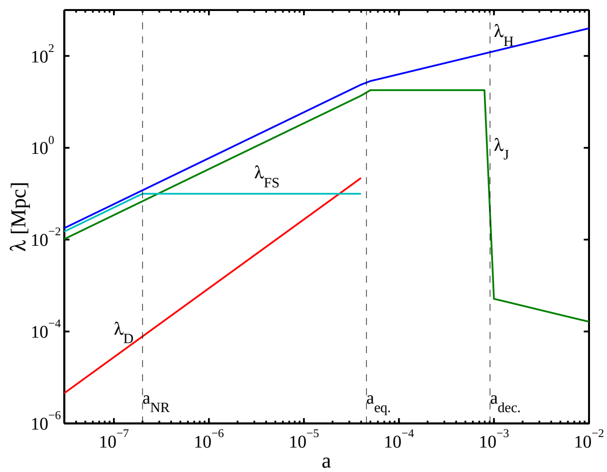

Figure 1: Characteristic length scales relevant for perturbation evolution. The values of cosmological parameters used in this figure are $\Omega = 1$, $h_{75} = 1$, $\Omega_b = 0.05$, and $mc^2 = 1$ keV. Scalings of various length-scales with the parameter values are given in Eqs. (2,5,6,38).

that there is no preferred direction in the universe, the average (taken over realizations of the random process) of the power spectrum, $< P(\mathbf{k}) >$, should depend only on the magnitude of $\mathbf{k}$ and not on its direction, $< P(\mathbf{k}) >= P_i(k)$. Since the number of modes in a logarithmic interval of k is $\propto k^3$, the total power of fluctuations on scales $\sim k$ is $\propto k^3 P(k)$. Thus, it is customary to use $[k^3 P(k)]^{1/2}$ as a measure for the amplitude of fluctuations on scales $\lambda \sim 2\pi/k$.

The results of previous sections allow us to determine the evolution of perturbations on various scales. In this section, will use these results to determine the present perturbations spectrum, $P(k)$, given the initial spectrum $P_i(k)$ at early times. Since we have shown that following decoupling perturbations are amplified, up to any given time $t > t_{\rm dec.}$, by a scale independent factor (the value of which depends on cosmology), the evolution following decoupling does not change the shape of the perturbation spectrum. We will therefore focus in this section on the evolution up to decoupling. Fig. 1, where the evolution is shown of various characteristic length scales relevant for our problem, may be useful for following the discussion below.

Perturbations on small scales are suppressed by the diffusion of photons and free streaming of dark-matter particles. Thus, we approximate $P_0 = 0$ on scales $\lambda < \max(\lambda_{D,{\rm eq.}}, \lambda_{FS,NR})$. Of course, if dark matter is absent, suppression is due to photon diffusion only.

Perturbations on large scale, of wavelength $\lambda > \lambda_{\rm eq.}$, grow as a^2 during radiation domination and as a^1 at $a \ll 1$ during matter domination. Since we are interested in the evolution for $a < a_{\rm dec.} \ll 1$, we may use the approximation $\delta \propto a$ for the matter dominated phase. Thus, perturbations of wavelength $\lambda > \lambda_{\rm eq.}$ are amplified by a factor $(a_{\rm eq.}/a_i)^2(a_{\rm dec.}/a_{\rm eq.})$ up to $a = a_{\rm dec.}$.

Let us now consider wavelengths $\max(\lambda_{FS,NR}, \lambda_{D,\rm eq.}) < \lambda < \lambda_{\rm eq.}$. Perturbations on these scales "enter the horizon", i.e. their wavelength becomes equal to λ_H, at $a = a_{\rm ent.} < a_{\rm eq.}$. After the perturbation enters the horizon, it does not grow during the radiation dominated phase, since the Jeans scale at this time is comparable to the horizon size. Thus, these perturbations are suppressed, compared to perturbations on larger scales $\lambda > \lambda_{\rm eq.}$, by a factor $(a_{\rm eq.}/a_{\rm ent.})^2$ by the time the scale factor reaches $a = a_{\rm eq.}$. During the time $a_{\rm eq.} < a < a_{\rm dec.}$, the evolution of such perturbations depends on whether or not Ω is dominated by dark matter. If Ω is dominated by dark matter, dark matter perturbations grow linearly with a during $a_{\rm eq.} < a < a_{\rm dec.}$, and although the baryon perturbations on scales $\lambda < \lambda_{\rm eq.}$ do not grow, but rather oscillate, during this time, the baryons fall into the dark-matter gravitational potential wells after decoupling (see §1.4). Thus, in the presence of a dominant dark matter component, perturbations on scales $\max(\lambda_{FS,NR}, \lambda_{D,\rm eq.}) < \lambda < \lambda_{\rm eq.}$ are suppressed, compared to perturbations on larger scales, by a factor $(a_{\rm eq.}/a_{\rm ent.})^2$ up to $a = a_{\rm dec.}$.

In the absence of dark-matter, baryon perturbations on scales $\lambda_{D,\rm eq.} < \lambda < \lambda_{\rm eq.}$ do not grow during $a_{\rm eq.} < a < a_{\rm dec.}$, and are therefore suppressed by an additional factor $a_{\rm dec.}/a_{\rm eq.}$. In this case, therefore, perturbations on scales $\lambda_{D,\rm eq.} < \lambda < \lambda_{\rm eq.}$ are suppressed by a factor $(a_{\rm eq.}/a_{\rm ent.})^2(a_{\rm dec.}/a_{\rm eq.})$ up to $a = a_{\rm dec.}$.

Since $\lambda_H \propto a$ for $a < a_{\rm eq.}$, $a_{\rm ent.} \propto \lambda$, i.e. $a_{\rm ent.}(\lambda) = a_{\rm eq.}(\lambda/\lambda_{\rm eq.})$. Thus, in the presence of dark-matter perturbations on scales $\max(\lambda_{FS,NR}, \lambda_{D,\rm eq.}) < \lambda < \lambda_{\rm eq.}$ are suppressed, compared to perturbations on larger scales, by a factor $(\lambda/\lambda_{\rm eq.})^{-2}$. In the absence of dark matter, perturbations on scales $\lambda_{D,\rm eq.} < \lambda < \lambda_{\rm eq.}$ are suppressed by a factor $(\lambda/\lambda_{\rm eq.})^{-2}(a_{\rm dec.}/a_{\rm eq.})$.

Let as consider, for example, a power law initial spectrum, $P_i(k) \propto k^n$. Following the above analysis, the evolved spectrum after decoupling would be

$$\left[k^3 P(k)\right]^{1/2} \propto \begin{cases} k^{(n+3)/2}, & \text{for } 2\pi/k > \lambda_{\rm eq.}; \\ k^{(n-1)/2}, & \text{for } \max(\lambda_{FS,NR}, \lambda_{D,\rm eq.}) < 2\pi/k < \lambda_{\rm eq.}; \\ 0, & \text{for } 2\pi/k < \max(\lambda_{FS,NR}, \lambda_{D,\rm eq.}). \end{cases} \quad (39)$$

In the presence of dark matter, the normalization of the intermediate wavelength spectrum, $\propto k^{(n-1)/2}$ is such that it matches the long wavelength component, $\propto k^{(n+3)/2}$, at $2\pi/k = \lambda_{\rm eq.}$. In the absence of dark-matter, the normalization of the intermediate component is lower by a factor $(a_{\rm dec.}/a_{\rm eq.})$. It should be noted that in reality there would be a smooth transition of $P(k)$ between the functional forms given in Eq. (39) for the three wavelength regimes.

2 CMB & LSS

The signatures of the inhomogeneities in the Universe at the time of decoupling are imprinted on the cosmic microwave background (CMB) radiation. In this chapter we will show that a comparison of the CMB anisotropy with the observed inhomogeneities in the present universe, i.e. with the large-scale structure (LSS) of the distribution of galaxies, provides, when combined with the results of the previous chapter on the evolution of the spectrum and amplitude of inhomogeneities, stringent constraints on the cosmological model.

The plan of this chapter is as follows. In the first section, we will derive the relation between observed LSS and the power spectrum $P(k)$. In §2.2 we will discuss the anisotropy imprinted on the CMB by inhomogeneities, and in §2.3 we will discuss the constraints imposed by observations on the cosmological model.

2.1 LSS and P(k)

Consider the observed distribution of galaxies at the present, $z = 0$ epoch. A common measure for the level of inhomogeneity on a scale λ is the variance, σ^2_λ, of the fractional mass fluctuations in spheres of radius λ. The fractional mass fluctuation in a sphere of radius λ centered at $\mathbf{x}$ is

$$\left[\frac{\delta M(\mathbf{x})}{< M >}\right]_\lambda \equiv \left(\frac{4\pi}{3}\lambda^3\right)^{-1} \int_{|\mathbf{x}'|<\lambda} d^3x' \, \delta(\mathbf{x}+\mathbf{x}') \quad , \tag{40}$$

and σ^2_λ is thus

$$\sigma^2_\lambda \equiv V^{-1} \int d^3x \, \left[\left(\frac{\delta M(\mathbf{x})}{< M >}\right)_\lambda\right]^2 \quad , \tag{41}$$

where V is the observed volume over which the spatial integration is taken.

In order to relate σ^2_λ to the power-spectrum $P(k)$, let us decompose δ to its Fourier modes,

$$\delta(\mathbf{x}) = \sum_{\mathbf{k}} \delta_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}. \tag{42}$$

σ^2_λ is given by

$$\sigma^2_\lambda = V^{-1} \int d^3x \left(\frac{4\pi}{3}\lambda^3\right)^{-2} \int_{|\mathbf{x}'|<\lambda} d^3x' \int_{|\mathbf{x}''|<\lambda} d^3x'' \sum_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{k}}\delta_{\mathbf{k}'} e^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}} e^{i(\mathbf{k}.\mathbf{x}'+\mathbf{k}'.\mathbf{x}'')} \quad . \tag{43}$$

The integral over $\mathbf{x}$ vanishes unless $\mathbf{k}+\mathbf{k}'=0$. Thus,

$$\sigma_\lambda^2=\sum_{\mathbf{k}}|\delta_{\mathbf{k}}|^2\left(\frac{4\pi}{3}\lambda^3\right)^{-2}\int_{|\mathbf{x}'|<\lambda}d^3x'\int_{|\mathbf{x}''|<\lambda}d^3x''\,e^{i\mathbf{k}.(\mathbf{x}'-\mathbf{x}'')} \quad . \tag{44}$$

For $k\ll 2\pi/\lambda$, the integrand, $e^{i\mathbf{k}.(\mathbf{x}'-\mathbf{x}'')}$, is close to unity over the integration volume, while for $k\gg 2\pi/\lambda$ the integrand oscillates rapidly within the integration volume. We may therefore approximate

$$\sigma_\lambda^2\simeq\sum_{|\mathbf{k}|<2\pi/\lambda}P(\mathbf{k}) \quad . \tag{45}$$

As mentioned in the last section of the previous chapter, the average of the power spectrum taken over realizations of the random process generating the perturbations, $<P(\mathbf{k})>$, should depend only on the magnitude of $\mathbf{k}$ and not on its direction. The number of modes within a sphere $k<k_0$ is proportional to k_0^3. Thus, if $<P(\mathbf{k})>\equiv P(k)$ does not decrease with k faster the k^{-3}, the sum in the last equation is dominated by the largest k's, i.e.

$$\sigma_\lambda^2\simeq\left[\frac{4\pi}{3}k^3P(k)\right]_{k=2\pi/\lambda} \quad . \tag{46}$$

One, somewhat subtle, point should be clarified here. $P(k)$ is the average of $P(\mathbf{k})$ over realizations of the random process generating the perturbations. In obtaining the last equation, we have assumed that the average of $P(\mathbf{k})$ over a sphere of fixed $|\mathbf{k}|$ in the universe we observe, i.e. *in a given realization of the random process*, is equal to the average over realizations $P(k)$. This assumption is valid for wavelengths λ much smaller than the linear scale L of the observed volume, provided the random process is such that modes with different $\mathbf{k}$'s are independent. In this case, there is a very large number of modes, $\sim(L/\lambda)^3$, over a sphere of fixed $|\mathbf{k}|\sim 1/\lambda$, and since these mode are independent, the average of $P(\mathbf{k})$ over the sphere is close to $P(k)$. It should be kept in mind, however, that even for random processes that satisfy the condition, that modes with different $\mathbf{k}$'s are independent, the spherical average of $P(\mathbf{k})$ may differ significantly from $P(k)$ for $\lambda\sim L$ due to the poor statistics, i.e. due to the small number of modes over which the average is taken.

Finally, observations of LSS imply $\sigma^2_{\lambda=8h^{-1}\mathrm{Mpc}}\approx 1$. Here h is the Hubble constant in units of 100km/Mps s (since distances are measured from redshift using Hubble's law, the absolute distance scale is inversely proportional to h).

2.2 The CMB anisotropy

2.2.1 Temperature anisotropy

Due to inhomogeneities in the universe at decoupling, the CMB temperature is different when observed in different directions on the sky. The temperature differences result from several different processes.

Variations in the radiation energy density, $\delta\rho_R$, imply variations in the radiation temperature, $\delta T/T = \frac{1}{4}\delta\rho_R/\rho_R$ since $\rho_R \propto T^4$. As long as radiation and matter are coupled, the entropy of the universe fluid is proportional to T^3/ρ_B, where ρ_B is the baryon density. For adiabatic perturbations, i.e. perturbations where the fluid entropy is everywhere unchanged from its homogeneous (unperturbed) value, the temperature perturbation is therefore related to the fractional perturbation in the baryon density, δ_B, through

$$\frac{\delta T}{T} = \frac{1}{3}\delta_B\,. \tag{47}$$

As explained in the previous chapter, the entropy of fluid elements is conserved during perturbations on scales $\lambda > \lambda_D$, and the evolution of growing modes on scales $\lambda \gg \lambda_J$ is independent of the entropy. The temperature perturbation of the growing modes is therefore given by Eq. (47). It should be noted, however, that if the mechanism generating the initial inhomogeneities produces non-adiabatic perturbations, then the relation between $\delta T/T$ and δ_B may be different than given in Eq. (47).

Another effect that creates temperature inhomogeneities is related to the fluctuations of the gravitational potential. Photons "climbing out" of over-dense regions suffer a higher than average redshift on their way to us, since the gravitational potential well of the over-dense region is deeper than average, and will therefore be "colder" than average. The modification of the gravitational potential due to mass fluctuation δM on (proper) scale $\lambda_{\rm prop.}$ is $\delta\phi \simeq 2G\delta M/\lambda_{\rm prop.} \simeq (8\pi G\rho/3)\lambda^2_{\rm prop.}\delta \simeq (H\lambda_{\rm prop.})^2\delta$, where $H \equiv \dot{a}/a$ is the Hubble "constant". The fractional temperature change due to the redshift is thus $\delta T/T \sim \delta\phi/c^2 \simeq (H\lambda_{\rm prop.}/c)^2\delta = (\lambda/\lambda_{\rm dec.})^2\delta$, where $\lambda = a^{-1}_{\rm dec.}\lambda_{\rm prop.}$ is the comoving wavelength and we have used $a^{-1}_{\rm dec.}(c/H) = \lambda_{\rm dec.}$. Thus, on scales $\lambda > \lambda_{\rm dec.}$, the gravitational redshift effect,

$$\left(\frac{\delta T}{T}\right)_\phi \sim \left(\frac{\lambda}{\lambda_{\rm dec.}}\right)^2\delta\,, \tag{48}$$

dominates over the intrinsic temperature change given by Eq. (47).

Finally, temperature variations will also result due to the peculiar velocity, i.e. due to the variation of velocity from the Hubble flow, associated with the perturbations, via the Doppler effect, $\delta T/T = \delta v/c$. Using eq. (23) we find $3\delta v/\lambda_{\rm prop.} \approx \dot{\delta} = H\delta$, implying

$$\left(\frac{\delta T}{T}\right)_v \sim \frac{1}{3}\left(\frac{\lambda}{\lambda_{\rm dec.}}\right)\delta\,. \tag{49}$$

The Doppler effect is thus dominated by the potential effect on large scales, and by the intrinsic variations on small scales.

2.2.2 The power spectrum

Let us consider the correlation function of the fractional temperature perturbations on scale λ, defined as

$$\xi_{T,\lambda} \equiv V^{-1}\int d^3x\,\frac{\delta T(\mathbf{x}+\lambda\hat{\mathbf{e}})\delta T(\mathbf{x})}{<T>^2} \quad . \tag{50}$$

Here $\hat{\mathbf{e}}$ is an arbitrary unit vector. The intrinsic contribution, due to changes in the radiation energy density, to ξ_T is given (for adiabatic perturbations) by

$$\xi_{T,\lambda} = \frac{1}{9}V^{-1}\int d^3x\,\delta_B(\mathbf{x}+\lambda\hat{\mathbf{e}})\delta_B(\mathbf{x}) \quad . \tag{51}$$

Decomposing δ_B into Fourier modes, we have

$$\begin{aligned}\xi_{T,\lambda} &= \frac{1}{9}V^{-1}\int d^3x \sum_{\mathbf{k},\mathbf{k}'}\delta_{B,\mathbf{k}}\delta_{B,\mathbf{k}'}e^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}}e^{i\mathbf{k}.\hat{\mathbf{e}}\lambda}\\ &= \frac{1}{9}\sum_{\mathbf{k}}|\delta_{B,\mathbf{k}}|^2 e^{i\mathbf{k}.\hat{\mathbf{e}}\lambda} \quad .\end{aligned} \tag{52}$$

For $k \ll 2\pi/\lambda$, the term $e^{i\mathbf{k}.\hat{\mathbf{e}}\lambda}$ is close to unity, while for $k \gg 2\pi/\lambda$ it oscillates rapidly. We may therefore approximate

$$\xi_{T,\lambda} \simeq \frac{1}{9}\sum_{k<2\pi/\lambda}|\delta_{B,\mathbf{k}}|^2 \quad , \tag{53}$$

and, under the same assumptions of §2.1,

$$\xi_{T,\lambda} \simeq \frac{1}{9}\left[\frac{4\pi}{3}k^3|\delta_{B,\mathbf{k}}|^2\right]_{k=2\pi/\lambda} = \frac{1}{9}\left[\frac{4\pi}{3}\left|\frac{\delta_{B,\mathbf{k}}}{\delta_{\mathbf{k}}}\right|^2 k^3 P(k)\right]_{k=2\pi/\lambda} \quad . \tag{54}$$

Recall, that in the presence of dark matter, the dark matter perturbation δ is larger at decoupling than δ_B by a scale independent factor $(a_{\rm dec.}/a_{\rm eq.})$.

The fluctuations of the CMB temperature are usually quantified in terms of the correlation function of the temperature fluctuations. Comparing Eq. (54) with Eq. (46), we find that the mass fluctuations on scale λ, σ_λ^2, and the temperature fluctuation correlation function, $\xi_{T,\lambda}$, both measure $k^3P(k)$ at $k = 2\pi/\lambda$.

Eq. (54) gives the contribution to $\xi_{T,\lambda}$ of the intrinsic temperature perturbations due to radiation energy density perturbations. It therefore holds only at $\lambda < \lambda_{\rm dec.}$, where this contribution dominates. On larger scale, $\lambda > \lambda_{\rm dec.}$, the temperature fluctuations are dominated by the gravitational redshift, which results in temperature perturbations larger by a factor $(\lambda/\lambda_{\rm dec.})^2$ than those given by Eq. (54). As discussed in §5 of the previous chapter, the evolution of perturbations leaves the perturbation power spectrum unchanged on scales $\lambda > \lambda_{\rm eq.}$, and suppresses perturbations on smaller scales, compared to those at $\lambda > \lambda_{\rm eq.}$, by a scale dependent factor $(\lambda/\lambda_{\rm eq.})^2$. Since the gravitational redshift effect amplifies the temperature perturbations on scales $\lambda > \lambda_{\rm dec.}$, compared to those at $\lambda < \lambda_{\rm dec.}$, by $(\lambda/\lambda_{\rm dec.})^2$, the CMB temperature fluctuation correlation function reflects the original power spectrum $P_i(k)$ at both small and large scales, $\xi_{T,k} \propto kP_i(k)$. In particular, for a power-law initial spectrum $P_i(k) \propto k^n$, we have

$$\xi_{T,k}^{1/2} \propto \begin{cases} k^{(n-1)/2}, & \text{for } 2\pi/k > \lambda_{\rm dec.}; \\ k^{(n-1)/2}, & \text{for } \max(\lambda_{FS,NR}, \lambda_{D,{\rm eq.}}) < 2\pi/k < \lambda_{\rm eq.}. \end{cases} \tag{55}$$

Note, that the value of $k^3P(k)$ at $\lambda = \lambda_{\rm eq.}$ is larger (in the presence of dark matter) than its value at $\lambda = \lambda_{\rm dec.}$ by a factor $(\lambda_{\rm dec.}/\lambda_{\rm eq.})^{(n+3)/2}$.

2.2.3 Angular and physical scales

By observing the CMB temperature fluctuations in different directions on the sky, the correlation function of temperature fluctuations is determined as a function of angular separation, θ, between different directions. This angular scale corresponds to a length scale at decoupling, $\lambda_{\rm prop.} = \theta d_{A,{\rm dec.}}$ where $d_{A,{\rm dec.}}$ is the angular diameter distance to the $z = z_{\rm dec.}$ last scattering surface. This length scale corresponds to a comoving scale $\lambda = \theta a_{\rm dec.}^{-1} d_{A,{\rm dec.}}$. In order to compare CMB measurements with the mass fluctuations in the LSS, which are observed on length scale λ, it is thus necessary to determine $d_{A,{\rm dec.}}$. The angular diameter distance is given by $d_A(a) = aR_0r(a)$, where $r(a)$ is determined by

$$\frac{c}{R_0}\int_a^1 \frac{da}{a\dot{a}} = \int_0^r \frac{dr}{\sqrt{1-kr^2}}\,. \tag{56}$$

Here, $k = 0$ for a flat universe, and for a curved universe, $k \neq 0$, we chose the normalization $|k| = 1$, for which the radial part of the present, $z = 0$, metric is $R_0 dr/\sqrt{1 - kr^2}$. Using approximations similar to those used in §2.2 of the previous chapter, it is straight forward to derive the following approximation for $d_A(a)$:

$$d_A(a) \approx \frac{2c}{H_0} a \times \begin{cases} \Omega^{-1}, & \text{for } \Lambda = 0; \\ \Omega^{-1/3}, & \text{for } \Omega \ll 1, \Omega + \Omega_\Lambda = 1. \end{cases} \tag{57}$$

Using the above approximation, we find that the angular scale corresponding to the size of the horizon at equality, $\lambda_{\text{eq.}} = c/\dot{a}_{\text{eq.}} = (c/H_0)(a_{\text{eq.}}/\Omega)^{1/2}$ is

$$\begin{aligned} \theta_{\text{eq.}} &\approx \frac{1}{2} a_{\text{eq.}}^{1/2} \times \begin{cases} \Omega^{1/2}, & \text{for } \Lambda = 0; \\ \Omega^{-1/6}, & \text{for } \Omega \ll 1, \Omega + \Omega_\Lambda = 1. \end{cases} \\ &= 0.2^\circ h_{75}^{-1} \times \begin{cases} \Omega^{0}, & \text{for } \Lambda = 0; \\ \Omega^{-2/3}, & \text{for } \Omega \ll 1, \Omega + \Omega_\Lambda = 1. \end{cases} \end{aligned} \tag{58}$$

The angular scale corresponding to the size of the horizon at decoupling, $\lambda_{\text{dec.}} = c/\dot{a}_{\text{dec.}} = (c/H_0)(a_{\text{dec.}}/\Omega)^{1/2}$, is given by

$$\begin{aligned} \theta_{\text{dec.}} &\approx \frac{1}{2} a_{\text{dec.}}^{1/2} \times \begin{cases} \Omega^{1/2}, & \text{for } \Lambda = 0; \\ \Omega^{-1/6}, & \text{for } \Omega \ll 1, \Omega + \Omega_\Lambda = 1. \end{cases} \\ &= 1^\circ \times \begin{cases} \Omega^{1/2}, & \text{for } \Lambda = 0; \\ \Omega^{-1/6}, & \text{for } \Omega \ll 1, \Omega + \Omega_\Lambda = 1. \end{cases} \end{aligned} \tag{59}$$

$\theta_{\text{dec.}}$ depends on the cosmological parameters Ω and Λ, which determine the geometry of the universe. From the analysis of the preceding section, we expect a strong feature in the CMB fluctuation spepctrum at $\theta_{\text{dec.}}$. Determination of $\theta_{\text{dec.}}$ from CMB observations will provide a stringent constraint on the value of these parameters, roughly on $\Omega + \Lambda$, and hence on the geometry. Note, that for a $\Lambda = 0$ universe, $\theta_{\text{dec.}}$ decreases for smaller Ω, while $\theta_{\text{dec.}}$ is only weakly dependent on Ω for $\Omega + \Omega_\Lambda = 1$ universe. Determination of $\theta_{\text{eq.}}$ from the shape on the fluctuation spectrum will provide a constraint on Ωh^2, since $\theta_{\text{dec.}}/\theta_{\text{eq.}} \propto (\Omega h^2)^{1/2}$.

In the range $\theta \sim \theta_{\text{eq.}} \equiv a_{\text{dec.}} \lambda_{\text{eq.}}/d_A(a = a_{\text{dec.}})$ to $\theta \sim \theta_{\text{dec.}}$, there is a transition from domination of intrinsic temperature fluctuations (at small θ) to domination of the gravitational redshift effect (at large θ). Moreover, at $\theta \sim \theta_{\text{dec.}}$ the Doppler redshift contribution to $\delta T/T$ is of the same magnitude as the contribution of the other effects. At scales smaller than the Jeans scale, $\theta < \theta_J \simeq \theta_{\text{eq.}}$, the photon and baryon fluid undergoes oscillations and

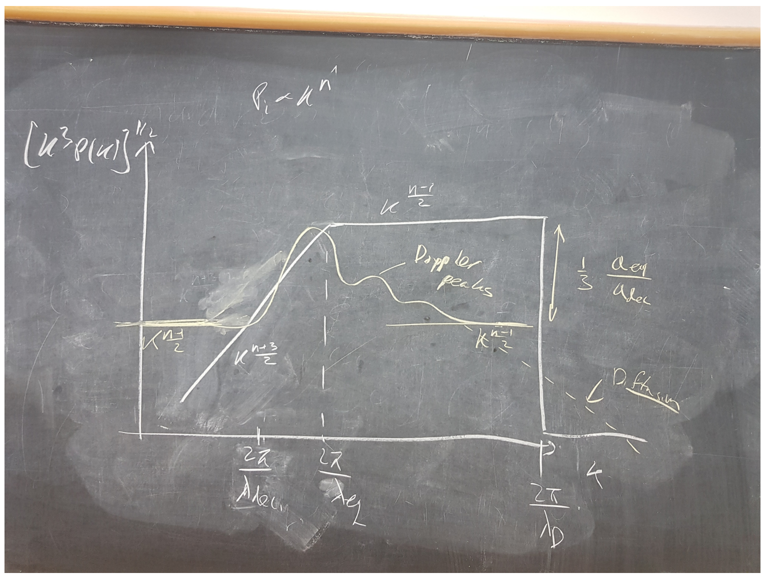

Figure 2: A schematic representation of the angular correlation function of the CMB temperature fluctuations.

the Doppler contribution to $\delta T/T$ oscillates as a function of θ, according to the phase of the oscillation accumulated from $a = a_{\rm ent.}$ up to $a = a_{\rm dec.}$. For a featureless power spectrum, e.g. $P_i(k) \propto k^n$, this introduces a series of peaks and deeps into the properly normalized temperature fluctuation correlation function, $\theta^{n-1}\xi_{T,\theta}$. The relative amplitude of the peaks decreases at smaller θ, as the Doppler effect becomes dominated by the intrinsic contribution to $\delta T/T$.

Note that the pressure effect introduces an oscillatory part also into the growing modes, $\lambda > \lambda_J$, as long as the mode wave-length is smaller than the horizon. Although the oscillation amplitude decreases as λ increases beyond λ_J, the pressure effects will be significant up to $\theta \sim \theta_{\rm dec.}$, since $\lambda_{\rm dec.}$ is not much larger than $\lambda_{\rm eq.}$. Fig. 1 presents a schematic temperature fluctuation spectrum, summarizing the conclusions of the discussion above.

2.3 Observations and cosmological parameters

For an initial power-law power spectrum, the observed CMB fluctuation spectrum is determined by 6 parameters: the densities of baryonic and dark matter particles at decoupling, ρ_B and ρ_{DM}, the amplitude and spectral index of the initial fluctuations, $P_i(k) = Ak^n$, the angular diameter distance to the $d_{A,\mathrm{dec.}}(\Omega, \Lambda, H_0)$ to the last scattering surface, and the Thomson optical depth, $\tau_T << 1$, to scattering of CMB photons by free electrons at low redshift due to the re-ionization of the plasma (as will be discussed later). At small angular scales the effect of Thomson scattering is a suppression of the amplitude by $e^{-2\tau_T}$, such the effect of τ_T is a 'normalization' of the amplitude to $Ae^{-2\tau_T}$. Note that the plasma condition at decoupling requires also a knowledge of the temperature at decoupling. However, given the current measured value of the CMB temperature, ρ_B determines the photon to baryon ratio and hence the decoupling temperature. Recalling that the physical density is given by ΩH_0^2, we may choose 5 parameters that determine the observed spectrum as $\{\Omega_B h^2, \Omega_{DM} h^2, d_{A,\mathrm{dec.}}, Ae^{-2\tau_T}, n\}$ ($\Omega = \Omega_B + \Omega_{DM}$, h is H_0 in units of 100km/s Mpc).

As discussed in the preceding section, the angular scales that determine the prominent features of the observed CMB power spectrum, and that can hence be simply determined from its shape, are $\theta_{\mathrm{eq.}} = a_{\mathrm{dec.}}\lambda_{\mathrm{eq.}}/d_{A,\mathrm{dec.}}$ and $\theta_{\mathrm{dec.}} = a_{\mathrm{dec.}}\lambda_{\mathrm{dec.}}/d_{A,\mathrm{dec.}}$. Their determination constrains $\Omega + \Lambda$ and Ωh^2. In addition, the relative amplitude of the 'Doppler peaks' at different wavelengths depends on Ω_B/Ω_{DM}, since the oscillating baryonic fluid is affected by the growing depth of the DM potential wells. Thus, an accurate measurement of the peaks' amplitudes enables a determination of this ratio.

Finally, the power spectrum is affected at (very) small scales by the suppression due to photon diffusion (we assume that this dominates 'free streaming') and due to the finite width of the last scattering 'surface'. The diffusion effect takes place on scales comparable to and smaller than the diffusion scale, $\theta_D = a_{\mathrm{dec.}}\lambda_{\mathrm{dec.}}/d_{A,\mathrm{dec.}} \approx 0.1'(\Omega/5\Omega_B)^{1/2}(\Omega h_{75}^2)^{-2}$, see eq. (6). The finite width effect takes place on a scale that depends on a different combination of $\{\Omega h^2, \Omega_B/\Omega_{DM}\}$, to be discussed in one of the seminars. Thus, an accurate measurement of the CMB spectrum, in which these effects are measured, provides consistency checks of the model (which is over constrained).

2.3.1 COBE and LSS

The COBE satellite measured (early 90's) the temperature fluctuations on scales of tens of degrees. The amplitude of the temperature fluctuations detected is $\delta T/T \simeq 10^{-5}$ and the spectrum is consistent with a power-law spectrum $P_i(k) \propto k^n$ with $n \approx 1$, i.e. the correlation function $\xi_{T,\theta}$ is consistent with being scale independent, $\xi_{T,\theta} \propto \theta^{n-1} = \theta^0$.

Angular scales of $\theta \sim \theta_{\mathrm{dec.}} \sim 1^\circ$ correspond to (comoving) length scales of hundreds of Mpc, about an order of magnitude larger than the largest scales over which the LSS is probed at present by galaxy surveys. Direct comparison of the COBE measurements of $P(k)$ with the $z = 0$ measurements based on LSS is therefore not possible. In order to make such comparison, an assumption must be made regarding the extension of the COBE power spectrum of the temperature fluctuations to small angles, corresponding to the scales over which the LSS is probed. Assuming that the initial power spectrum extends as $P_i(k) \propto k^n$ with $n \approx 1$ down to such small scales, implies $\xi_{T,\theta} \propto \theta^0$ and hence that the amplitude of fluctuations is $\delta T/T \simeq 10^{-5}$ on all scales.

As mentioned in §2.1, the amplitude of perturbations today is of order unity on $8h^{-1}$ Mpc. Let us consider the implications of this observation, adopting the assumption that $P_i(k)$ extends as k^1 to small scales. Let us first consider a universe with $\lambda = 0$ and no dark matter. For such a universe, which has a low density $\Omega \ll 1$ given the nucleosynthesis constraint $\Omega_B h^2 = 0.027$, perturbations are amplified from decoupling till today by a factor $\approx 2.5\Omega \times 10^3$ (see § 1.2.2). The temperature fluctuations on small scales give the perturbation in baryon density at decoupling, $\delta_B = 3\delta T/T \approx 3 \times 10^{-5}$. Multiplying by the growth factor, $2.5\Omega \times 10^3$, we find that perturbations at decoupling would have grown to a present day amplitude of $\approx 0.1\Omega \ll 1$. This is inconsistent with the measured LSS perturbation amplitude, ≈ 1 at ≈ 10 Mpc. A $\lambda = 0$, no dark matter universe is therefore ruled out by the observations.

Let us consider next a $\lambda = 0$ universe with a dominant component of dark matter, $\Omega_{DM} \gg \Omega_B$. In this case, perturbations are amplified from decoupling till today by a factor $\approx 10^3\Omega$ (see §2.2 of previous chapter). The dark matter density perturbation amplitude is larger than the baryon density perturbation amplitude by a factor $a_{\rm dec.}/a_{\rm eq.} = 20\Omega h_{75}^2$ (see §4 of previous chapter; here h_{75} is H_0 in units of 75km/s Mpc). The dark matter perturbation amplitude is the relevant quantity, since the baryons fall into the dark matter potential wells after decoupling. In this case we have, therefore, $\delta = 20\Omega h_{75}^2 \delta_B \approx 6 \times 10^{-4}\Omega h_{75}^2$. Multiplying by the growth factor, we find that perturbations at decoupling would have grown to a present day amplitude of $\approx 1\Omega^2 h_{75}^2$. Such a universe may therefore be consistent with observations, provided $\Omega_{DM} \approx 1$.

We have mentioned in the discussion of observational evidence for dark matter existence in stellar and galactic systems, that such observations do not provide information on the nature of the dark matter particles, which may be baryonic. The dark matter may be composed, for example, of black-holes or faint stars. It is the comparison of CMB and LSS observations discussed above, combined with constraint on Ω_B from nucleosynthesis, that suggests the existence of a non-baryonic component of dark matter.

Finally, let us consider an $\Omega + \Omega_\Lambda = 1$ universe with $\Omega_{DM} \gg \Omega_B$. In this case, the amplification factor is $\approx 10^3$ with weak dependence on Ω, and perturbations at decoupling would grow to a present day amplitude of $\approx 1\Omega h_{75}^2$. Compared to the $\Lambda = 0$ universe, the dependence of present day amplitude on Ω is weaker. A $\Lambda > 0$ model would therefore be consistent with observations for values of Ω_{DM} which are lower than those allowed by observations for a $\Lambda = 0$ universe.

2.3.2 Boomerang and Maxima

The balloon experiments Bommerang and Maxima have recently (2001) measured $\xi_{T,\theta}$ on 1 degree scale. The main conclusions from these measurements are:

- Comparing with the COBE measurement on larger scale, the amplitude of $\xi_{T,\theta}$ is consistent with $\xi_{T,\theta} \propto \theta^{n-1}$ with $n = 1 \pm 0.1$.
- The first Doppler peak at $\theta \sim \theta_{\rm dec.}$ is clearly detected. Hints for a second peak on smaller scale are present as well.
- The location of the first peak at $\theta \simeq 1^\circ$ implies [see Eq. (59)] that the universe geometry is nearly flat, with $\Omega + \Omega_\Lambda \approx 1$.

2.3.3 WMAP

The WMAP satellite provided precision measurements of the temperature fluctuations on a wide range of angular scales, down to $\sim 0.3^\circ$. It confirmed the results described above, and provided high accuracy determination of cosmological parameters (see table 2 in Dunkley et al. 2009, ApJS 180, 306). As discussed above, the values of $\Omega + \Lambda$ is determined by the location of the first peak of the CMB fluctuations. WMAP data imply

$$\Omega + \Lambda = 1.02 \pm 0.02, \quad \text{where} \quad \Omega \equiv \frac{\rho_{m0}}{\rho_c}, \quad \Lambda \equiv \frac{\rho_\lambda}{\rho_c}. \tag{60}$$

This implies that the geometry is nearly flat,

$$\frac{k}{R_0^2} = (0.02 \pm 0.02)\frac{H_0^2}{c^2}. \tag{61}$$

The spectral index is inferred to be $n = 0.96 \pm 0.02$.

Adopting a flat universe, $k = 0$, the subsequent peaks of the CMB fluctuations give

$$\Omega_m h^2 = 0.13 \pm 0.01, \quad \Omega_b h^2 = 0.022 + 0.001, \tag{62}$$

with $h \equiv H_0/(100\text{km/s/Mpc})$. Taking $h = 0.72 \pm 0.08$ (based on Cepheid distance ladder, see intro), this gives

$$t_0 = 14\,\text{Gyr}, \quad \Omega_m = 0.25 \pm 0.4, \tag{63}$$

implying also $\Lambda \simeq 0.7$. The baryon density is consistent with BBN. This model fits very well the observed CMB fluctuations, see slide 3.

Given $\Omega + \Lambda$ from CMB, measuring

$$d_L(z) = (c/H_0)[z + (1 + \Lambda - \Omega/2)z^2/2] \tag{64}$$

using SNIa gives $\Omega - \Lambda$ (slides 4,5). Combining with the CMB $\Omega + \Lambda = 1$ yields $\Omega \simeq 0.3$, $\Lambda \simeq 0.7$ (slide 6, consistent with the results above based on h).

2.3.4 Planck

The high precision Planck measurements of the CMB spectrum (see slide 4 and 2016 A&A 594, A13) constrain $|\Omega + \Lambda - 1| < 0.005$. For $\Omega + \Lambda = 1$, it determines $n = 0.97 \pm 0.01$, $h = 0.68 \pm 0.01$, $\Omega = 0.032 \pm 0.02$.

2.4 Polarization & re-ionization

To be completed.

A Hydrodynamics of relativistic fluids

A.1 Eqs.

We consider here the flow of a relativistic ideal fluid, neglecting gravity. The assumption of "ideal" fluid implies that the plasma is everywhere in local thermal equilibrium, and that dissipative processes (diffusion, heat conduction etc.) that increase the entropy are negligible. The energy momentum tensor of an ideal fluid was derived in the discussion of GR & FRW, and shown to be given by

$$T^{\mu\nu} = p\eta^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}(p+e). \tag{65}$$

Here u^μ is the 4-velocity, $u^0 = \gamma c$ and $u^i = \gamma v^i$ where v^i is the 3-velocity and $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ the Lorentz factor, p is the pressure and e is the energy density. In our notations $x^0 = ct$ and η is diagonal with $\eta^{00} = -1$ and $\eta^{ii} = 1$. Conservation of energy and momentum are given by

$$\partial_\mu T^{\mu\nu} = 0. \tag{66}$$

The 0-th component of eq. (66) may be written as

$$\partial_t p = \partial_\mu \left[\gamma u^\mu (e+p)\right] . \tag{67}$$

Substituting this in the i=th eq. we have

$$(\partial_t + \mathbf{v}.\nabla)\mathbf{v} = -\frac{c^2}{\gamma^2(e+p)}\left(\nabla p + \frac{1}{c^2}\mathbf{v}\partial_t p\right). \tag{68}$$

We have 4 equations for the 5 variables, $\mathbf{v}$, e and p. The 5-th equation is the conservation of entropy along a fluid element's path,

$$u^\mu \partial_\mu s = (\partial_t + \mathbf{v}.\nabla)s = 0. \tag{69}$$

Note that an equation of state, $e(s,p)$, is required to close the equations.

If the fluid contains particles that are not created or destroyed, the conservation of these particles may be written as

$$\partial_\mu(u^\mu n) = \partial_t(\gamma n) + \nabla(\gamma n\mathbf{v}) = 0, \tag{70}$$

where n is the (proper) density of the particles. In this case, using the conservation of particles and $u_\nu \partial_m u T^{\mu\nu} = 0$ gives

$$u^\mu\left[\partial_\mu\left(\frac{e}{n}\right) + p\partial_\mu\left(\frac{1}{n}\right)\right] = 0, \tag{71}$$

which is a particular realization of eq. (69) for the case where a conserved number exists.

A.2 Sound waves

Consider a small perturbation in a homogeneous fluid at rest: $p \to p + \delta p$, $s \to s + \delta s$, and small (first order in a sense to be defined below) $\mathbf{v}$. The linearized eqs. of motion are

$$\partial_t \delta e = -(e+p)\nabla \mathbf{v}\,, \tag{72}$$

$$\partial_t \mathbf{v} = -\frac{c^2}{(e+p)}\nabla \delta p, \tag{73}$$

and

$$\partial_t \delta s = 0. \tag{74}$$

Note that $\gamma = 1 + O(v^2)$, so that $\gamma = 1$ in the linearized equations.

We can separate the perturbation into non-adiabatic and adiabatic parts: $\{\delta s \neq 0, \delta p = 0, \mathbf{v} = 0\}$ and $\{\delta s = 0, \delta p \neq 0, \mathbf{v} \neq 0\}$. The evolution of the non-adiabatic part is simple, it is time independent. Next, we decompose the adiabatic part in Fourier modes, $f(\mathbf{x}, t) \propto \exp[i(\mathbf{k}.\mathbf{x} - \omega t)]$, writing eqs. (73) and (75) as

$$\omega \delta e = (e+p)\mathbf{k}.\mathbf{v}\,, \tag{75}$$

and

$$\omega \mathbf{v} = \frac{c^2}{(e+p)}\mathbf{k}\delta p. \tag{76}$$

We now separate the adiabatic perturbation into compressible ($\mathbf{v} \parallel \mathbf{k}$, $\nabla \times \mathbf{v} = 0$) and incompressible ($\mathbf{v} \perp \mathbf{k}$, $\nabla.\mathbf{v} = 0$), parts: $\{\delta p \neq 0, \mathbf{v}_{\text{comp.}} = (\mathbf{k}.\mathbf{v}/k^2)\mathbf{k}\}$ and $\{\delta p = 0, \mathbf{v}_{\text{incomp.}} = \mathbf{v} - (\mathbf{k}.\mathbf{v}/k^2)\mathbf{k}\}$. The evolution of the incompressible mode is again simple, it is time independent ($\omega = 0$). For this mode, $\mathbf{v}$ and $\mathbf{k}$ are perpendicular, i.e. it describes shear flow with no compression of the fluid, $\nabla \mathbf{v} = 0$ and no pressure variation.

Finally, for the adiabatic compressible ($\{\delta s = 0, \mathbf{v} \parallel \mathbf{k}\}$) component we have

$$\frac{\delta e}{(e+p)} = \frac{kv}{\omega}, \tag{77}$$

$$\frac{\delta p}{(e+p)} = \frac{\omega v}{kc^2}. \tag{78}$$

Dividing the two eqs. we finally obtain

$$c_s^2 \equiv \left(\frac{w}{k}\right)^2 = c^2 \left(\frac{\partial p}{\partial e}\right)_s. \tag{79}$$

The waves therefore satisfy a linear dispersion relation, $\omega = kc_s$, so that their group and phase velocity is the speed of sound, c_s. Finally, $\delta e/(e+p) = v/c_s$ so that our assumption of a "small velocity" corresponds to first order expansion in v/c_s.