Problem set #6

[1] In our analysis of cosmic microwave background (CMB) fluctuations we have assumed that all photons have been last scattered at the decoupling redshift $z_{\rm dec.} = 10^3$, that is, that the last scattering surface is infinitesimally thin and defined by $z = z_{\rm dec.}$. In reality, the last scattering surface has a finite thickness, that is, we receive CMB photons that have been last scattered over a finite range of redshifts Δz around $z = z_{\rm dec.}$. Define the optical depth to redshift z, $\tau(z)$, as the average number of Thomson scatterings a CMB photon suffers propagating from the time $t = t(z)$ corresponding to redshift z till today, $t = t_0$:

$$\tau(z) = \int_{t(z)}^{t_0} dt\, c\, n_e\, \sigma_T \quad .$$

n_e is the number density of free electrons, and is given by $n_e = X_e n_b$, where n_b is the baryon number density and X_e is the fractional ionization.

[1.1] Show that the probability that a CMB photon we observe had originated from the redshift interval $[z, z + dz]$ is

$$f(z)dz = e^{-\tau(z)} \frac{d\tau}{dz} dz.$$

[1.2] Show that the thickness of the last scattering surface is $\Delta z \approx 100$. Hints: you may assume here that $X_e \ll 1$, and use $\eta = 10^{-9}$, $\Omega_b h^2 = 0.02$.

[1.3] How would the finite thickness of the last scattering surface affect the observed CMB anisotropy $\delta T/T$? On what angular scales?

[2] We have mentioned in the introductory lectures that the Universe plasma has been re-ionized at some redshift $z_{\rm reion.} \sim 10$. CMB photons may be Thomson scattered on their way to us by the free electrons at $z < z_{\rm reion.}$. Assume, for simplicity, that re-ionizations is instantaneous, i.e. that the value of X_e "jumps" from its value after decoupling to $X_e = 1$ at $z = z_{\rm reion.}$.

[2.1] Using the above definition of τ, derive an approximate expression for $\tau(z = z_{\rm reion.})$ for the 3 cosmological models $\{\Omega = 1, \Lambda = 0\}$, $\{\Omega \ll 1, \Lambda = 0\}$, $\{\Omega \ll 1, \Omega + \Lambda = 1\}$. You may use the approximate expressions we derived for $\dot{a}$.

[2.2] If $\tau(z = z_{\rm reion.}) > 1$, then CMB photons do not propagate freely, but rather diffuse, from $z = z_{\rm reion.}$ till today. What effect would this have on the observed CMB anisotropy $\delta T/T$? On what angular scales?

[3] Let us consider the possibility that the CMB is not a relic of the big-bang, but rather is produced by a large number of faint point sources (of microwave photons), which we can not individually resolve. For simplicity, let us assume that these sources are all identical (each producing the same luminosity of photons), and that they are homogeneously distributed throughout the Universe. Due to the redshift effects, the

flux of sources at distances $> c/H_0$ is suppressed (why?). Let us therefore further assume that we observe the flux from a homogeneous distribution of sources in a sphere (of radius $\sim c/H_0$) around us with Euclidean geometry.

[**3.1**] Show that the number of sources with (observed) flux $> f$ is $N(> f) \propto f^{-3/2}$ for $f > f_{\rm min.}$.

[**3.2**] Consider a telescope of angular resolution $\Delta\Omega$, which measures the flux in bins of solid angle $\Delta\Omega \ll 4\pi$ at different positions over the sky. Show that for a random distribution of the sources, i.e. assuming that their positions in space are not correlated, the standard deviation $\delta F(\Delta\Omega)$ of the distribution of fluxes measured in different bins is $\delta F(\Delta\Omega)/F(\Delta\Omega) = 3^{-1/2}[N(> f_{\rm min.})\Delta\Omega/4\pi]^{-1/3}$, where $F(\Delta\Omega)$ is the average flux over all bins.

[**3.3**] Given that $\delta F/F \sim 10^{-5}$ on 1 degree scale, what is the minimum number of sources? What is their minimum space density? Is this a reasonable scenario for the production of the CMB (remember that the galaxy density is $\approx 0.005{\rm Mpc}^{-3}$)?