I. THE FRW UNIVERSE

The great simplifying fact of cosmology is that the universe appears to be homogeneous (the same at every point) and isotropic (the same in every direction) along a preferred set of spatial hypersurfaces. Of course homogeneity and isotropy are only approximate, but they become increasingly good approximations on larger length scales. The assumptions of homogeneity and isotropy of the 3-space singles out a preferred class of observers, *i.e.*, those observers for whom the universe appears homogeneous and isotropic. Another observer, who is moving with a uniform velocity with respect to this fundamental class of observers, will find the universe to be anisotropic. The description of physics will be simplest if we use a coordinate system appropriate for this fundamental class of observers.

Let us determine the form of spacetime metric in such a coordinate system (t, x^i) in which homogeneity and isotropy are self-evident. The general spacetime interval (ds^2) can be separated out as

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j \\ &= g_{00}dt^2 + 2g_{0i}dtdx^i - \sigma_{ij}dx^i dx^j, \end{aligned} \tag{1.1}$$

where σ_{ij} is a positive definite spatial metric. Isotropy of space implies that g_{0i} terms must vanish; otherwise they identify a particular direction in space related to the 3-vector v_i with components g_{0i}. Further, in the coordinate system determined by fundamental observers, we may use the proper time of clocks carried by these observers to label the spacelike surfaces. This choice for the time coordinate t implies that $g_{00} = 1$, bringing the interval to the form:

$$ds^2 = dt^2 - \sigma_{ij}dx^i dx^j \equiv dt^2 - dl^2. \tag{1.2}$$

The problem now reduces to determining the 3-metric σ_{ij} of a 3-space which, at any instant of time, is homogeneous and isotropic.

The curvature of the 3-space is determined by the 3-dimensional Riemann tensor P_{ijkl}. For a space which is homogeneous and isotropic, P_{ijkl} cannot depend on the derivatives of the metric σ_{ij} and must be constructed out of the components σ_{ij} alone. The symmetry properties of P_{ijkl} uniquely determine this form to be

$$P_{ijkl} = q(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{kj}) \tag{1.3}$$

where q is related to the scalar 3-curvature $P = \sigma^{ik}\sigma^{jl}P_{ijkl} = 6q$. Since homogeneity implies the constancy of P, it follows that q is a constant.

The most general 3-space for which the property (1.3) holds is described by the line element

$$dl^2 = a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right] \tag{1.4}$$

where $k = 0, +1$ or -1 and a is independent of spatial coordinates and related to the curvature by $P = (6k/a^2)$. Notice that the assumption of isotropy, which implies spherical symmetry, can be used to write the line element in the form

$$dl^2 = a^2\left[\lambda^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]. \tag{1.5}$$

Computing the scalar curvature P for this space and equating it to a constant determines the form of λ to be $\lambda(r) = (1 - kr^2)^{-1/2}$.

The prefactor a determines the overall scale of the spatial metric and, in general, can be a function of time: $a = a(t)$. Combining (1.4) and (1.2), we arrive at the full spacetime metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]. \tag{1.6}$$

This metric describes a universe that is spatially homogeneous and isotropic at each instant of time. The value of k and the form of the function $a(t)$ (called the 'expansion factor') can be determined using Einstein's equations once the matter distribution is specified.

The coordinates in (1.6) are, of course, chosen in such a way as to make the symmetries of spacetime self-evident. This coordinate system is called the 'comoving' coordinate system.

The spatial hypersurfaces of the FRW universe have positive, zero and negative curvatures for $k = +1, 0$ and -1, respectively. These possibilities are more informally known as "closed", "flat", and "open" universes, in reference to the spatial topology, but there are problems with such designations. First, the flat and negatively-curved spaces may in fact be compact manifolds obtained by global identifications of their noncompact relatives. Second, there is a confusion between the use of "open"/"closed" to refer to spatial topology and the evolution of the universe; if such universes are dominated by matter or radiation, the negatively curved ones will expand forever and the positively curved ones will recollapse, but more general sources of energy/momentum will not respect this relationship.

A. $D = 2 + 1$ intuition

In order to illustrate the construction of the metric, consider the simpler case of two spatial dimensions. Examples of two-dimensional spaces that are homogeneous and isotropic are: (i) the flat "$x - y$" plane (R^2); (ii) the positively curved two-sphere (S^2); and (iii) the negatively curved hyperbolic plane (H^2).

First consider the two sphere. To visualize the two sphere it is convenient to introduce an extra fictitious spatial dimension and to embed this two-dimensional curved space in a three-dimensional Euclidean space with cartesian coordinates x_1, x_2, x_3. The equation of the two sphere of radius R is

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{1.7}$$

The element of length in the three-dimensional Euclidean space is

$$\vec{\mathbf{dl}}^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{1.8}$$

If x_3 is taken as the fictitious third spatial coordinate, it can be eliminated from $\vec{\mathbf{dl}}^2$ by use of (1.7):

$$\vec{\mathbf{dl}}^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2}. \tag{1.9}$$

Now introduce the coordinates r' and θ defined by

$$x_1 = r' \cos\theta, \quad x_2 = r' \sin\theta. \tag{1.10}$$

Physically, r' and θ correspond to the polar coordinates in the x_3 plane ($x_3^2 = R^2 - r'^2$). In terms of the new coordinates, (1.9) becomes

$$\vec{\mathbf{dl}}^2 = \frac{R^2 dr'^2}{R^2 - r'^2} + r'^2 d\theta^2. \tag{1.11}$$

Finally, with the definition of a dimensionless coordinate $r \equiv r'/R$ ($0 \le r \le 1$), the spatial metric becomes

$$\vec{\mathbf{dl}}^2 = R^2 \left(\frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right). \tag{1.12}$$

Note the similarity between this metric and the $k = 1$ RW metric. It should also be clear now that $a(t)$ is the radius of the space.

The two sphere is homogeneous and isotropic. Every point in space is equivalent to every other point, and there is no preferred direction. In other words, the space embodies the *Cosmological Principle*, i.e., no observer (especially us) occupies a preferred position in the universe. Note that this space is unbounded; there are no edges on the two sphere. It is possible to circumnavigate the two sphere, but it is impossible to fall off. Although the space is unbounded, the volume is finite ($V = 4\pi R^2$).

The expansion (or contraction) of this $d = 2$ universe is equivalent to an increase (or decrease) in the radius of the two sphere R. Since the universe is spatially homogeneous and isotropic, the scale factor R can only be a function of time. As the two sphere expands or contracts, the coordinates (r and θ) remain unchanged; they are "comoving." Also note that the physical distance between any two comoving points in the space scales with R (hence the name scale factor).

The equivalent formulae for a space of constant negative curvature can be obtained with the replacement $R \to iR$ in (1.7). In this case the embedding in in three dimensional Minkowski space. The metric corresponding to the form of (1.12) for the negative curvature case is

$$\vec{\mathbf{dl}}^2 = R^2 \left(\frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right). \tag{1.13}$$

The hyperbolic plane is unbounded with infinite volume since $0 \le \theta \le \infty$.

The spatially-flat model can be obtained from either of the above examples by taking $R \to \infty$. The flat model is unbounded with infinite volume. For the flat model the scale factor does not represent any physical radius as in the closed case, or an imaginary radius as in the open case, but merely represents how the physical distance between comoving points scales as the space expands or contracts.

II. KINEMATIC PROPERTIES OF THE FRW UNIVERSE

The FRW metric in (1.6) is nonstatic because of the time dependence of $a(t)$. Since $a(t)$ multiplies the spatial coordinates, any proper distance $l(t)$ will change with time in proportion to $a(t)$:

$$l(t) = l_0 a(t) \propto a(t). \tag{2.1}$$

In particular, the proper separation between any two observers, located at constant comoving coordinates, will change with time. Let the coordinate separation between two such observers be δr, so that the proper separation is $\delta l = a(t)\delta r$. Each of the two observers will attribute to the other a velocity

$$\delta v = \frac{d}{dt}\delta l = \dot{a}\delta r = \left(\frac{\dot{a}}{a}\right)\delta l. \tag{2.2}$$

This leads to several important physical consequences of rather generic nature.

To begin with, consider a narrow pencil of electromagnetic radiation which crosses any two comoving observers separated by proper distance δl. The time for transit will be $\delta t = \delta l$. Let the frequency of the radiation measured by the first observer be ω. Since the first observer sees the second one to be receding with velocity δv, he will expect the second observer to measure a Doppler shifted frequency $(\omega + \delta\omega)$ where

$$\frac{\delta\omega}{\omega} = -\frac{\delta v}{c} = -\delta v = -\frac{\dot{a}}{a}\delta l = -\frac{\dot{a}}{a}\delta t = -\frac{\delta a}{a}. \tag{2.3}$$

(Since the observers are separated by an infinitesimal distance of first order, δl, we can introduce a locally inertial frame encompassing both the observers. The laws of special relativity can be applied in this frame.) This equation can be integrated to give

$$\omega(t)a(t) = \text{constant}. \tag{2.4}$$

In other words, the frequency of electromagnetic radiation changes due to expansion of the universe according to the law $\omega \propto a^{-1}$. We have made implicit use of the homogeneity of the spacetime in extending the local result to a global context.

The same results can be obtained by solving Maxwell's equations for wave propagation in the FRW universe. Notice that, if $\dot{a} > 0$, the frequency will be redshifted irrespective of whether the wave is travelling from a location A to location B or from B to A. Suppose that an observer receives, at time t_0, radiation from a distant source which was emitted at some earlier time $t_1 (< t_0)$. The observed and emitted frequencies are related by

$$\frac{\omega_e}{\omega_0} = \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} \equiv 1 + z \tag{2.5}$$

where the last equation defines the 'redshift' z. It is usual to say that the particular source has the redshift z. Thus, the expansion of the FRW model allows a natural explanation for the redshift of light emitted by distant sources.

This relation is often used to characterize a particular time (especially in the early universe) at which an event took place; that is, we *define* a function $z(t)$ by $1+z(t) \equiv a_0/a(t)$, where $a_0 \equiv a(t_0)$, and use it instead of the time coordinate t or the value of the expansion factor $a(t)$ at that time.

The expansion of the universe also affects the motion of material particles. Consider again two comoving observers separated by proper distance δl. Let a matter particle pass the first observer with velocity v. When it has crossed the proper distance δl (in time interval δt), it passes the second observer whose velocity (relative to the first one) is

$$\delta u = \frac{\dot{a}}{a}\delta l = \frac{\dot{a}}{a} v \delta t = v \frac{\delta a}{a}. \tag{2.6}$$

The second observer will attribute to the particle the velocity

$$v' = \frac{v - \delta u}{1 - v\delta u} = v - (1 - v^2)\delta u + \mathcal{O}[(\delta u)^2] = v - (1 - v^2) v \frac{\delta a}{a}. \tag{2.7}$$

This follows from the special relativistic formula for addition of velocities which is valid in the infinitesimal region around the first observer. Rewriting this equation as

$$\delta v = -v(1 - v^2)\frac{\delta a}{a} \tag{2.8}$$

and integrating, we obtain

$$p = \frac{v}{\sqrt{1 - v^2}} = \frac{\text{constant}}{a}. \tag{2.9}$$

In other words, the magnitude of the 3-momentum decreases as a^{-1} due to the expansion. if the particle is non-relativistic, then $v \propto p$ and velocity itself decays as a^{-1}.

One conclusion which can be reached immediately from (2.9) is the following: consider a stream of particles propagating freely in spacetime. At some time t, a comoving observer finds dN particles in a proper volume $d\mathcal{V}$, all having momentum in the range $(\mathbf{p}, \mathbf{p} + d^3\mathbf{p})$. The phase space distribution function $f(\mathbf{x}, \mathbf{p}, t)$ for the particles is defined by the relation $dN = f\ d\mathcal{V}\ d^3\mathbf{p}$. At a later instant $(t + \delta t)$ the proper volume occupied by these particles would have increased by a factor $[a(t + \delta t)/a(t)]^3$ while the volume in the momentum space will be redshifted by $[a(t)/a(t + \delta t)]^3$, showing that the phase volume occupied by the particles does not change during the free propagation. Since the number of particles dN is also conserved, it follows that f is conserved along the streamline.

The above considerations have been purely classical. It is, however, sometimes convenient to think of electromagnetic radiation as consisting of photons with zero rest mass, for which $E = \hbar\omega = p$. The decay law $p \propto a^{-1}$ then implies the redshift for radiation. A few other properties of radiation can also be easily derived by using the photon concept. If $f(\mathbf{x}, \mathbf{p}, t)$ is the phase space density of photons, then the energy density du of photons in the frequency band $(\omega, \omega + d\omega)$ flowing into a particular solid angle $d\Omega$ is proportional to $f(\hbar\omega)(p^2 dp\ d\Omega)$. The quantity 'spectral energy density' or the 'specific density', $I(\omega)$, of the radiation (which is the energy density per unit bandwidth) is defined by the relation

$$du \equiv I(\omega)\ d\Omega\ d\omega = \hbar\omega\ f\ p^2 dp\ d\Omega. \tag{2.10}$$

Since $p^2dp \propto \omega^2 d\omega$ for photons, we find that $I(\omega) \propto f\omega^3$; and further, since f is conserved during the propagation, it follows that the quantity $I(\omega)/\omega^3$ is conserved during the propagation of radiation.

This quantity $I(\omega)$ will have units (for example) erg cm^{-3} Hz^{-1} (steradian)$^{-1}$. The 'energy flux' of radiation is sometimes defined as $F(\omega) = (c/4\pi)I(\omega)$ which will have the units (erg s^{-1}) cm^{-2} Hz^{-1} (steradian)$^{-1}$. It is clear from the invariance of I/ω^3 that I and F vary as a^{-3} in the expanding universe. more precisely,

$$F[\omega(1+z); z] = F[\omega; 0](1+z)^3. \tag{2.11}$$

The total flux of radiation, obtained by integrating over all frequencies, varies as $(1+z)^4$:

$$F_{\text{total}} = \int_0^\infty F d\omega \propto (1+z)^4. \tag{2.12}$$

Radiation which has intensity distribution of the form $I(\omega) = \omega^3 G(\omega/T)$ will retain the spectral shape during the expansion, with the parameter T varying with expansion as $T \propto a^{-1}$. The planck spectrum has this form in which T corresponds to the temperature; it follows that the temperature of the radiation, which has a Planck spectrum, decreases with expansion as $T(t) \propto a^{-1}$.

Cosmological observations are mostly based on electromagnetic radiation which is received from far away sources. Let an observer, located at $r = 0$, receive at time $t = t_0$ radiation from a source located at $r = r_1$. This radiation must have been emitted at some earlier time t_1 such that the events (t_1, r_1) and $(t_0, 0)$ are connected by a null geodesics. Taking the propagation of the ray to be along $\theta =$ constant, $\phi =$ constant, we can write the equation for the null geodesic to be

$$0 = ds^2 = dt^2 - a^2(t)\frac{dr^2}{1-kr^2}. \tag{2.13}$$

Integrating this, we can find the relation between r_1 and t_1:

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}}. \tag{2.14}$$

Suppose that the source emits two photons at t_1 and $t_1 + dt_1$, which are received by the observer at times t_0 and $t_0 + dt_0$. The right hand side of (2.14) is the same for both these photons implying

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{a(t)}. \tag{2.15}$$

This equation can be rewritten as

$$\int_{t_1}^{t_1+dt_1} \frac{dt}{a(t)} = \int_{t_0}^{t_0+dt_0} \frac{dt}{a(t)}, \tag{2.16}$$

giving

$$\left(\frac{dt_0}{dt_1}\right) = \frac{a(t_0)}{a(t_1)}. \tag{2.17}$$

If the source has an intrinsic luminosity L, then it will emit an energy $L\ dt_1$ in a time interval dt_1. This energy, which will be received by the observer in a time interval $dt_0 = dt_1[a(t_0)/a(t_1)]$, would have undergone a redshift by a factor $[a(t_1)/a(t_0)]$ and will be distributed over a sphere of area $4\pi a^2(t_0)r_1^2$. The observed flux, therefore, will be

$$l = L\left(\frac{dt_1}{dt_0}\right)\frac{a(t_1)}{a(t_0)}\frac{1}{4\pi a^2(t_0)r_1^2} = \frac{L}{4\pi a_0^2 r_1^2}\left(\frac{a(t_1)}{a_0}\right)^2 = \frac{L}{4\pi a_0^2 r_1^2(1+z)^2}. \tag{2.18}$$

It is usual to define a 'luminosity distance' $d_L(z)$ to a source at redshift z through the relation $l \equiv (L/4\pi d_L^2)$. We get

$$d_L(z) = a_0 r_1(t_1)(1+z). \tag{2.19}$$

Another observable parameter for distant sources is the angular diameter. If D is the physical size of the object which subtends an angle δ to the observer, then, for small δ, we have $D = r_1 a(t_1)\delta$. The 'angular diameter distance' $d_A(z)$ for the source is defined via the relation $\delta = (D/d_A)$; so we find that

$$d_A(z) = r_1 a(t_1) = a_0 r_1(t_1)(1+z)^{-1}. \tag{2.20}$$

Quite clearly $d_L = (1+z)^2 d_A$.

In these formulae, r_1 is related to t_1 via (2.14) and hence can be expressed as a function of z. To do this exactly, we need to know the functional form of $a(t)$. If, however, we are only interested in small z (*i.e.* in a small time interval $(t_0 - t_1)$) then we can Taylor expand $a(t)$ around t_0 and parametrize it by the coefficients of the Taylor expansion. It is conventional to write this expansion in the following form:

$$\begin{aligned} a(t) &= a(t_0)\left[1 + \left(\frac{\dot{a}}{a}\right)_0 (t-t_0) + \frac{1}{2}\left(\frac{\ddot{a}}{a}\right)_0 (t-t_0)^2 + \cdots\right] \\ &= a(t_0)\left[1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \cdots\right] \end{aligned} \tag{2.21}$$

where

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0}; \qquad q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_{t=t_0}. \tag{2.22}$$

Substituting this expansion in (2.14), expanding up to quadratic order in $(t-t_0)$ and r_1 and integrating, we get the relation between r_1 and t_1:

$$r_1 = \frac{1}{a_0}\left[(t_0 - t_1) + \frac{1}{2}H_0(t_0-t_1)^2 + \cdots\right]. \tag{2.23}$$

Inverting (2.21), we can express $(t-t_0)$ in terms of $(1+z) = a_0/a$:

$$(t - t_0) = -H_0^{-1}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \cdots\right]. \tag{2.24}$$

Finally, substituting (2.24) into (2.23) we can express r_1 in terms of z:

$$a_0 r_1 = H_0^{-1}\left[z - \frac{1}{2}(1+q_0)z^2 + \cdots\right]. \tag{2.25}$$

This shows that, to first order in z, the 'redshift velocity' $v \equiv cz$ is proportional to the proper distance $a_0 r_1$. We can now use this relation to express d_L (and d_A) in terms of z; we find, for example,

$$d_L(z) = a_0 r_1(1+z) = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2 + \cdots\right]. \tag{2.26}$$

The quantity d_L can be determined by measuring the flux l for a class of objects for which the intrinsic luminosity L is known. If we can also measure the redshift z for these objects, then a plot of d_L against z will allow us to determine the parameters H_0 and q_0. In particular, observations suggest that

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}; \quad 0.6 \lesssim h \lesssim 0.8. \tag{2.27}$$

The quantity H_0, called the 'Hubble constant', is of primary significance in cosmology. Its value determines the rate at which the universe is expanding today. From H_0 we can construct the time scale, $t_{\rm univ} \equiv H_0^{-1} = 9.8 \times 10^9 h^{-1}$ yr and the length scale $l_{\rm univ} \equiv ct_{\rm univ} \simeq 3000h^{-1}$ Mpc. These are the characteristic scales over which global effects of cosmological expansion will be important today.

III. THE DYNAMICS OF THE FRW UNIVERSE

The FRW metric contains a constant k and a function $a(t)$, both of which can be determined via Einstein's equations

$$G^\alpha_\beta = R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R = 8\pi G T^\alpha_\beta \tag{3.1}$$

if the stress-energy tensor T^α_β for the source is specified. (Here G^α_β is the Einstein tensor, R^α_β is the Ricci tensor and R is the Ricci scalar.) The assumption of homogeneity and isotropy implies that $T^i_0 = 0$ and that the spatial components of T^j_k must have a diagonal form with $T^1_1 = T^2_2 = T^3_3$. It is conventional to write such a stress-tensor as

$$T^\alpha_\beta = \text{diag}\left[\rho(t), -p(t), -p(t), -p(t)\right]. \tag{3.2}$$

This notation is suggested by the following consideration: If the source was an ideal fluid with pressure p and density ρ then the stress-tensor for an ideal fluid, $T^\alpha_\beta = (p+\rho)u^\alpha u_\beta - p\delta^\alpha_\beta$, will have the above form in the rest frame of the fluid. The nature of the source is completely specified once the relation between ρ and p is given in the form of an equation of state $p = p(\rho)$.

The constituents of the *present day* universe are essentially matter, radiation and vacuum energy. "Matter" (also called "dust") is used by cosmologists to refer to any set of non-relativistic, non-interacting particles; the pressure is then negligible, and the energy density is dominated by the rest mass of the particles, and it redshifts away as the volume increases. "Radiation" includes any species of relativistic particles, for which the individual particle energies will redshift as $1/a$ in addition to the volume dilution factor. (Coherent electromagnetic fields will also obey this equation of state.) The vacuum energy density, equivalent to a cosmological constant Λ via $\rho_\Lambda = \Lambda/8\pi G$, is by definition the energy remaining when all other forms of energy and momentum have been cleared away.

Non-relativistic matter is well approximated as pressureless 'dust', $p \simeq 0$. This follows from the fact that, for non-relativistic particles of mass m and number density n, $\rho \simeq nmc^2$ while $p \simeq nmv^2$ with $v \ll c$. Radiation has the equation of state $p = (1/3)\rho$. For vacuum energy, $p = -\rho$. The nature - and hence the equation of state - of the constituents will change with time in an evolving universe. The equation of state at any given epoch has to be determined by studying the physical processes at that particular epoch.

The tensor G^α_β can be computed from the FRW metric in a straightforward manner. As is to be expected, we find that G^i_0 vanish while G^i_j is proportional to a unit matrix. The two nontrivial components are:

$$G^0_0 = \frac{3}{a^2}(\dot{a}^2 + k), \qquad G^i_j = \frac{1}{a^2}(2a\ddot{a} + \dot{a}^2 + k)\delta^i_j. \tag{3.3}$$

Thus (3.1) gives two independent equations,

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho, \tag{3.4}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi G p. \tag{3.5}$$

These two equations, combined with the equation of state $p = p(\rho)$, completely determine the three functions $a(t)$, $\rho(t)$ and $p(t)$.

The value of k, which determines the curvature of spatial sections of the universe, canbe determined from (3.4). This equation implies that, at the present epoch ($t = t_0$),

$$\frac{k}{a_0^2} = \frac{8\pi G}{3}\rho_0 - \left(\frac{\dot{a}}{a}\right)_0^2 \equiv \frac{8\pi G}{3}(\rho_0 - \rho_c) \equiv H_0^2(\Omega - 1) \tag{3.6}$$

where we have introduced two important definitions:

$$\rho_c = \text{critical density} \equiv \frac{3H_0^2}{8\pi G} \simeq \begin{cases} 1.88 \times 10^{-29} h^2 & \text{g cm}^{-3} \\ 1.06 \times 10^4 h^2 & \text{eV cm}^{-3} \end{cases} \tag{3.7}$$

and

$$\Omega \equiv \rho_0/\rho_c. \tag{3.8}$$

It follows from (3.6) that the universe will be closed, flat or open depending on whether Ω is greater than, equal to or less than unity.

Equations (3.4) and (3.5) can be combined to give

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{3.9}$$

For the usual kind of matter, $(\rho + 3p) > 0$ implying that $\ddot{a} < 0$. The $a(t)$ curve (which has a positive $\dot{a}$ at the present epoch t_0) must be convex; in other words, a will be smaller in the past and will become zero at sometime (in the past), say, at $t = t_{\rm sing}$. It is also clear that $(t_0 - t_{\rm sing})$ must be less than the value of the intercept $(\dot{a}/a)_0^{-1} = H_0^{-1}$. For convenience, we will choose the time coordinate such that $t_{\rm sing} = 0$, *i.e.* we take $a = 0$ at $t = 0$. In that case, the present 'age' of the universe satisfies the inequality $t_0 < t_{\rm univ}$ where

$$t_{\rm univ} \equiv H_0^{-1} = 3.1 \times 10^{17} h^{-1}{\rm s} = 9.8 \times 10^9 h^{-1}{\rm yr}. \tag{3.10}$$

As a becomes smaller, the components of the curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ become larger and when $a = 0$ these components diverge. Such a divergence ('singularity') is an artifact of our theory. When the radius of curvature of the spacetime becomes comparable to the fundamental length $(G\hbar/c^3)^{1/2} \simeq 10^{-33}$ cm constructed out of G, $\hbar$ and c, quantum effects of gravity will become important, rendering the classical Einstein equations invalid. So, in reality, t_0 is the time which has elapsed from the moment at which the classical equations become valid.

The quantities ρ and p are *defined* in (3.2) as the T^0_0 and (say) T^1_1 components of the stress tensor. The interpretation of p as 'pressure' depends on treating the source as an ideal fluid. The source for a FRW model should *always* have the form (3.2); but if the source is not an ideal fluid then it is not possible to interpret the spatial components of T^{α}_{β} as pressure. It is, therefore, quite possible that the equation of state for matter at high energies does not obey the condition $(\rho + 3p) > 0$. The violation of this condition may occur much before (*i.e.* at larger value of a) the quantum gravitational effects become important. If this happens, then 'the age of the universe' refers to the time interval since the breakdown of the condition $(\rho + 3p) > 0$.

From equation (3.4), we see that $\rho a^3 = (3/8\pi G)a(\dot{a}^2 + k)$; differentiating this expression and using equation (3.5) we get

$$\frac{d}{dt}(\rho a^3) = -3a^2\dot{a}p = -p\frac{da^3}{dt}, \tag{3.11}$$

or, equivalently,

$$\frac{d}{da}(\rho a^3) = -3a^2 p. \tag{3.12}$$

Given the equation of state $p = p(\rho)$, we can integrate (3.12) to obtain $\rho = \rho(a)$. Substituting this relation into (3.4) we can determine $a(t)$.

For an equation of state $p = \omega\rho$, (3.12) gives $\rho \propto a^{-3(1+\omega)}$; in particular,

$$p = \omega\rho \implies \rho \propto a^{-3(1+\omega)} \implies \begin{cases} \rho_m \propto a^{-3} & [\omega = 0] \\ \rho_r \propto a^{-4} & [\omega = 1/3] \\ \rho_v \propto a^0 & [\omega = -1] \end{cases} \tag{3.13}$$

where ρ_m is the energy density in non-relativistic matter, ρ_r is the energy density in radiation and ρ_v is the vacuum energy density. Thus

$$\rho_m(t) = \rho_m(t_0)\left(\frac{a_0}{a}\right)^3 = \rho_c\Omega_m(1+z)^3, \tag{3.14}$$

$$\rho_r(t) = \rho_r(t_0)\left(\frac{a_0}{a}\right)^4 = \rho_c\Omega_r(1+z)^4, \tag{3.15}$$

$$\rho_v(t) = \rho_v(t_0). \tag{3.16}$$

Observations suggest that, at the present epoch,

$$\Omega_{\text{total}} \equiv \Omega \sim 1; \quad \Omega_m \sim 0.3; \quad \Omega_r h^2 \simeq 2.56 \times 10^{-5}. \tag{3.17}$$

Thus, at present matter dominates over radiation. But from (3.14) and (3.15) it follows that the radiation density grows faster than the matter density as we go to earlier phases of the universe (*i.e.* to higher redshifts). At some time $t = t_{\text{eq}}$ in the past (corresponding to a value $a = a_{\text{eq}}$ and redshift $z = z_{\text{eq}}$) the radiation and matter will have equal energy densities. From (3.15) and (3.14) we get

$$(1+z_{\text{eq}}) = \frac{a_0}{a_{\text{eq}}} = \frac{\Omega_m}{\Omega_r} \simeq 1.2 \times 10^4 h^2. \tag{3.18}$$

Since the temperature of the radiation grows as a^{-1}, the temperature of the universe at this epoch will be

$$T_{\text{eq}} = T_0(1+z_{\text{eq}}) = 2.8\ h^2\ eV. \tag{3.19}$$

For $t \ll t_{\text{eq}}$ the energy density of the universe is dominated by radiation (with $p \simeq \rho/3$) while for $t \gg t_{\text{eq}}$ the energy density is dominated by matter (with $p \simeq 0$).

In deriving the above expressions for T_{eq} and z_{eq}, we have assumed that the energy density of relativistic particles today, $\rho_r(t_0)$, arises essentially from CMBR photons. Among all electromagnetic radiation present in the universe, CMBR definitely makes the dominant contribution. However, it is quite possible that there exists other relativistic particles (say, massless neutrinos) in the universe today contributing to ρ_r. In that case, the numerical values for Ω_r, T_{eq} and z_{eq} will change.

Let us now solve (3.4) to determine $a(t)$. The simplest solutions are those for flat universes, those with $k = 0$. In a flat universe dominated by a single energy density source, the scaling of the source is directly related to the expansion history:

$$k = 0, \quad \rho \propto a^{-n} \quad \Longrightarrow \quad a \propto t^{2/n}. \tag{3.20}$$

(For $n = 0$, we get exponential growth.) Thus, a matter-dominated flat universe expands as $a \propto t^{2/3}$, and a radiation-dominated flat universe as $a \propto t^{1/2}$.

When $k \neq 0$, the solution for matter- and radiation-dominated universes are slightly more complicated, but may still be expressed in closed forms. Though it can be done for the general case, some of the facts noted above allow us to simplify the analysis considerably. At any redshift z, the curvature term ka^{-2} in the left hand side of (3.4) has the magnitude

$$\frac{1}{a^2} = \frac{1}{a_0^2}(1+z)^2 = H_0^2\,|1-\Omega|\,(1+z)^2, \tag{3.21}$$

while the radiation density on the right hand side of (3.4) contributes

$$\frac{8\pi G}{3}\rho_r = \frac{8\pi G}{3}\rho_c\Omega_r(1+z)^4 = H_0^2\Omega_r(1+z)^4. \tag{3.22}$$

Comparing these two expressions we see that the curvature term is completely negligible (compared to the radiation energy density) at redshifts $z \gg z_c$ where

$$(1+z_c) = \left[\Omega_r^{-1}|1-\Omega|\right]^{1/2} \simeq 200|1-\Omega|^{1/2}h. \tag{3.23}$$

Since both factors in $|1-\Omega|^{1/2}h$ are less than unity, z_c is less than 200. This redshift is quite small compared to $z_{\rm eq}$ in (3.18). So we can ignore the curvature term in (3.4) for $t \lesssim t_{\rm eq}$ without incurring any significant error.

This does not necessarily mean that the curvature term dominates for $z < z_c$. It is, of course, true that the curvature dominates over *radiation* for $z < z_c$. But, since $z_c \ll z_{\rm eq}$, the universe is *matter dominated* near $z \simeq z_c$ and we should really compare the *matter density* with the curvature to decide which term is dominant.

During the matter dominated phase, the energy density varies as $\rho = \rho_c\Omega(1+z)^3$ while the curvature term grows as $a^{-2} = H_0^2|\Omega-1|(1+z)^2$. If $\Omega < 1$, then the two terms on the right hand side of Einstein's equations become comparable at some $z = z_{\rm flat} = z_f$ where z_f is determined by equating the two terms on the right hand side:

$$a_0^{-2}(1+z_f)^2 = H_0^2(1-\Omega)(1+z_f)^2 = \frac{8\pi G}{3}\rho = H_0^2\Omega(1+z_f)^3. \tag{3.24}$$

This gives $z_f = \Omega^{-1} - 2$. It is possible for the curvature term to dominate over matter density at sufficiently small z; say, for $z \ll z_f$ provided $\Omega < 0.5$; then the curvature term can dominate over matter for $0 < z \ll z_f$. During this phase, the effect of ρ can be ignored and the expansion parameter $a(t)$ grows as t. For $z \gg z_f$ we can again ignore the curvature term. Since observations suggest that $\Omega \gtrsim 0.2$, we must have $z_f < 3$. So, except at small redshifts (say, in the range $z \simeq 1-10$) the curvature term is negligible in Einstein's equations.

Equation (3.4) can therefore be written as

$$\frac{\dot a^2}{a^2} + \frac{k}{a^2} \simeq \frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_{\rm eq}\left[\left(\frac{a_{\rm eq}}{a}\right)^4 + \left(\frac{a_{\rm eq}}{a}\right)^3\right] \tag{3.25}$$

for $z \gg z_c$. In terms of the dimensionless variables $x = (a/a_{\rm eq})$ and $y = t(8\pi G\rho_{\rm eq}/3)^{1/2} \equiv (tH_{\rm eq}/\sqrt{2})$, this equation becomes

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{x^2} + \frac{1}{x} = \frac{1+x}{x^2}. \tag{3.26}$$

Integrating and using the initial conditions $x = 0$ at $y = 0$, we find the solution to be

$$H_{\rm eq}t = \frac{2\sqrt{2}}{3}\left[\left(\frac{a}{a_{\rm eq}} - 2\right)\left(\frac{a}{a_{\rm eq}} + 1\right)^{1/2} + 2\right]. \tag{3.27}$$

This gives the expansion factor of the universe $a(t)$ in terms of the two (known) parameters:

$$\begin{aligned} H_{\rm eq}^2 &= \frac{16\pi G}{3}\rho_{\rm eq} = \frac{16\pi G}{3}\rho_c\Omega_r(1+z_{\rm eq})^4 = 2H_0^2\Omega_r(1+z_{\rm eq})^4 \\ &= 2H_0^2\Omega(1+z_{\rm eq})^3, \end{aligned} \tag{3.28}$$

$$a_{\rm eq} = a_0(1+z_{\rm eq})^{-1} = H_0^{-1}|\Omega - 1|^{-1/2}(\Omega_r/\Omega). \tag{3.29}$$

From (3.27) we can find the value of $t_{\rm eq}$; setting $a = a_{\rm eq}$ gives $H_{\rm eq}t_{\rm eq} \simeq 0.552$, or

$$\begin{aligned} t_{\rm eq} &= \frac{2\sqrt{2}}{3}H_{\rm eq}^{-1}(2-\sqrt{2}) \simeq 0.39H_0^{-1}\Omega^{-1/2}(1+z_{\rm eq})^{-3/2} \\ &= 1.57\times 10^{10}(\Omega h^2)^{-2}{\rm s}. \end{aligned} \tag{3.30}$$

From (3.27) we can also find the two limiting cases for $a(t)$ valid for $t \gg t_{\rm eq}$ and $t \ll t_{\rm eq}$:

$$\left(\frac{a}{a_{\rm eq}}\right) = \begin{cases} (3/2\sqrt{2})^{2/3}(H_{\rm eq}t)^{2/3} & t \gg t_{\rm eq} \\ (3/\sqrt{2})^{1/2}(H_{\rm eq}t)^{1/2} & t \ll t_{\rm eq} \end{cases} \tag{3.31}$$

Thus $a \propto t^{2/3}$ in the matter dominated phase (when the curvature is negligible) and $a \propto t^{1/2}$ in the radiation dominated phase.

If we use the $a(t)$ relation for the matter-dominated phase and define $t_{\rm eq}$ as the time at which $a = a_{\rm eq}$, then we will get

$$t_{\rm eq} \simeq \frac{2}{3}H_0^{-1}\Omega^{-1/2}(1+z_{\rm eq})^{-3/2} = 1.71(t_{\rm eq})_{\rm exact} = 2.7\times 10^{10}(\Omega h^2)^{-2}{\rm s}. \tag{3.32}$$

The values of $t_{\rm eq}$ in (3.32) and (3.30) differ slightly. For practical calculations we will use the expression in (3.32) rather than (3.30).

The formulas derived above are valid for $z \gtrsim 10$. A different approximation is useful for $z \ll z_{\rm eq}$ when we can ignore the contribution due to radiation and treat the universe as matter dominated. For lower redshifts, however, we need to take into account the curvature term (k/a^2). From (3.4) we get

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 = H_0^2\Omega a_0^2\left(\frac{a_0}{a}\right), \tag{3.33}$$

or, using the dimensionless variables $x = a/a_0$ and $y = H_0 t$,

$$\left(\frac{dy}{dx}\right) = (1 - \Omega + \Omega x^{-1})^{1/2}. \tag{3.34}$$

Integrating and using the fact that $a/a_0 = x = (1+z)^{-1}$, we get

$$H_0 t = \int_0^{(1+z)^{-1}} \frac{dx}{[1-\Omega+\Omega x^{-1}]^{1/2}}. \tag{3.35}$$

To be precise, we should set the lower limit of integration to be $(1+z_{\rm eq})^{-1}$, when matter domination begins; but we will be using (3.35) for $z \ll z_{\rm eq}$. Since $z_{\rm eq}^{-1} \ll z^{-1}$, the lower limit can be taken to be zero with sufficient accuracy. The value of the integral depends on Ω. We get, for $\Omega > 1$,

$$H_0 t = \frac{\Omega}{2(\Omega-1)^{3/2}}\left[\arccos\left(\frac{\Omega z - \Omega + 2}{\Omega z + \Omega}\right) - \frac{2[(\Omega-1)(\Omega z+1)]^{1/2}}{\Omega(1+z)}\right], \tag{3.36}$$

and, for $\Omega < 1$,

$$H_0 t = \frac{\Omega}{2(1-\Omega)^{3/2}}\left[\frac{2[(1-\Omega)(\Omega z+1)]^{1/2}}{\Omega(1+z)} - \operatorname{arccosh}\left(\frac{\Omega z - \Omega + 2}{\Omega z + \Omega}\right)\right]. \tag{3.37}$$

These expressions, together with the relation $a = a_0(1+z)^{-1} = H_0^{-1}|1-\Omega|^{-1/2}(1+z)^{-1}$, completely determine $a(t)$.

Setting z to zero in the expressions derived above will give us the *present* age of the universe for a given value of Ω:

$$\begin{aligned} t_0 &= \frac{1}{H_0}\frac{\Omega}{2(\Omega-1)^{3/2}}\left[\arccos\left(2\Omega^{-1}-1\right) - 2\Omega^{-1}(\Omega-1)^{1/2}\right] \quad (\text{for } \Omega > 1), \\ t_0 &= \frac{1}{H_0}\frac{\Omega}{2(1-\Omega)^{3/2}}\left[2\Omega^{-1}(1-\Omega)^{1/2} - \operatorname{arccosh}\left(2\Omega^{-1}-1\right)\right] \quad (\text{for } \Omega < 1), \end{aligned} \tag{3.38}$$

and $t_0 = (2/3)H_0^{-1}$ for $\Omega = 1$. We can use this expression, written in the form

$$t_0 = H_0^{-1}A(\Omega) = 9.78h^{-1}A(\Omega)\ \text{Gyr} \tag{3.39}$$

(where $A(\Omega)$ is the dimensionless 'age function') to constrain the parameters Ω and h. The function $A(\Omega)$ decreases monotonically with Ω; thus, for a fixed h, models with larger Ω will have younger age.

Observations suggest that the age of the universe is in the range (10-20) Gyr, and that $h \geq 0.5$. Taking $t_0 \geq 10$ Gyr, we get the bound $\Omega < 3.1$; if $t_0 \geq 12$ Gyr and $h \geq 0.5$ then $\Omega < 1.5$. For $\Omega = 1$, we get $t_0 = 6.52 \times 10^9 h^{-1}$ yr which is consistent with the condition $t_0 \geq 10$ Gyr if $h < 0.65$; if $\Omega = 1$ and $t_0 \geq 12$ Gyr we must have $h < 0.54$.

The mass density contributed by a particular species-x today is $\rho_x = \Omega_x \rho_c = 1.88 \times 10^{-29}(\Omega_x h^2)$ g cm^{-3}, in which the combination $(\Omega_x h^2)$ occurs. It is therefore convenient to have a bound on (Ωh^2) from observations. From (3.39) it follows that

$$B(\Omega) \equiv \Omega A^2(\Omega) = (\Omega h^2)\left(\frac{t_0}{9.78\ \text{Gyr}}\right)^2. \tag{3.40}$$

The function $B(\Omega)$ increases monotonically and reaches the value $\pi^2/4 \simeq 2.47$ as $\Omega \to \infty$; thus $\Omega h^2 < 2.47(t_0/9.78\ \text{Gyr})^{-2}$. Since $t_0 \geq 10$ Gyr, this gives the bound $\Omega h^2 \leq 2.4$ irrespective of the value of h. If we further use the constraint $h \geq 0.5$, we can improve this bound and get $\Omega h^2 \leq 0.8$. This result can be used to put powerful constraints on ρ_x contributed by various species.

A. Nonvanishing vacuum energy

Let us now consider universes with a nonvanishing vacuum energy, with all other energy set to zero. Unlike ordinary energy, a cosmological constant $\Lambda = 8\pi G\rho_\Lambda$ can be either positive or negative. When $\Lambda > 0$, we can find solutions for any spatial curvature:

$$\begin{aligned} k = -1 &\rightarrow a \propto \sinh\left[(\Lambda/3)^{1/2}t\right] , \\ k = 0 &\rightarrow a \propto \exp\left[(\Lambda/3)^{1/2}t\right] , \\ k = +1 &\rightarrow a \propto \cosh\left[(\Lambda/3)^{1/2}t\right] . \end{aligned} \tag{3.41}$$

In fact, all of these solutions are the same spacetime, "de Sitter space", just expressed in different coordinates. Any given spacetime can (locally) be foliated into spacelike hypersurfaces in infinitely many ways, although typically such hypersurfaces will be wildly inhomogeneous; de Sitter space has the property that it admits Robertson-Walker foliations with any of the three spatial geometries (just as Minkowski space can be foliated either by surfaces of constant negative curvature to obtain the Milne universe, or more conventionally by flat hypersurfaces). In general such foliations will not cover the entire spacetime: the $k = +1$ coordinates cover all of de Sitter (which has global topology $\mathbf{R} \times S^3$), while the others do not. However, this does not imply that the $k = +1$ RW metric is a "better" representation of de Sitter. For different purposes, it might be useful to model a patch of some spacetime by a patch of de Sitter in certain coordinates.

When $\Lambda < 0$, eq. (3.4) implies that the universe must have $k = -1$. For this case the solution is

$$a \propto \sin\left[(-\Lambda/3)^{1/2}t\right] . \tag{3.42}$$

This universe is known as "anti-de Sitter space", or "AdS" for short. The RW coordinates describe an open universe which expands from a Big Bang, reaches a maximum value of the scale factor, and recontracts to a Big Crunch (recall that for a nonzero Λ the traditional relationship between spatial curvature and temporal evolution does not hold). Again, however, these coordinates do not cover the entire spacetime (which has global topology $\mathbf{R}^4$). There are a number of differenet coordinates that are useful on AdS, and they have been explored by string theorists in the context of the celebrated correspondence between string theory on AdS in n dimensions and conformal field theory in $n-1$ dimensions. One of the reasons why AdS plays a featured role in string theory is that unbroken supersymmetry implies that the cosmological constant is either negative or zero. Of course, in our low-energy world supersymmetry is broken if it exists at all, and SUSY breaking generally contributes a positive vacuum energy, so one might think that it is not so surprising that we observe a positive cosmological constant. The surprise is more quantitative; the scale of SUSY breaking is at least 10^3 GeV, while that of the vacuum energy is 10^{-12} GeV.

de Sitter and anti-de Sitter, along with Minkowski space, have the largest possible number os isometries for a Lorentzian manifold of the appropriate dimension; they are therefore known as "maximally symmetric" (and are the only such spacetimes). In an n-dimensional maximally symmetric space, the Riemann tensor satisfies

$$R_{\mu\nu\rho\sigma} = \frac{1}{n(n-1)} R(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{3.43}$$

where R is the Ricci scalar, which in this case is constant over the entire manifold. The well-known symmetries of Minkowski space include the Lorentz group $\mathrm{SO}(n-1,1)$ and the translations $\mathbf{R}^4$, together known as the Poincaré group. de Sitter space possesses an $\mathrm{SO}(n,1)$ symmetry, while AdS has an $\mathrm{SO}(n-1,2)$ symmetry. All of these groups are of dimension $n(n+1)/2$.

B. The Ricci Tensor and Ricci Scalar

The affine connection is defined by

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}, \tag{3.44}$$

where ξ^{α} is a free falling coordinate system and x^{μ} is any other coordinate system. The affine connection serves in the equation of motion, $\frac{d^2\xi}{d\tau^2} = 0$:

$$\frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0. \tag{3.45}$$

For the FRW metric, the non-vanishing components of $\Gamma^{\lambda}_{\mu\nu}$ are

$$\begin{aligned}
\Gamma^{i}_{jk} &= \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \\
\Gamma^{0}_{ij} &= -\frac{\dot{a}}{a} g_{ij}, \\
\Gamma^{i}_{0j} &= \frac{\dot{a}}{a} \delta^{i}_{j}.
\end{aligned} \tag{3.46}$$

The Riemann-Christoffel curvature tensor $R^{\lambda}_{\mu\nu\kappa}$, the Ricci tensor $R_{\mu\kappa}$ and the Ricci scalar R are defined by

$$\begin{aligned}
R^{\lambda}_{\mu\nu\kappa} &\equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}, \\
R_{\mu\kappa} &\equiv R^{\lambda}_{\mu\lambda\kappa}, \quad R \equiv g^{\mu\kappa} R_{\mu\kappa}.
\end{aligned} \tag{3.47}$$

For the FRW metric, we obtain:

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{ij} &= -\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) g_{ij}, \\
R &= -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right).
\end{aligned} \tag{3.48}$$