Problem set #5

[1] Consider the volution of perturbations on scales much smaller than the horizon, $\lambda \ll d_H$, in a matter-dominated universe. Define "comoving coordinates" $\mathbf{x}$, related to the physical coordinates $\mathbf{r}$ by $\mathbf{r} = a(t)\mathbf{x}$, where a is the scale factor.

[1.1] The "peculiar velocity" $\mathbf{u}$ is defined as the difference between the matter velocity $\mathbf{v}$ and the Hubble flow $\dot{a}\mathbf{x}$, $\mathbf{u} = \mathbf{v} - \dot{a}\mathbf{x}$. Show that the evolution of the peculiar velocity in linear theory is

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} + \frac{\dot{a}}{a}\mathbf{u} = -a^{-1}\nabla_{\mathbf{x}}\delta\Phi,$$

where $\delta\Phi$ is the gravitational potential perturbation.

[1.2] Decompose the peculiar velocity to Fourier modes, $\mathbf{u}(\mathbf{k}, t)$, and separate each mode to $\mathbf{u}(\mathbf{k}, t) = \mathbf{u}_1 + \mathbf{u}_2$ such that $\mathbf{k} \cdot \mathbf{u}_1 = 0$ and $\mathbf{k} \times \mathbf{u}_2 = 0$. Show that the "rotational mode" decays in linear theory as $\mathbf{u}_1 \propto a^{-1}$.

[1.3] Show that on sufficiently large scale, corresponding to small $\mathbf{k}$ where mode growth rate is independent of $\mathbf{k}$ (how large is "sufficiently large"?),

$$\mathbf{u} = -\frac{2}{3H\Omega}\frac{d\ln\delta}{d\ln a}\nabla\delta\Phi,$$

where δ is the fractional density perturbation $(\delta\rho/\rho)$ and $H = \dot{a}/a$. How can this result be used to constrain cosmological parameters?

[2] We have shown that the equations describing the evolution of perturbations in the Newtonian limit are

$$\delta\dot{\rho} = -3H\delta\rho - 3\rho\delta H,$$

and

$$\delta\dot{H} = -2H\delta H - \frac{4\pi}{3}G\delta\rho - \frac{1}{3\rho}\nabla^2\delta p.$$

In the general relativistic case, these equations are replaced by

$$\delta\dot{\rho} = -3H\delta\rho - 3(\rho + p)\delta H,$$

and

$$\delta\dot{H} = -2H\delta H - \frac{4\pi}{3}G\delta\rho - \frac{1}{3(\rho + p)}\nabla^2\delta p$$

(where ρ is the energy density).

[2.1] Consider perturbations on scale $\lambda \gg d_H$. Show that $\delta\rho/\rho \propto a^\alpha$ with $\alpha = 1, \quad 2$ for the matter, radiation dominated epochs.

[2.2] Consider perturbations on scale $\lambda \ll d_H$ during radiation domination. Show that the perturbations exhibit an oscillatory behavior, and derive the oscillation period.

[3] Consider a steady, spherically symmetric distribution of gravitationally interacting particles. Assume that the velocity distribution of the particles is isotropic and independent of position. Derive the density distribution $\rho(r)$ as a function of $\overline{v^{-2}}$ (where the average is over the particle velocity distribution). How can this result be used to infer the mass of a galaxy cluster?