

Theory of Figures to the 7th order and the interiors of Jupiter and Saturn

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ABSTRACT

Interior modeling of Jupiter and Saturn has advanced to a state where thousands of models are generated that cover the uncertainty space of many parameters. This approach demands a fast method of computing their gravity field and shape. Moreover, the Cassini mission at Saturn and the ongoing Juno mission delivered gravitational harmonics up to J_{12} . Here, we report the expansion of the Theory of Figures, which is a fast method for gravity field and shape computation, to the 7th-order (ToF7), which allows for computation of up to J_{14} . We apply three different codes to compare the accuracy using polytropic models. We apply ToF7 to Jupiter and Saturn interior models in conjunction with CMS-19 H/He-EOS. For Jupiter, we find that J_6 is best matched by a transition from He-depleted to He-enriched envelope at 2–2.5 Mbar. However, the atmospheric metallicity reaches $1\times$ solar only if the adiabat is perturbed toward lower densities, or if the surface temperature is enhanced by $\sim 14K$ from the Galileo value. Our Saturn models imply a largely homogeneous-in-Z envelope at $1.5\text{--}4\times$ solar atop a small core. Perturbing the adiabat yields metallicity profiles with extended, heavy-element enriched deep interior (diffuse core) out to $0.4 R_{\text{Sat}}$, as for Jupiter. Classical models with compact, dilute, or no core are possible as long as the deep interior is enriched in heavy-elements. Including a thermal wind fitted to the observed wind speeds, representative Jupiter and Saturn models are consistent with all observed J_n values.

1. INTRODUCTION

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Since the era of the Voyager 1 and 2 gravity field determinations and shape measurements of the outer planets, only two methods have extensively been employed to calculate the shape and the gravity field from interior models to compare with the data. These methods are the Theory of Figures (ToF) (Zharkov & Trubitsyn 1978) and the Concentric Maclaurin Spheroids (CMS) method (Hubbard 2012, 2013). ToF has served that purpose before the advent of accurate gravity data from Juno at Jupiter and from the Cassini Grand Finale Tour at Saturn. Beforehand, only the gravitational moments J_2 , J_4 , and J_6 were measured, and the smallest given uncertainty in Jupiter’s J_6 of 10% was still rather large (Jacobson 2003).

The low-order gravitational harmonics are important observables as they constrain the density profile about midway into the planetary interior. They are expansion coefficients of the external planetary gravity field evaluated at a reference radius in the equatorial plane, R_{eq} , which encompasses the planet’s total mass. They are defined as integrals over density $\rho(r)$ in the planet’s interior,

$$J_n = -\frac{1}{M R_{\text{eq}}^n} \int d^3r \rho(\vec{r}) r^n P_n(\cos \vartheta), \quad (1)$$

where P_{2n} are the Legendre polynomials, and ϑ is co-latitude. Thanks to the Juno and Cassini missions, the observational accuracy in the even harmonics J_{2n} have seen significant improvement (Iess et al. 2018; Durante et al. 2020; Iess et al. 2019). For both Jupiter and Saturn, the uncertainties in the low-order harmonics reduced to a level that can be considered exact from the perspective of adjusting internal density distributions to reproduce the data. However, significant spread in the deep interior density distributions is still possible, see Movshovitz et al. (2020) for Saturn, as the sensitivity of the J_{2n} toward the center fades with $(r/R_{\text{eq}})^{2n}$, see Eq. (1). This spread is a residual uncertainty related to other causes such as the positioning of internal helium gradients due to uncertainty in the H-He phase diagram, the temperature profile in stably stratified regions, the H-He equations of state, or the positioning of heavy element gradients due to uncertainties in planet formation and evolution.

ToF to the fourth order (ToF4) has been deemed sufficiently accurate for computation of J_2 and J_4 usually used to constrain the density distribution (Nettelmann 2017). Higher order moments beyond J_4 have been provided to a precision in J_6 to J_{10} of better than $(0.01-0.1) \times 10^{-6}$ for Jupiter (Durante et al. 2020) and $(0.1-1) \times 10^{-6}$ for Saturn (Iess et al. 2019), but as the order increases so does the influence by the zonal flows on the harmonics. At present, this is where the limitations of the ToF method become evident. ToF is an expansion method. An n th-order expansion (ToF $_n$) allows to compute up to J_{2n} and to an error of the order of q_{rot}^{n+1} , where $q_{\text{rot}} = \omega^2 R_{\text{eq}}^3 / GM$ is the ratio of centrifugal to gravitational force at the equatorial radius. The highest presented ToF-order so far is 5 (Zharkov & Trubitsyn 1975). It has recently been applied to compute Saturn’s J_2 – J_{10} values (Ni 2020), however, its accuracy has not been validated yet.

Although the CMS method is also an expansion method, it can conveniently be carried out to the order of 15-20 or higher (Hubbard 2013). Therefore, it enables high-accuracy computation of the high-order J_{2n} up to the order of the measurements (Wahl et al. 2017b; Militzer et al. 2019). The CMS method provides further advantages such as its expansion to 3D to account for tidal shape and gravity field perturbations (Wahl et al. 2017a) and brevity in its formulation (Hubbard 2013). Its only drawback is that CMS method goes along with high computational cost even in its accelerated version (Militzer et al. 2019). This is because CMS method explicitly solves for the 2D planetary shape not only taking the sum over radial spheroids but also by integrating over latitude. Even if making use of Gaussian quadrature, typically several tens of angular grid points are required. If

the integrand were a polynomial, only $N_{\text{lat}} = n + 1$ grid points would be required to evaluate the integrals over P_{2n} , which for J_{12} ($n = 6$) amounts to only $N_{\text{lat}} = 7$, or even 4 points when accounting for hemispheric symmetry. However, the integrands are functions of the non-polynomial shape itself. In practice, 48 grid points (Wahl et al. 2017a) are used. Obtaining the shape to sufficient accuracy at these grid points is the most time-consuming part of the CMS method. In contrast, ToF solves for the shape explicitly only in the equatorial plane while the shape at higher latitudes is obtained by spherical harmonics expansion, and the required precision of the shape is only the one wanted for the J_{2n} . One may thus see a benefit in using ToF for computation of the high-order moments.

Here, we introduce ToF7 tables, which allow one to calculate up to J_{14} . In Section 2, we **give an overview of the ToF method while for further details we refer to Appendix A**. In Section 3, we assess the accuracy of the ToF method by comparing to the analytic $n = 1$ polytrope solution. In Section 4, we apply the new tables to Jupiter models, and in Section 5 to Saturn. In Section 6, we connect representative interior models to thermal wind models to predict the wind decay depth profiles. **Observing notoriously low atmospheric metallicities of our Jupiter models, we discuss further influences in Section 7. Section 8 concludes the main body of the paper. In Appendix A.3 we introduce the ToF7 tables for public usage.**

2. THEORY OF FIGURES

The Theory of Figures is described in Zharkov & Trubitsyn (1978) and the coefficients up to the 3rd order presented therein. Nettelmann (2017) followed their notation and calculated the 4th-order coefficients. We note that 5th-order coefficients were presented in Zharkov & Trubitsyn (1975) and adopted by Ni (2020) for application to Saturn. Building upon the work of Nettelmann (2017), we here conduct the expansion of ToF to the 7th-order, meaning that the even harmonics up to J_{14} can be calculated.

Both the ToF and CMS methods assume that surfaces of equal potential U exist on which density and pressure are constant. One can show that this assumption holds for planets in hydrostatic equilibrium that rotate along cylinders, e.g., if their rotation rate can be expressed as $\vec{\omega} = \omega(s) \vec{e}_\omega$ with axis distance s . Rigid rotation and no rotation meet this condition. For rotation along cylinders, the odd harmonics J_{2n+1} disappear. However, the Juno measurements at Jupiter (Iess et al. 2018; Durante et al. 2020) and Cassini Grand Finale at Saturn (Iess et al. 2019) revealed that these planets' odd harmonics are non-zero. Instead, they are of the order of 0.1×10^6 , comparable to the values of J_{10} , J_{12} , and Saturn's uncertainty in J_6 . Based on the commonly used approach of using the thermal wind equation (TWE) to infer the density anomalies (Kaspi et al. 2010; Kaspi 2013), the depth of the wind-induced deviation from cylinder-rotation has been inferred to be about 3000 km ($\sim 0.035 R_{\text{Jup}}$) in Jupiter (Kaspi et al. 2018) and 9000 km ($\sim 0.14 R_{\text{Sat}}$) in Saturn (Galanti et al. 2019). These results are consistent with the tangent-cylinder model of Dietrich et al. (2021), which goes beyond the TWE simplification by including not only the wind-induced density perturbation but also the associated gravitational perturbation, what has long been argued to be significant (Zhang et al. 2015). Taking into account also constraints from the observed secular variation of the magnetic field on deep flows (Moore et al. 2019), suggests a somewhat steeper decay function for the winds, with the zonal flow extending inward on cylinders almost barotropically to a depth of about 2000 km on Jupiter and 8000 km on Saturn and then the winds decay abruptly within then next 1000 km (Galanti & Kaspi 2021). While the non-asymmetric gravity field is important for these width depth issues, here we have to neglect the asymmetries, as otherwise neither the ToF nor the CMS method could be applied.

120 In the absence of tides, the problem at hand is axisymmetric and thus 2D: r, ϑ . In ToF, the
 121 description is further reduced to 1D by introducing the mean radius coordinate l . Spheres of radius
 122 l are defined by the condition that they enclose the same volume as the equipotential surface $r_l(\vartheta)$,

$$123 \quad \frac{4\pi}{3} l^3 = 2\pi \int_0^\pi \sin \vartheta d\vartheta \int_0^{r_l(\vartheta)} dr' r'^2. \quad (2)$$

124 On the surface of the planet,

$$125 \quad \frac{4\pi}{3} R_m^3 = 2\pi \int_0^\pi \sin \vartheta d\vartheta \int_0^{R(\vartheta)} dr' r'^2$$

126 with $R(\pi/2) = R_{eq}$. In ToF, the potential is thus constant on spheres. Both the total potential $U(l)$
 127 and the axisymmetric 2D-shape $r_l(\vartheta)$ are expanded into Legendre polynomials. One can write

$$128 \quad U(l) = \frac{GM}{R_m} \left(\frac{l}{R_m} \right)^2 \sum_{k=0}^O A_{2k}(l) P_{2k}(\cos \vartheta), \quad (3)$$

129 while the shape of an equipotential surface, also called level surface, is given by

$$130 \quad r_l(\vartheta) = l \left(1 + \sum_{k=0}^O s_{2k}(l) P_{2k}(\cos \vartheta) \right), \quad (4)$$

131 where the $s_{2k}(l)$ are the figure functions. The condition that $U(l)$ is constant on spheres of radius l
 132 implies that $A_{2k} = 0$ for $k > 0$. These expansions are carried out up to an order O . In the absence
 133 of tides, U is a superposition of only the gravitational potential V and the centrifugal potential Q
 134 so that $U = V + Q$ and $A_{2k} = A_{2k}^{(V)} + A_{2k}^{(Q)}$. By definition, the gravitational harmonics J_{2n} can be
 135 obtained in the ToF as

$$136 \quad J_{2i} = -(R_m/R_{eq})^{2i} S_{2i}(1), \quad (5)$$

137 where the integrals S_{2i} , not to be confused with the figure functions s_{2i} , are given by Eq. (A7) in the
 138 Appendix. However, using Eq. (A7) for the S_n and the ToF-expansion coefficients to calculate the
 139 functions f_n on which the S_n depend, see Eqs. (A7,A9) in the Appendix, implies that only information
 140 on the equatorial radius $r_l(\pi/2)$ enters the computation of the J_{2n} , while information on the full
 141 shape $r(l, \theta)$ is reduced to the order of the expansion, that is up to P_{14} . An alternative method is to
 142 calculate the integrals over latitude explicitly. In Section 3.3 we compare both methods. **For details**
 143 **on how the ToF-coefficients are computed and for an example of the machine-readable**
 144 **asci-tables that contain their values for public usage, see Appendix A.3. Moreover, to**
 145 **facilitate the application of our ToF7 tables, we share computer routines for read-in of**
 146 **the tables and documentation at <https://doi.org/10.6084/m9.figshare.16822252>.**

147 3. VALIDATION AGAINST THE $N = 1$ POLYTROPE

148 The $n = 1$ polytropic planet is specified by a number of conditions. First, the polytropic EOS
 149 $P = K \rho^2$. Furthermore, the gravity field of the rotating polytrope depends on the values of q_{rot} ,
 150 equatorial radius R_{eq} , and planet mass M . The density profile $\rho(r)$ is not known in advance but is
 151 obtained from solution of the equation of hydrostatic equilibrium, $\nabla P/\rho = \nabla U$. In ToF, the radial

coordinate is taken the level surface l , and the **equation of hydrostatic equilibrium** reduces to $dP/dl = \rho dU(l)$. The internal m - l relation is obtained by integrating the equation of mass conservation, $m(l) = 4\pi \int_0^l dl \rho(l)l^2$. The latter is a source of numerical inaccuracy. We employ three different codes to compute the solution to the rotating polytrope. All polytropic models use $q_{\text{rot}} = 0.089195487$ and $GM = 126686536.1 \times 10^9 \text{ km}^3/\text{s}$ as in [Wisdom & Hubbard \(2016\)](#).

Before we compare the results of our application of three different codes and different orders of expansion of ToF to the analytic, Bessel-functions based method of [Wisdom & Hubbard \(2016\)](#) in [Figure 1](#), we describe each of the three employed methods in [Sections 3.1–3.3](#).

3.1. *Polytrope with MOGROP*

In the MOGROP code ([Nettelmann 2017](#)), the constant K is adjusted to fit the mass M . The mean radius R_m is adjusted to fit R_{eq} . The radial grid, for this application, is split into N grid points, out of which $N/2$ are equally distributed over 0 – $0.95 R_m$, and the other half equally over 0.95 – $1 R_m$. Such a choice was found to give a better match to the analytic solution than a split at $0.9 R_m$ or deeper. **Indeed with MOGROP, we find that the accuracy increases the farther out the separation is made, with a difference up to an order of magnitude compared to a flat distribution.** The integrals in [Eqs. \(A7,A8\)](#) are converted into integrals over density by partial integration and solved by simple trapezoidal rule. The integration of the equations of mass conservation, $dm/dl = 4\pi l^2 \rho(l)$, and hydrostatic equilibrium, $(1/\rho(l)) dP/dl = dU/dl$, is performed by the RungeKutta 4th-order method. The J_{2n} are computed using [Eq. \(5\)](#) and denoted by ‘4,7/Ne’ and shown as green curves in [Figure 1](#).

3.2. *Polytrope with TOF-PLANET*

The second code we use in our $n = 1$ polytrope comparison test case is an independent implementation of the ToF algorithm using the same coefficients but otherwise unrelated to the MOGROP code. The two codes are therefore expected to reproduce very similar solutions if given the same conditions. TOF-PLANET has previously been applied in a Bayesian study of Saturn’s possible interior ([Movshovitz et al. 2020](#)). Since for that purpose it was necessary to run tens of millions of density models to draw representative statistical samples, the code had to be optimized for speed and memory usage. An optional feature allows the shape functions to be explicitly calculated on a subset of level surfaces, while the shape of the rest can be spline-interpolated in the radial direction. This “skip-n-spline” trick can provide a significant speed advantage when high resolution density profiles are needed. We find that, even when a very high resolution of the density profile is required to accurately calculate integrals over density, there is no advantage in calculating the shape functions for more than a few hundred level surfaces. **The speed advantage of this optimization applies mainly to high-resolution ToF7 calculations. For lower resolution, and for most ToF4 runs the interpolation overhead ruins the effort. (ToF7 is much slower than ToF4 for a given N owing to the many more terms appearing in each of the shape function equations.)**

To validate TOF-PLANET with both ToF4 and ToF7 coefficients, we use it to reproduce the $n = 1$ polytrope test of [Wisdom & Hubbard \(2016\)](#). To make a direct comparison in a consistent way, some care is needed. The mass and equatorial radius are taken as in [Wisdom & Hubbard \(2016\)](#) and remain fixed for the duration of the calculation. (The mass is taken from the reported GM and with $G = 6.6738480 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$). However, in [Wisdom & Hubbard \(2016\)](#) the rotation state is given

194 by the parameter q_{rot} whereas the ToF algorithm needs the related parameter m_{rot} . The conversion
 195 needs the ratio R_{eq}/R_m , which is only available after the equilibrium shape is solved. To obtain a self-
 196 consistent solution we fix the planet’s rotation frequency ω using the value $q_{\text{rot}} = 0.089195487$. We
 197 then proceed with a guess for R_{eq}/R_m and therefore m_{rot} , solve for the shape function and gravity
 198 field, integrate the hydrostatic equilibrium equation to solve for pressure everywhere, update the
 199 density everywhere to match the polytropic relation, renormalize the level radii grid to match the
 200 reference equatorial radius, renormalize the density to match the reference mass, recalculate m_{rot} for
 201 the updated $R_{\text{eq}}/R_{\text{mean}}$ ratio, and rerun all the steps until a self consistent solution is found.

202 **In this test, for both ToF7 and ToF4, we compute all integrals with the trapezoidal rule**
 203 **and constant grid spacing. With TOF-planet, we experimented and found that different**
 204 **integration schemes and grid spacing schemes did not reduce substantially the number**
 205 **of grid points required for a given precision. This should not discourage, however, future**
 206 **users of our ToF7 tables from optimizing their grids and integration schemes for their**
 207 **particular cases.** The resultant J_{2n} values appear in blue and are denoted by ‘4,7/Mo’ in Figure
 208 1.

209 3.3. *Polytrope with CEPAM*

210 As in previous work (Ni 2020) we apply the CEPAM code (Guillot & Morel 1995) to calculate the
 211 gravity field and shape using ToF5 (Zharkov & Trubitsyn 1975). Here, we have expanded the code
 212 to address the case of the rotating $n = 1$ polytrope.

213 For the $n = 1$ polytropic EOS $P = K\rho^2$, the constant K is determined in terms of mass conservation
 214 and the mean radius R_m is adjusted to reproduce the equatorial radius $R_{\text{eq}} = 71492$ km. The initial
 215 density distribution is firstly given by that of a nonrotating $n = 1$ polytrope $\rho(z) = \rho_c \sin \pi z / \pi z$.
 216 The figure function $s_{2k}(z)$ and total potential $U(z)$ are computed using the ToF5 as described in
 217 Zharkov & Trubitsyn (1975) and Ni (2019). **In its original version, CEPAM uses an automatic**
 218 **grid refinement method that distributes the grid points in a way that a distribution**
 219 **function of the variables pressure, temperature, mass, radius, and luminosity changes**
 220 **by a constant amount at the grid points. This method requires smooth behavior of**
 221 **the variables and their derivatives. B-splines are used as the interpolating polynomials,**
 222 **which exhibit the desired properties. However, for number of grid points larger than**
 223 **10^3 , we did not obtain stable solutions with CEPAM. Therefore, for higher number of**
 224 **grid points we switched to our own solution of the pressure profile using the trapozoidal**
 225 **rule**

$$226 \quad P(z_j) = P(z_{j-1}) + 0.5[\rho(z_j) + \rho(z_{j-1})][U(z_j) - U(z_{j-1})] \quad (6)$$

227 with the outer boundary condition $P(R_m) = 0$ Mbar. **In this case and** in view of the fact that
 228 gravitational harmonics show greater sensitivities to the external levels of a planet, more radial grid
 229 points are taken for the outer part: $N/2$ equally distributed over $0.85-1 R_m$ and the other half equally
 230 over $0-0.85 R_m$. **With CEPAM, we find this choice yields a modest optimum in accuracy.**

231 Finally, the new density distribution is obtained from the $n = 1$ polytropic EOS $\rho(z_j) = \sqrt{P(z_j)/K}$.
 232 This procedure is iteratively performed until all changes in the density distribution are reduced to
 233 within a specified tolerance.

234 In the work of Ni (2020), the gravitational zonal harmonics are calculated as weighted integrals over

Table 1. Comparison of the J_{2n} obtained from Eq. (7) and from Eq. (5)

Method	$J_2 \times 10^6$	$J_4 \times 10^6$	$J_6 \times 10^6$	$J_8 \times 10^6$	$J_{10} \times 10^6$
Bessel	13988.51	-531.8281	30.11832	-2.13212	0.17407
Eq. (7)	13988.54	-531.8292	30.11989	-2.13048	0.17446
$ \Delta J_{2i}/J_{2i}^{\text{Bessel}} $	2.42×10^{-6}	1.99×10^{-6}	5.22×10^{-5}	7.67×10^{-4}	2.23×10^{-3}
Eq. (5)	13988.55	-531.8207	30.13506	-2.10486	0.19555
$ \Delta J_{2i}/J_{2i}^{\text{Bessel}} $	2.52×10^{-6}	1.40×10^{-5}	5.56×10^{-4}	1.28×10^{-2}	1.23×10^{-1}

NOTE—The numerical values in this Table are for a typical number of grid points $N = 2000$ and using the ToF5 coefficients of Zharkov & Trubitsyn (1975). The Bessel solution is taken from Wisdom & Hubbard (2016).

235 the internal density distribution $\rho(z)$ using the resulting figure functions $s_{2k}(z)$,

$$236 \quad J_{2i} = -\frac{2\pi}{MR_{eq}^{2i}} \int_{-1}^{+1} d\cos\theta P_{2i}(\cos\theta) T(R_m, \theta), \quad (7)$$

$$237 \quad T(R_m, \theta) = \int_0^{R_m} dl \rho(l, \theta) r^{2i+2} \left(\frac{dr}{dl} \right). \quad (8)$$

239 Using the scaled mean radius $z = l/R_m$ and abbreviating the level surface Eq. (4) as $r = zR_m[1 + \Sigma(z, \theta)]$, one can express the function $T(R_m, \theta)$ as

$$241 \quad T(R_m, \theta) = \frac{3M R_m^{2i}}{4\pi} \left\{ \frac{[1 + \Sigma(1, \theta)]^{2i+3}}{2i+3} \frac{\rho(1, \theta)}{\bar{\rho}} - \int_0^1 dz \frac{[1 + \Sigma(z, \theta)]^{2i+3}}{2i+3} z^{2i+3} \frac{d\rho(z, \theta)/\bar{\rho}}{dz} \right\}. \quad (9)$$

242 Table 1 shows a comparison of the even harmonics obtained from Eq. (7) and from Eq. (5) for a
 243 typical number of grid points $N = 2000$. The numerical accuracy in J_2 is almost the same for both of
 244 them. But the results from Eq. (7) are in better agreement with the analytic Bessel-functions based
 245 solution for J_4 – J_{10} , where the numerical accuracy for Eq. (7) is about 1–2 orders of magnitude better
 246 than that for Eq. (5).

247 3.4. Comparison of the J_{2n} of the uniformly rotating polytrope

248 In Figure 1 we show the relative deviations of the calculated even harmonics from the analytic
 249 solutions of Wisdom & Hubbard (2016) as a function of the number N of radial grid points.

250 With CEPAM code and ToF5, denoted by 5/Ni in the Figure, the numerical accuracy of all the
 251 calculated J_{2i} shows good convergence with an increased number of grid points. When the number
 252 of grid points is increased beyond $\sim 10^3$, the numerical accuracy in J_4 – J_{10} falls below the current
 253 observational uncertainty (Juno D20) reported in Durante et al. (2020). Moreover, the accuracy in
 254 all the harmonics J_2 – J_{10} is better than the CEPAM-WH16 results from Guillot et al. (2018), who
 255 reportedly applied ToF4, by a factor of roughly 5–100.

256 Using ToF4 and ToF7 in conjunction with the MOGROP code, denoted by 4/Ne and 7/Ne in the
 257 Figure, the accuracy significantly improves with denser grid points. Apparently, this code requires
 258 a factor of 100 more radial grid points than CEPAM to obtain the same accuracy in J_2 and J_4 .
 259 For these low-order harmonics, ToF7 vs. ToF5 provides a negligible improvement in accuracy. The

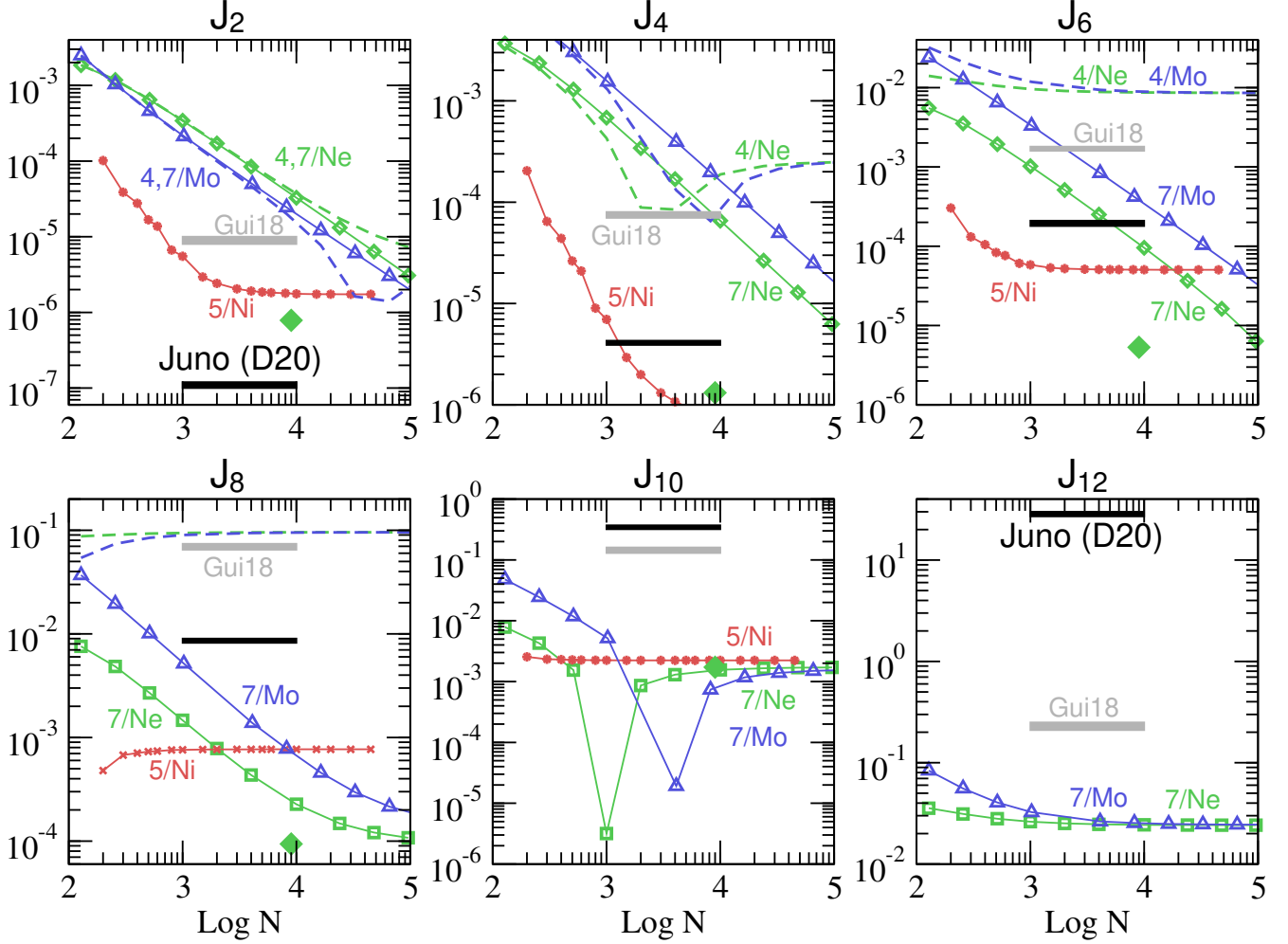


Figure 1. Relative differences $|\Delta J_n/J_n|$ between ToF solutions and the analytic Bessel-functions based solution (Wisdom & Hubbard 2016). Dashed lines (green and blue): ToF4; lines with open symbols (green and blue): ToF7; red lines with stars: ToF5 using CEPAM and labeled 5/Ni, green lines and labeled 4/Ne or 7/Ne: using MOGROP, blue lines and labeled 4/Mo or 7/Mo: using TOF-planet; grey bars labeled Gui18: CEPAM-WH16 from Guillot et al. (2018), black bars labeled Juno (Du20): obs. uncertainty (Durante et al. 2020). X-axis shows number N of radial grid points used in this work. **Green diamonds placed at $N = 9000$ are for extrapolated J_{2n} values based on linear regression on the three computed values at $N = 1000, 2000, 4000$.**

260 situation changes with J_6 . Here, CEPAM with ToF5 levels off at a relative uncertainty of $\sim 5 \times 10^{-5}$,
 261 while the higher accuracy of ToF7 vs. ToF5 becomes evident as N increases beyond 20,000. However,
 262 the typical number of grid points used for planet interior models ranges between 2000 and 4000. For
 263 such N values, the numerical accuracy in J_6 with 7/Ne is about the same as the current observational
 264 uncertainty reported in Durante et al. (2020). ToF7 begins to pay off with J_8 and higher, even with
 265 MOGROP, where J_8 becomes an order of magnitude better than the observational one, 2.5 orders of
 266 mag in J_{10} , and 3 orders of mag in J_{12} . We conclude that ToF7 is sufficiently accurate to address
 267 the influence of the winds on J_6 and higher, given current observational uncertainties.

268 Using the independent TOF-PLANET code of Movshovitz et al. (2020), we obtained similar J_{2n}
 269 values as with the MOGROP code, compare the blue and the green lines in Figure 1. In particular,

270 the results for J_4 – J_8 with ToF4 after convergence with grid point number N agree, indicating that
 271 the remaining errors $\Delta J_n/J_n$ of 2.5×10^{-4} in J_4 , 10^{-2} in J_6 , and 10^{-1} in J_8 are due to the truncation
 272 of the expansion of ToF4. The ToF7 values also agree when convergence is reached, though this
 273 applies only to J_{10} and J_{12} for high values of $N > 10,000$. Before convergence with N is reached, the
 274 ToF7 errors deviate by a factor of a few, suggesting that the radial grid discretization error matters,
 275 which can differ between different implementations even if they use the same trapezoidal rule.

276 **The similar accuracy of Mogrop and TOF-planet is due to similar methods for the**
 277 **numerical integration over density (trapezoidal rule) and of the differential equations**
 278 **dm/dr and dP/dr (Runge-Kutta). There is room for improvement. As an example, we**
 279 **extrapolate the J_{2n} values using linear regression on the three solutions for $N = 1000$,**
 280 **$N = 2000$, and $N = 4000$ for each J_{2n} . In Figure 1, the resulting accuracy is conservatively**
 281 **compared to the result for $N = 9000$ corresponding to a the computational cost that**
 282 **scales linearly with N and a small offset for each run. Apparently, the gain in accuracy**
 283 **amounts two orders of magnitude in J_2 and J_4 , and is still better than compared to**
 284 **using 10^5 grid points. This suggests that methods other than simply sky-rocketing the**
 285 **number of grid points may help to improve the accuracy of numerical J_{2n} computations.**

286 We note that **at present**, ToF is entirely outperformed by the CMS method in regard to accuracy.,
 287 [Militzer et al. \(2019\)](#) reported relative inaccuracies of only 7.3×10^{-9} in J_2 , 2.1×10^{-10} in J_4 , 3.6×10^{-8}
 288 in J_6 , 4.2×10^{-8} in J_8 , 1.1×10^{-7} in J_{10} , and 6.7×10^{-9} in J_{12} for $N = 2^{17} = 131072$ CMS layers,
 289 out of which only 512 are treated explicitly, while the shape of intermediate ones is obtained by
 290 interpolation.

291 4. APPLICATION TO JUPITER

292 4.1. *Jupiter models*

293 The models of this work assume a four-layer structure. By Y_i we denote the helium mass fraction
 294 in layer number i with respect to the H/He system. Layer 4 is a compact rocky core. Layer 1
 295 is an atmosphere with a helium mass fraction of $Y_1 = 0.238$ as measured by the Galileo entry
 296 probe. Layers 2 and 3 have the same helium abundance ($Y_2 = Y_3$), which is adjusted to yield an
 297 overall helium mass fraction $Y = 0.2700(4)$. A possible He-rain region in Jupiter is represented
 298 by a jump in helium abundance between layers 1 and 2. Transition pressures of 1–4 Mbar for
 299 P_{12} are considered, which are typical pressures where the Jupiter adiabat reaches the closest point
 300 to the H/He-demixing boundary of H/He phase diagrams for protosolar H/He ratios as predicted
 301 by first-principles simulations ([Lorenzen et al. 2011](#); [Morales et al. 2013](#); [Hubbard & Militzer 2016](#);
 302 [Schoettler & Redmer 2018](#)). Adjusting the local He abundance to the local P - T conditions along the
 303 phase boundary yields an approximately linear increase in Y ; however, the gradient and width of the
 304 He-rain region depends on the temperature profile assumed therein ([Nettelmann et al. 2015](#)), which
 305 may range in Jupiter from adiabatic to modest superadiabaticity ([Mankovich & Fortney 2021](#)). A
 306 recent analysis of reflectivity data obtained for H/He samples that were pre-compressed to 2–4 GPa in
 307 Diamond Anvil Cells and further shock-compressed to 60–180 GPa using the OMEGA layer indicate
 308 that an even larger portion of the Jupiter adiabat may intersect with the H/He phase boundary, as
 309 at the highest pressure where evidence of demixing is seen, 150 GPa, the measured temperatures
 310 were 10,000 K ([Brygoo et al. 2021](#)). Assuming a flat $T(P)$ phase curve at Mbar pressures, such a
 311 temperature corresponds to ~ 8 Mbar along the Jupiter adiabat ([Hubbard & Militzer 2016](#)).

312 Although He droplets may carry specific elements such as Ne with them downward
 313 (Wilson & Militzer 2010) and affect the metallicity between the He-depleted outer and He-enriched
 314 inner region, we assume constant heavy element mass fractions across that boundary ($Z_1 = Z_2$).
 315 Finally, between layers 2 and 3, the heavy element mass fraction is allowed to change. We either use
 316 a constant Z_3 -value implying a jump in Z at a transition pressure P_{23} , or a Gaussian Z_3 -profile that
 317 starts with $Z(P_{23}) = Z_2$ and smoothly increases toward a maximum $Z_{3,max}$ at $P = 38$ Mbar near the
 318 core. The choice of 38 Mbar is arbitrary and was taken to be just atop usual core-mantle-boundary
 319 pressures, which are found to be around 40 Mbar in Jupiter. The two free parameters in that setup
 320 to adjust J_2 and J_4 are Z_1 and Z_3 or $Z_{3,max}$.

321 We employ the CMS-2019 equations of state (EOS) for H and He (Chabrier et al. 2019) and mix
 322 them with the water EOS H2O-REOS with respect to density only. The T - P profile is that of
 323 the H/He adiabat, which begins at $T=166.1$ K at 1 bar. We construct curves of constant entropy
 324 (adiabats) by using the specific entropy values $s_H(P, T)$ and $s_{He}(P, T)$ provided in the tables for
 325 hydrogen and helium (Chabrier et al. 2019) after adding a composition-dependent entropy of mixing
 326 term $s_{mix}(X_{H_2}, X_{He})$. For the concentrations, we assume that helium is non-ionized and that hydrogen
 327 is either molecular or ionized, taking the degree of ionization as in Nettelmann et al. (2008). Since
 328 we found these H/He adiabats to be too dense to yield Jupiter models with non-negative atmospheric
 329 metallicity, we also perturb that adiabat toward lower densities as described in Section 4.3.

330 4.2. Results for Jupiter's even harmonics

331 In Figure 2 we show the even J_2 - J_{10} values from rigidly rotating Jupiter models. Models adjusted
 332 to the Juno-observations of J_2 and J_4 (Durante et al. 2020) are shown in bluish color, while models
 333 adjusted to the wind-corrected J_2 and J_4 values by the corrections of Kaspi et al. (2018) applied to
 334 the J_2 , J_4 values of Durante et al. (2020) are shown in reddish color.

335 Due to imperfect fit to the J_2 , J_4 values, the scatter in model J_2 and J_4 values is larger than
 336 the observational uncertainty. For J_4 , the scatter $\Delta J_4/J_4$ is about $\pm 2 \times 10^{-4}$ and of same size as
 337 the relative uncertainty due to using ToF7 in the MOGROP code, while for J_2 the latter relative
 338 uncertainty is with 7×10^{-5} overwhelming. Still, these relative deviations are too small to matter for
 339 the inferred metallicities. Guillot et al. (2018) allowed for a similarly wide scatter in J_2 model values
 340 of $\pm 3.4 \times 10^{-5}$ and a wider scatter in J_4 of $\pm 10^{-3}$ relative deviations. They found that nevertheless,
 341 the high-order moments J_8 vs. J_6 and J_{10} vs. J_8 were strictly confined to a straight line. We confirm
 342 that behavior.

343 Notably, the model $|J_8|$ values are higher than the observed value, and a trend in that direction
 344 is also seen for J_{10} although the model J_{10} values are still within the 3σ observational uncertainty.
 345 Wind models assuming rotation along cylinders indeed predict a slight decrease in $|J_8|$ and J_{10} if the
 346 observed wind profile of the southern hemisphere is applied to the entire surface, while they predict
 347 an enhancement if the wind profile of the northern hemisphere is used (Hubbard 1999). The wind
 348 model by Kaspi et al. (2018) that was adjusted to explain the odd moments of Jupiter observed by
 349 Juno yields a correction qualitatively in the direction as predicted for the southern winds and seen in
 350 the model values for rigid rotation, albeit quantitatively stronger by a factor of two. This deviation
 351 may have many reasons; clearly, further exploration of the connection between interior and wind
 352 models is desirable.

353 For J_6 , the uncertainties from the application of ToF7 in the MOGROP code, from observations, and
 354 from the wind contribution are all small and of same size. In contrast, model assumptions such as the

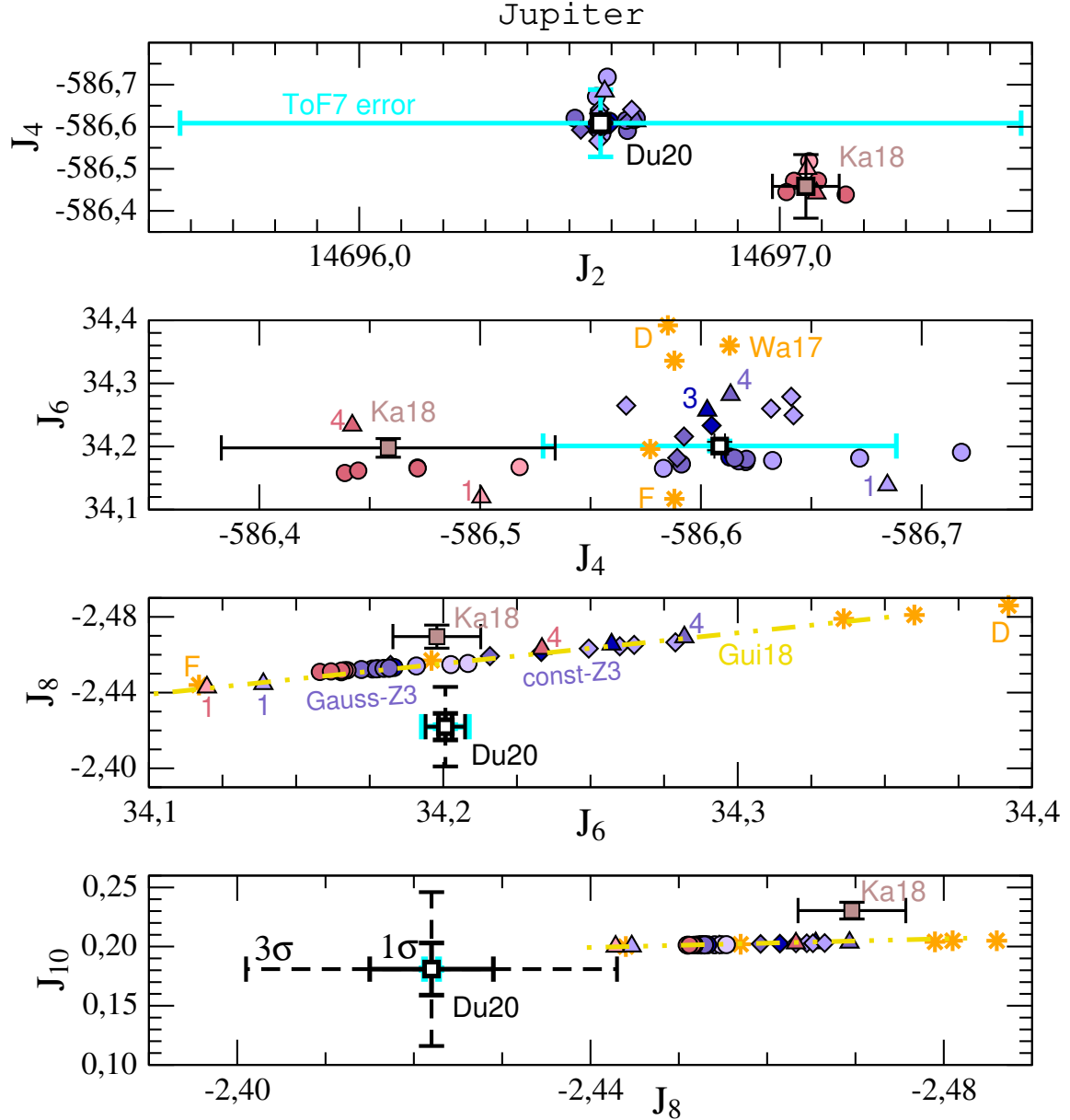


Figure 2. J_{2n} values observed by Juno (white squares), corrected for latitude-dependent winds (brown squares; Kaspi et al. 2018), models with MH13 EOS (orange stars, Wahl et al. 2017b), models in Extended Data Figure 1 of Guillot et al. (2018) (yellow), models with CMS19 H/He-Eos and $P_{12} = 1, 3, 4$ Mbar (triangles), $P_{12} = 2$ Mbar and P_{23} varied from 5 to 20 Mbar and constant Z_3 (diamonds) or Gaussian Z_3 (circles). All J_{2n} values are multiplied by 10^6 .

355 location of layer boundaries not only have a larger influence on J_6 but also yield a scatter *around* the
 356 observed value. Hence, we conclude that Jupiter's J_6 is unique in that it is neither adjusted nor seems
 357 to be significantly influenced by the winds, and therefore offers an additional parameter to further
 358 constrain interior models. We find that models with a Gaussian- Z_3 and an abrupt He-poor/He-rich
 359 transition at $P_{12} = 1\text{--}2$ Mbar yield $J_6 = 34.11\text{--}34.18$ slightly below the observed value, while models
 360 with that transition deeper inside at $P_{12} = 2.5\text{--}3$ Mbar yield $J_6 = 34.23\text{--}34.28$ slightly above the
 361 observed value. Constant- Z_3 profiles and $P_{12} = 2$ Mbar stretch from $J_6 = 34.18$ to 34.28 around the

observed value 34.2007 ± 0.0067 upon shifting P_{23} from 5 Mbar deeper down to 18 Mbar. Taken at face value, Jupiter’s observed J_6 value indicates that the He-depleted/He-enriched transition occurs at around 2–2.5 Mbar.

4.3. Z -profiles for Jupiter

In Figure 3 we show the radial heavy element distribution of some of the models with Gaussian- Z in layer No. 3. Models with unmodified H/He-adiabat appear in light-blue and are described in Section 4.3.1, while models with modified H/He-adiabat appear in red and are described in Section 4.3.2.

4.3.1. Unmodified H/He-adiabat

All our models with CMS-19 EOS that fit J_2 and J_4 have negative Z_1 values between -0.005 and -0.020 (-0.33 to $-1.33 \times$ solar). This is consistent with Hubbard & Militzer (2016), who obtained $-0.6 M_E$ of heavy elements in the molecular region for their MH13 EOS based model DFT-MD7.15, which, with $J_4 = 587$, is the one that comes closest to the Juno value of 586.61. For a conservative estimate of their Z_1 value we take 1 Mbar, the entry of their Jovian adiabat into the H/He demixing boundary of Morales et al 2013, or 2 Mbar, the pressure-medium in their H/He-rain region. With a corresponding molecular envelope mass of $\sim 30 M_E$ and $\sim 53 M_E$, respectively, we obtain Z_1 between -0.011 and -0.02 for model DFT-MD7.15. In contrast, Debras & Chabrier (2019) found a variety of models for non-negative atmospheric Z values of $1 \times Z_{\text{Gal}} = 0.0167$ using CMS-19 H/He EOS. We cannot reproduce the results of Debras & Chabrier (2019) quantitatively.

The deeper the layer boundary for heavy elements is placed, the higher will the deep interior heavy element abundance become, and the smaller the core mass (Nettelmann et al. 2012). With CMS-19 EOS, the response of M_{core} to P_{23} is comparably weak, so that P_{23} can be placed as deep as 20 Mbar before the compact core disappears. The thick blue model in Fig. 3 is for $P_{23} = 21$ Mbar and has a core mass of only $0.25 M_E$.

The non-exclusive compact core mass values of our models range from 0.2 to $4.8 M_E$ for $P_{12} = 2$ Mbar and Gaussian- Z envelopes, from 0.6 to $6.0 M_E$ for constant- Z envelopes, and from 1.2 to $3.8 M_E$ for wind-corrected models.

For Gaussian- Z deep envelopes, $Z_{3,\text{max}}$ can become quite large toward the center. We obtained $Z_{3,\text{max}}$ values up to 0.5, although larger values may be possible if the maximum of the Gaussian curve is placed at the center, while we placed it slightly off at 38 Mbar.

The mass of heavy elements in the deep interior below the negative- Z envelope amounts to 7.5 – $10.1 M_E$. Assuming a $1 \times$ solar instead of negative- Z envelope would add another $3.8 M_E$ of heavy elements. A total of 11.3 – $13.9 M_E$ of heavy elements is consistent with the Jupiter core accretion formation models A and C of Lozovsky et al. (2017). These models assume solids surface densities of 6 and 10 g/cm^2 and planetesimal sizes of 100 km and 1 km, respectively. Lozovsky et al. (2017) find that a total amount of heavy elements of $9.3 M_E$ (A) and $16.4 M_E$ (C) is accreted. Correspondingly, for the average Z -value after final mass accretion, Helled & Stevenson (2017) find ~ 0.03 – $0.05 \times M_J = 9.5$ – $16 M_E$ of heavy elements for model A and a third model D, which assumes a solids surface density of 10 g/cm^2 like model C. In Figure 3, models A and C are shown after settling of heavy elements but before possible convective mixing. Settling takes place if the partial pressure of ablated incoming material exceeds its vapor pressure. The resulting $Z(m)$ profiles after formation resemble our interior models with Gaussian- Z_3 , although the Z -gradient in the post-formation models begins farther out at $\sim 0.5 M_J$ than at ~ 0.2 – $0.3 M_J$ as in our models. On the other hand, a shallow, primordial

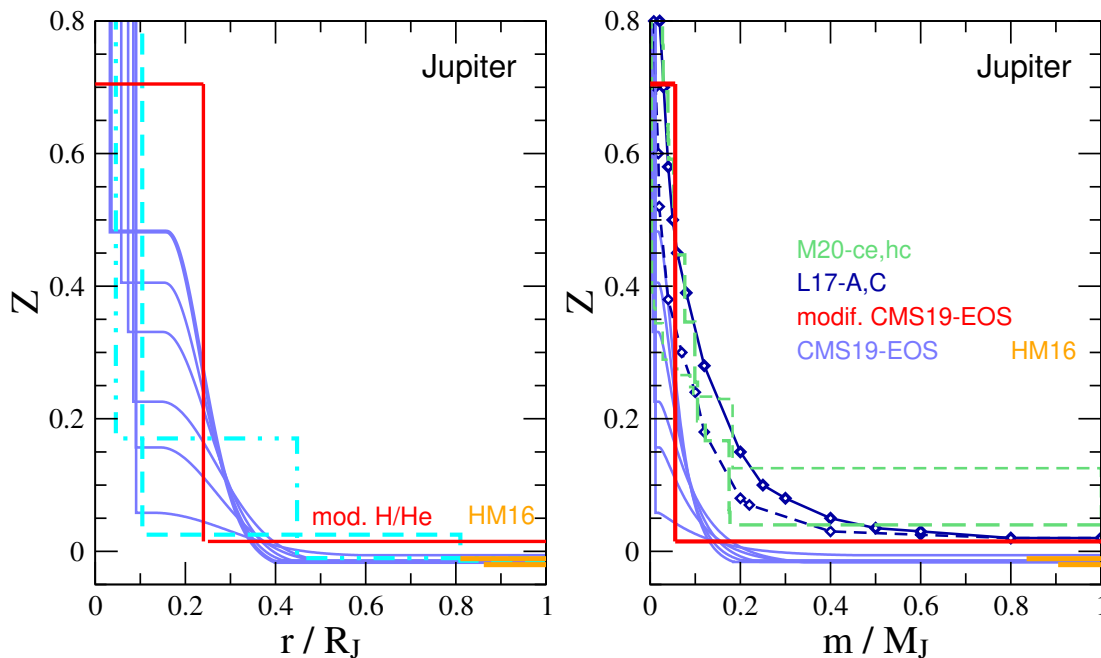


Figure 3. Internal heavy element abundance profiles over radius (left panel) and over mass (right panel) of some of the Jupiter models in Figure 2 with Gaussian- Z_3 (blue), with constant- Z_3 (cyan) and for a model with $\rho(P)$ along the adiabat modified to yield $1 \times$ solar Z (red). Overplotted are the outer envelope Z -level of model DFT-MD7.15 from Hubbard & Militzer (2016) (orange), the $Z(m)$ profiles of the formation models A and C (dark blue) of Lozovsky et al. (2017) at the final stage of mass accretion, after settling but before possible homogenization by mixing, and $Z(m)$ profiles of Müller et al. (2020) for their envelope accretion models assuming a hot-compact or a cold-extended state at the onset of gas accretion (green).

404 compositional gradient has been found to erode and to be erased in present Jupiter if vigorous
 405 convection takes place in the envelope (Müller et al. 2020), while a step compositional gradient may
 406 still persist within $0.2M_J$. The present-state models for Jupiter of Müller et al. (2020) are similar to
 407 our models with either Gaussian- Z_3 or constant- Z_3 when the heavy element-enriched deep interior
 408 (or dilute core) is assumed to begin deep inside at > 15 Mbar, except that our models underestimate
 409 the outer envelope metallicity while the evolution models of Müller et al. (2020) overestimate it, as
 410 they yield too small a present-day radius.

411 4.3.2. H/He adiabat modification

412 Models with negative Z -values are, needless to say, not considered a viable solution. There are two
 413 obvious ways how negative Z -values in the atmosphere and outer envelope can be circumvented. One
 414 possibility is to assume a superadiabatic region above the region where J_4 is most sensitive, which
 415 –for a polytropic Jupiter model– is in the molecular envelope at around 50 GPa ($0.9 R_J$). A super-
 416 adiabatic temperature profile may result from stable stratification. Christensen et al. (2020) showed
 417 that meridional flows in a stably stratified, slightly conducting region slow down the strong zonal
 418 flows and suggested the existence of such a region in Jupiter as an explanation for the truncation
 419 of the zonal flows, which, according to recent combined analysis of magnetic field and gravity field
 420 data, occurs rather sharply at $0.97 R_{Jup}$ (Galanti & Kaspi 2021).

421 However, stable stratification does not necessarily result in a super-adiabatic temperature profile.
 422 Depending on its origin, stable stratification can also be accompanied by a sub-adiabatic temperature

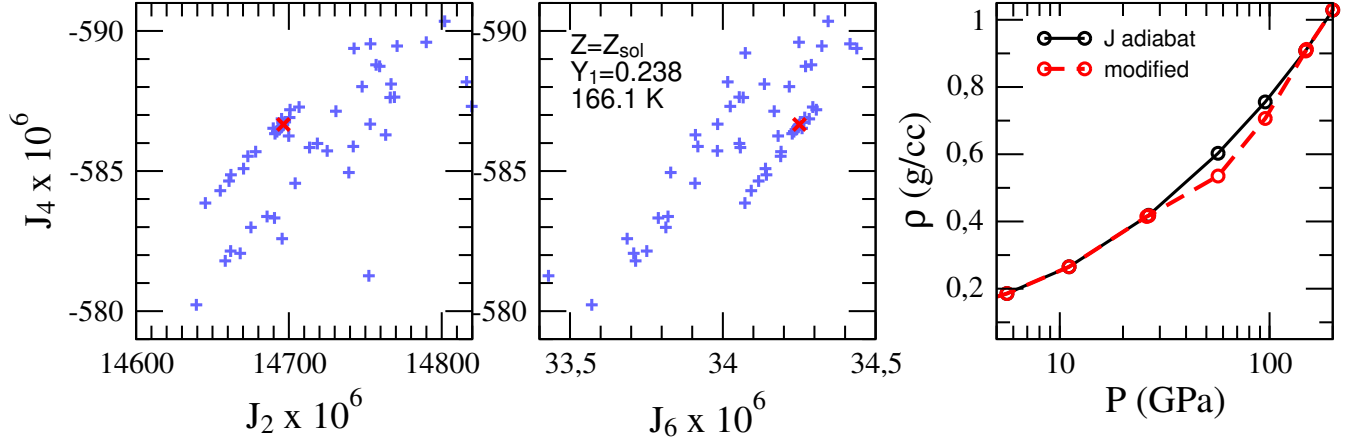


Figure 4. J_2 – J_4 (left) and J_6 – J_4 (middle) values of three-layer models with $Z_1 = 1 \times$ solar and modified Jupiter adiabats. Only the red-highlighted model meets the Juno constraints. Its adiabat is shown in the right panel (red curve).

profile. For instance, in the absence of alkali metals the opacity of the H/He fluid becomes sufficiently low for the intrinsic heat to be transported by radiation along a sub-adiabatic radiative gradient (Guillot et al. 1994; Guillot et al. 2004), leading to sub-adiabatic stable stratification according to the Schwarzschild criterion. Super-adiabatic gradients are predicted in a Ledoux-stable, inhomogeneous medium of upward decreasing mean molecular weight. Clouds formed by condensibles of higher molecular weight than the background composition can induce Ledoux-stability if their abundance is high enough. Such a scenario has been proposed for the presumably water-rich atmospheres of the ice giants (Leconte et al. 2017). Water clouds may occur in Jupiter at 100 bar, silicate clouds at 1000 bar, both below the level that so far could be probed by the Galileo entry probe (22 bar) and Juno MWR remote sensing (Li et al. 2020). Therefore, clouds are candidate causes for super-adiabatic stable stratification.

Another possibility to avoid negative metallicities is to perturb the H/He-adiabat toward lower densities. The CMS-19 hydrogen EOS shows excellent agreement with a variety of experimental data ranging from shock compression experiments for H and D at various initial conditions to isentropic compression (Chabrier et al. 2019). At 50 GPa, the H EOS is even slightly stiffer than the experimental data. Only the helium EOS shows significantly higher densities in the 20 GPa to 150 GPa area than inferred from the shock compression experiments (Chabrier et al. 2019). Although the good agreement between the theoretical P - ρ relations and the experiments as well as between CMS-19 H/He adiabats and MH13 EOS adiabats is far from suggesting that the CMS-19 H/He EOS would significantly overestimate the density along the Jupiter adiabat, we here perturb it toward lower densities. We conduct a three-parameter study where we vary the maximum deviation $\delta\rho_{\max}$, the pressure entry point P_{start} and the pressure exit point P_{end} . Between P_{start} and P_{end} , $\delta\rho$ adopts its maximum at the logarithmic mean pressure value and is otherwise linearly interpolated as $\delta\rho(\log P)$. We explore P_{start} values between 1 and 50 GPa while P_{end} values between 50 and 150 GPa. The smaller P_{start} and P_{end} , the lower the $|J_4|$ and $|J_4|/J_2$ ratio. We find $P_{\text{start}} \leq 30$ GPa necessary in order to have a noticeable influence on J_4 . Conversely, higher P_{end} values lead to a stronger reduction on J_2 . The question we ask is, for what values of $\delta\rho_{\max}$, P_{start} , and P_{end} can a model be found with a $1 \times$ solar homogeneous Z ?

For a homogeneous, unperturbed adiabat at $Y_1 = 0.238$ and Y_2 adjusted to meet $Y = 0.27$, both $|J_4|$ and J_2 turn out significantly too large. This contrasts the result by [Debras & Chabrier \(2019\)](#), who could match J_2 at $1 \times Z_{\text{Gal}}$. Thus, we need P_{start} to be sufficiently low for J_4 and P_{end} sufficiently high for J_2 . We find such an optimized solution for $P_{\text{start}} = 26$ GPa, $P_{\text{end}} = 150$ GPa, and $\delta\rho = -0.1257$, i.e., a maximum reduction of the H/He adiabat by 12.57%. The resulting P - ρ profile is shown in [Figure 4c](#), while the ensemble of models in the J_2 - J_4 and J_6 - J_4 space is shown in [Figure 4a,b](#). Notably, among the wide spread of intermediate models in the J_2 - J_4 - J_6 space, the one model (red cross) that meets the Juno J_2 , J_4 values, yields $J_6 = 34.20$, in excellent agreement with the Juno observation. For this model, the total Z amounts to $15.6 M_{\text{E}}$ in good agreement with the formation model of [Lozovsky et al. \(2017\)](#). This exploration suggests that winds on Jupiter have a negligible influence on J_6 .

4.3.3. Models for enhanced 1-bar temperature

In this Section, we present Jupiter models for $T_{1\text{bar}} = 175$ K and $T_{1\text{bar}} = 180$ K. Here, we do not modify the adiabat or EOS, and adjust the J_2 , J_4 model values to the wind-corrected observed values using the corrections of [Kaspi et al. \(2018\)](#).

Such warmer models are not preferred, first, because these 1-bar-temperatures significantly exceed the Galileo entry probe measurement of 166.1 K. This would not pose a problem if a mechanism had been studied that predicted a superadiabatic region underneath the 22-bar region, wherein temperatures and thus entropy would rise to the level corresponding to these or even higher 1-bar temperatures. Clouds may have a warming effect if they stabilize the region of condensation ([Leconte et al. 2017](#)); however, latent heat release from condensation opposes this effect and leads to a cooler interior underneath the cloud region, as has been discussed for ice giant atmospheres ([Kurosaki & Ikoma 2017](#)). Second, jovian adiabats for different H/He EOS tend to intersect with H/He demixing curves at best in a small region at 1–3 Mbar. At present, only the rather cool MH13-EOS based Jupiter adiabat for 166.1 K shows a clear intersection by about 450 K ([Hubbard & Militzer 2016](#)) with the state-of-the-art first-principles based H/He demixing curve of [Morales et al. \(2013\)](#), while the intersection with the lower demixing curve $T_{\text{dmx}}(P)$ of [Schoettler & Redmer \(2018\)](#) is only marginal [Mankovich & Fortney \(2021\)](#). Since enhancing $T_{1\text{bar}}$ from the Galileo value of 166.1 K by only 14 K leads to an enhancement by ~ 350 K at 1 Mbar and even by 460 K at 2 Mbar according to our CMS-19 EOS based Jupiter adiabats, higher surface temperatures might let the demixing region in Jupiter entirely disappear. We stress that although our CMS19 Jupiter adiabat for $T_{1\text{bar}} = 166.1$ is rather dense, it is also rather warm and with 5700 K at 1 Mbar and 6840 K at 2 Mbar outside of the first-principles based demixing regions ([Morales et al. 2013](#); [Schoettler & Redmer 2018](#)).

On the other hand, the recent experimentally predicted phase boundary inferred from an observed upward jump in reflectivity at 0.93 Mbar, and downward jump at 1.5 Mbar, which are interpreted as entry and exit of the compressed H/He-sample in and out from the demixing region ([Brygoo et al. 2021](#)) suggest high demixing temperatures of 10,000 K. Primarily it is this finding which motivates to allow for higher surface temperatures and for higher transition pressures, which we allow to reach the maximum where the core mass disappears, or for practical reasons drops below $1 M_{\text{E}}$.

In [Figure 5](#) we show the resulting outer envelope metallicity Z_{atm} and J_6 values. Obtaining 1x solar metallicity requires $T_{1\text{bar}}$ of 180 K (orange curve) or higher. While not negligible, additional uncertainties in the atmospheric helium abundance and J_4 are small and not considered here. Notably, for $T_{1\text{bar}} = 180$ K and at transition pressure between the He-poor and the He-rich region at $P_{12} = 6$

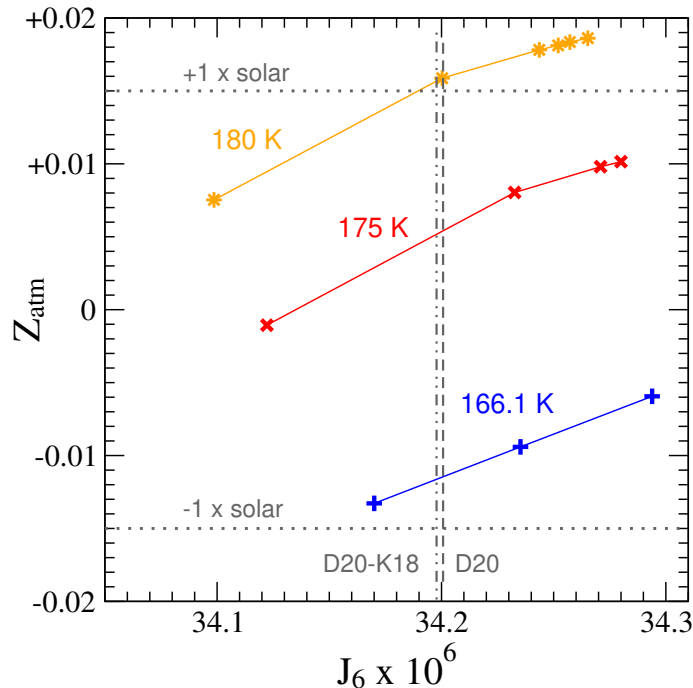


Figure 5. Outer envelope heavy element mass fraction (water) and J_6 values of Jupiter models with 1-bar temperature of 166.1 K for transition pressures from left to right of $P_{12} = 2, 3, 6$ Mbars (blue), 175 K for P_{12} of 2,6,8,8.5 Mbars (red), and 180 K for P_{12} of 2,6,8–10 Mbars (orange). The models are calculated using the unmodified H/He-CMS19 EOS and fit the wind-corrected J_2 , J_4 values using the corrections of (Kaspi et al. 2018). Horizontal dotted lines indicate $\pm 1 \times$ solar metallicity $Z_{\text{solar}} = 0.015$, while vertical lines indicate the observed J_6 value of Durante et al. (2020) (dashed) and its wind-corrected value (dot-dashed).

494 Mbar, we obtain 1x solar metallicity throughout the interior down to $\sim 0.4R_J$, thus a largely solar-
 495 metallicity envelope. At 6 Mbar, the temperature amounts to 10,400 K at is thus at the upper limit
 496 of the experimentally inferred demixing temperature (Brygoo et al. 2021). For that model, the static
 497 J_6 value is consistent with the observed value and its small wind-correction according to (Kaspi et al.
 498 2018). As is well known (Nettelmann et al. 2012), Z_1 rises with P_{12} .

499 If there were no uncertainties in the H/He EOS, these models would suggest that the internal
 500 Jupiter adiabat lies at higher entropy than the observed adiabat down to 22 bars, and that Jupiter’s
 501 envelope metallicity is not much higher than $1 \times$ solar.

502 5. APPLICATION TO SATURN

503 5.1. Saturn models

504 The Saturn models of this work are built in the same manner as the Jupiter models described in
 505 Section 4.1, although in the real planets, helium rain may induce a dichotomy (Mankovich & Fortney
 506 2021). We fit the Saturn models to the observed J_2 , J_4 values without accounting for the wind
 507 corrections. We assume a rotation rate of 10:32:45 hr as suggested by Helled et al. (2015) would
 508 yield a best match of interior models to the observed pre-Cassini Grand Finale gravity and Pioneer
 509 and Voyager shape data. Within the given uncertainty of 46s this value is consistent with the more
 510 recently suggested rotation rates of 10:33:34 hr (Militzer et al. 2019), using the Cassini Grand Finale
 511 gravity and same shape data, and with the rotation rate of $10:33:38 \text{ hr}_{-1m.19s}^{+1m.52s}$ inferred from the

512 comparison of Saturn ring wave frequencies observed by Cassini with theoretical predictions for f-
 513 mode frequencies as a function of the planet’s rigid-body rotation rate (Mankovich et al. 2019). We
 514 set the 1-bar surface temperature to 135 K in accordance with the Voyager measurement of 135 ± 5 K
 515 (Lindal 1992). The outer boundary is placed at a reference radius for the J_n of 60330 km, which
 516 corresponds to the 0.1-bar level.

517 Saturn’s atmospheric He abundance can be considered poorly known, as different estimates only
 518 agree in finding depletion compared to the protosolar value $Y_{\text{proto}} \sim 0.27$ but disagree about the
 519 level of depletion. The lowest estimate $Y_1 = 0.06 \pm 0.05$ stems from a combined Voyager radio
 520 occultation and infrared spectra analysis (Conrath et al. 1984), while the highest estimate $Y_1 = 0.18$ –
 521 0.25 from a reanalysis of only the Voyager infrared data (Conrath & Gautier 2000). More recent
 522 Cassini data-based estimates fall in between, ranging from $Y_1 = 0.075$ –0.13 from Cassini infrared
 523 remote sensing (Achterberg & Flasar 2020) to $Y_1 = 0.158$ –0.217 from Cassini stellar occultations and
 524 infrared spectra (Koskinen & Guerlet 2018). A low value of 0.07 ± 0.01 is independently inferred
 525 from shifting the most recent H/He phase diagram of Schoettler & Redmer (2018) to reproduce the
 526 He abundance measurement by the Galileo entry probe on Jupiter, in conjunction with the MH13
 527 H/He-EOS and adiabat (Mankovich & Fortney 2021). When the same procedure is applied to the
 528 H/He phase diagram of Lorenzen et al. (2011) in conjunction with the SCvH-H/He EOS, which both
 529 are now outdated, the yield is $Y_1 = 0.13$ –0.16 (Nettelmann et al. 2015), in between the most recent
 530 observational estimates (Achterberg & Flasar 2020; Koskinen & Guerlet 2018). Here we construct
 531 models for the two moderate depletion values $Y_1 = 0.14$ and $Y_1 = 0.18$ using the CMS-19 EOS and
 532 allow for a wider spread of 0.06–0.18 when using the modified CMS19-adiabat. For comparison,
 533 Galanti et al. (2019) used $Y_1 = 0.18 \pm 0.07$, Militzer et al. (2019) used $Y_1 = 0.18$ –0.26, and Ni (2020)
 534 used $Y_1 = 0.12$ –0.23. Lower Y_1 values yield higher atmospheric and higher maximum deep interior
 535 metallicities (Militzer et al. 2019; Ni 2020).

536 Saturn thermal evolution models with H/He phase separation and helium rain predict that in
 537 Saturn, helium rains down to the core, forming a He-rich shell of 0.90-0.95 mass percent helium atop
 538 the core (Püstow et al. 2016; Mankovich & Fortney 2021). If the helium abundance between the
 539 onset pressure of H/He phase separation and the He-shell follows a H/He phase diagram, it increases
 540 gradually with depth. However, these thermal evolution models assume for simplicity a constant low
 541 envelope metallicity. How deep He droplets can sink in a high- Z and thus higher-density deep interior,
 542 as predicted by Saturn models constrained by gravity and ring-seismology data (Mankovich & Fuller
 543 2021), remains to be investigated. For simplicity, we here represent the He-gradient by a jump in
 544 the He abundance at a pressure of $P_{12} = 2$ to 4 Mbars deep within the He-rain region, for which
 545 predicted onset pressures in present Saturn range from 0.8 Mbar (Militzer et al. 2019) to 2 Mbar
 546 (Mankovich & Fortney 2021). Below P_{12} , we keep the He/H ratio constant with depth.

547 Typical core-envelope pressures are around 15 Mbar. We vary P_{23} between 4 and 7 Mbar, and let
 548 the Gaussian- Z_3 profiles adopt its maximum at 12 Mbar if the core is compact ($Z = 1$) or farther
 549 out at 6–12 Mbar if the core is dilute and thus more extended. Dilute cores are created by setting
 550 $Z = 0.6$ –0.7 in the core, the remaining constituent being the H/He/ Z mix from envelope layer No. 3
 551 above.

552 5.2. Results for Saturn’s even harmonics

553 Figure 6 shows the even J_n values from our Saturn models, from the two Saturn models of uniform
 554 rotation (UR) in Iess et al. (2019), and the Cassini Grand Finale observed values (Iess et al. 2019).

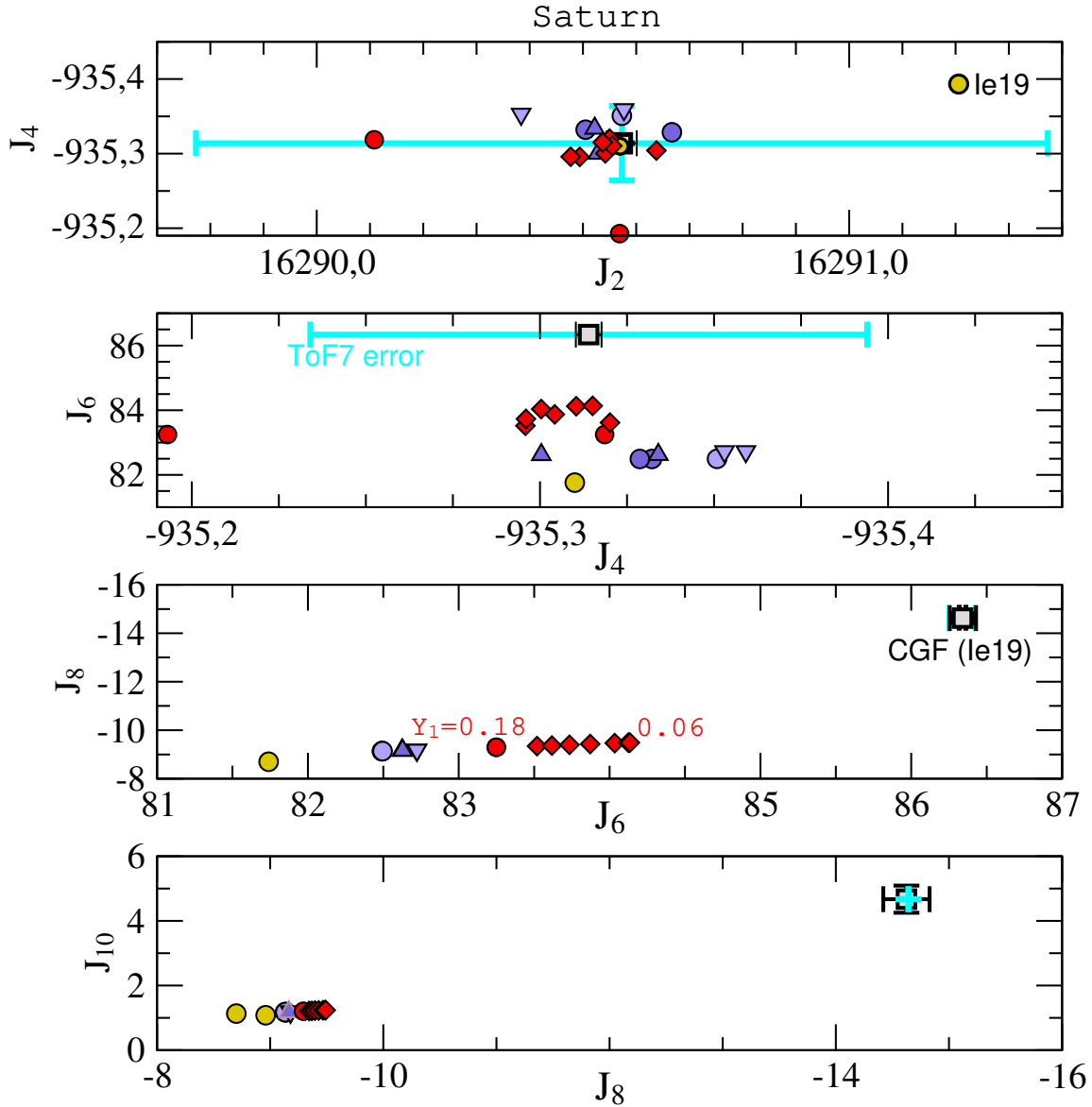


Figure 6. J_{2n} values multiplied by 10^6 for Saturn: Observed J_{2n} values from the Cassini Grand Finale (CGR, grey squares, [Iess et al. 2019](#)), ToF7 uncertainties overplotted to the observed values (cyan), interior model results assuming uniform rotation between 10h32m44s and 10h47m06s ([Iess et al. 2019](#)) (golden). This work’s models: blue triangles-up: adiabatic, $Y_1 = 0.14$, Gaussian- Z_3 ; blue triangles-down: adiabatic, $Y_1 = 0.14$, constant- Z_3 ; blue circles: $Y_1 = 0.18$, super-adiabatic; red circles: modified H/He-adiabat, dilute core, $Y_1 = 0.18$; red diamonds: modified H/He-EOS and Y_1 between 0.16 and 0.06, dilute core, Gaussian- Z_3 .

555 Unlike the case of Jupiter, Saturn’s observed even $|J_n|$ values are clearly enhanced over the model
 556 predictions for $n \geq 6$. The enhancement can be explained by rotation along cylinders that rotate
 557 approximately but not exactly with the observed speeds of clouds in the equatorial region and mid-
 558 latitude region up to $\pm 40^\circ$ ([Militzer et al. 2019](#)). The observed J_n can also be explained by a thermal
 559 wind if a little deviation of the deeper wind speeds from the cloud speeds is allowed ([Galanti et al.](#)
 560 [2019](#)). The enhancement of the even J_n by the zonal winds is quite substantial. Already for J_6 , we
 561 find a 4.2–5.3% influence, although it diminishes to 2.4–3.7% for the modified adiabat. [Iess et al.](#)

(2019) obtain slightly lower UR model J_6 values and a 5.5% effect, while Galanti et al. (2019), who allow for a much wider scatter in model J_4 values of $\pm 40 \times 10^{-6}$, obtain UR model J_6 values up to 87×10^6 , which encompasses the observed value. However, the mean of their distribution for fast, uniform rotation lies at 82×10^{-6} implying a 5.5% influence of the winds on J_6 , consistent with this work. The strong influence of the winds seen in J_6 suggests that also J_4 and J_2 are affected by the winds. In Section 6, we investigate whether the observed winds are consistent with a smaller influence on J_6 than found in previous work (Iess et al. 2019).

5.3. Z - and ρ -profiles for Saturn

Saturn interior models allow for higher atmospheric metallicities than Jupiter interior models when using the same H/He-EOS. For instance, Nettelmann et al. (2013) obtains $1.5\text{--}6\times$ solar for Saturn and fast rotation of 10:32:00 hr, but only $0\text{--}2.5\times$ solar for Jupiter when applying H-REOS.2 EOS. Wahl et al. (2017b) obtained only $0\text{--}0.7\times$ solar for Jupiter using MH13 EOS, while Militzer et al. (2019) obtained $1\text{--}4\times$ solar for Saturn, consistent with Ni (2020) who obtained $0\text{--}6\times$ for Saturn by considering a wide range of atmospheric He abundances, rotation rates, and wind corrections.

Here, we obtain $Z_1 = 0.02\text{--}0.06$ ($1.5\text{--}4\times$ solar) for nominal He-abundances Y_1 of $0.14\text{--}0.18$, and we obtain compact non-exclusive core masses of $5\text{--}8.6 M_E$, meaning that solutions with lower core masses are expected to be possible for deeper transition pressures than considered here. Representative Z - and ρ -profiles are shown in Figure 7. Compact rocky cores yield higher central densities than suggested by the 16-84% percentile probability range of models of Movshovitz et al. (2020), which are constrained by the gravity data. The latter models agree well with density distributions with inhomogeneous Z -profiles constrained by Cassini ring-seismology data (Mankovich & Fuller 2021). However, we were not able to find a model with a low-density core and the original H/He-EOS as such cores extend far out and yield J_2 values that are too large. Applying the same modification to the Saturn adiabat as for the Jupiter-optimized adiabat (see Section 4.3.2), we are able to obtain Saturn models with extended, low-density cores, in agreement with the likelihood distributions. As we mix H/He, with little addition of ice, into the rocky core region, we need rather high H/He amounts of 30–40% by mass (Figure 7, right panel) to reduce the core densities to $6\text{--}7 \text{ cm}^{-3}$ (left panel). This is consistent with Mankovich & Fuller (2021) who need 30–40% of H/He for rocky cores but only 0–10% for icy cores. When also allowing the atmospheric He abundance to decrease down to 0.06, we obtain up to $7\times$ solar atmospheric Z .

Even in our dilute core models, the high- Z part is concentrated in the innermost $0.4 R_{\text{Sat}}$ at pressures above 4 Mbar. For comparison, seismic constraints required (Mankovich & Fuller 2021) to extend the Z -gradient zone, depending on the assumed functional form of $Z(r)$, out to $0.6\text{--}0.7 R_{\text{Sat}}$, where the pressure is around 1 Mbar. While we did not explore such extended Z -gradients in order to keep them separated from the He-gradient zone, which we placed at $2\text{--}4$ Mbar, this comparison suggests that in the real Saturn, heavy element and helium gradients overlap. Together, the Z - and the He abundance profiles of our models suggest that the compositional gradient required to explain the ring seismology data could be due to both a diffuse core (inner region) and helium rain (outer region).

In comparison to Jupiter, the total amount of heavy elements is clearly higher in Saturn. Models with the unperturbed H/He-adiabat yield $M_Z \sim 12.6\text{--}13.6 M_E$ for Saturn while $7.5\text{--}10.1 M_E$ for Jupiter (Section 4.3.1). Lower densities along the H/He adiabat and warmer interior temperatures would increase these values.

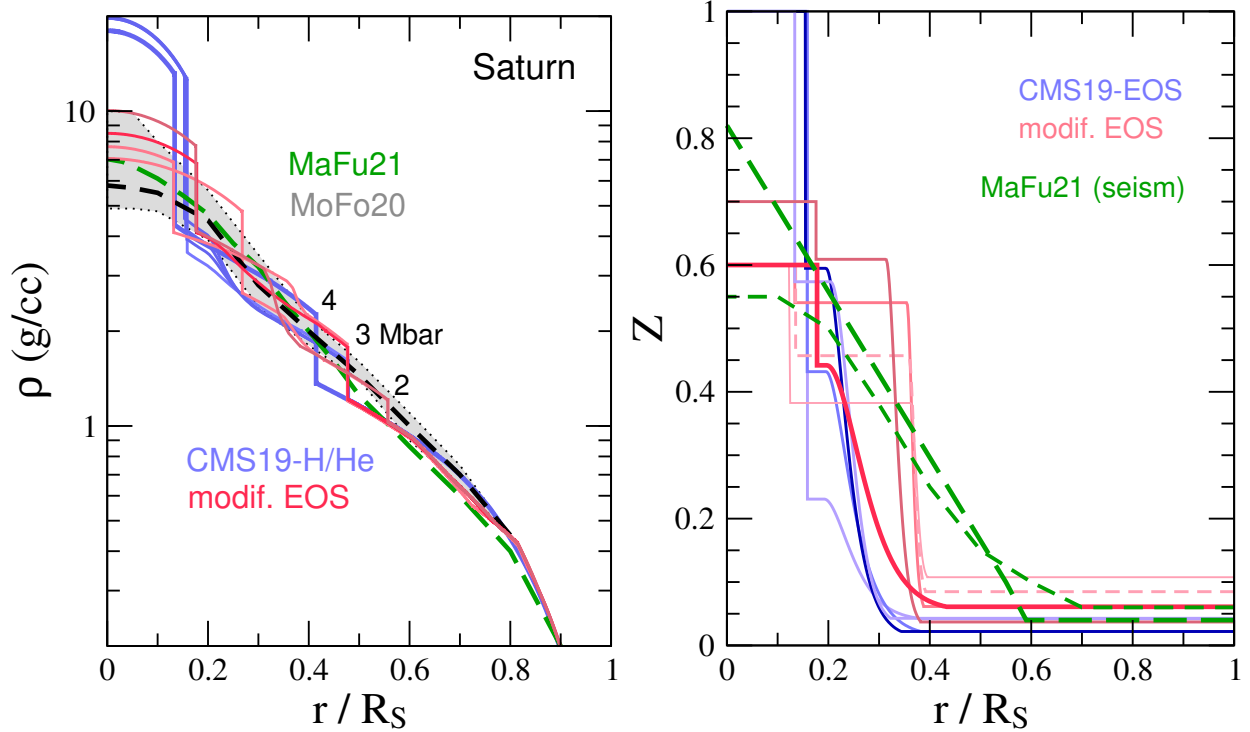


Figure 7. Density profiles (left panel) and radial Z -profiles (right panel) of Saturn models with a compact core (blue) or a dilute core of $x_R = 0.6$ – 0.7 and using the Jupiter-optimized adiabat (red), approximate 2σ -likelihood distribution of [Movshovitz et al. \(2020\)](#) (grey), and likelihood-mean from seismic constraints (green) adapted from [Mankovich & Fuller \(2021\)](#) assuming a linear $Z(r)$ (thick dashed) or a sigmoidal $Z(r)$ (right panel only). The highest atmospheric Z levels of the red curves are for lowest atmospheric $Y = 0.06$.

6. INCLUDING THE ZONAL WIND PROFILES

Our Jupiter models with a modified or unmodified H/He adiabat allow for a largely homogeneous interior down to $0.4 R_J$ with a small rock core (Figure 3) and for J_6 unaffected by the winds (Figure 2). Our Saturn models allow for larger static J_6 values in the range $(82.5\text{--}84.0)\times 10^{-6}$ than previous work (81.8×10^{-6}), see Figure 6.

Here, we investigate if such models for Jupiter and Saturn are consistent with the observed wind profiles and odd and even J_n values. We pick one representative Jupiter model (unmodified adiabat, $P_{12} = 2$, $P_{23} = 20$ Mbar, compact core of mass $1.26 M_E$) and the Saturn model highlighted in red in Figure 7 (modified H/He adiabat, dilute core, $Y = 0.14$).

The wind modeling approach is the same as described in [Galanti & Kaspi \(2021\)](#), except that the Saturn rotation period to which the wind speeds refer is adjusted to the value of the interior model, 10h32m45s. We either use only the gravity data to constrain the wind decay depth profiles as in [Kaspi et al. \(2018\)](#); [Galanti et al. \(2019\)](#), or both the gravity and magnetic field data combined (grav+MHD) as in [Galanti & Kaspi \(2021\)](#). As an uncertainty in the static J_{2n} values we take the Tof7 errors produced by MOGROP for $N = 4000$.

With this approach, we are able to find fits that reproduce all the observed J_n within the observational uncertainties. The wind profiles and decay depths are shown in Figure 8. This implies that extending the wind profiles, roughly as they appear at the cloud level, gives a good match to the difference between the Juno and Cassini measurements and our preferred models. Nonetheless there

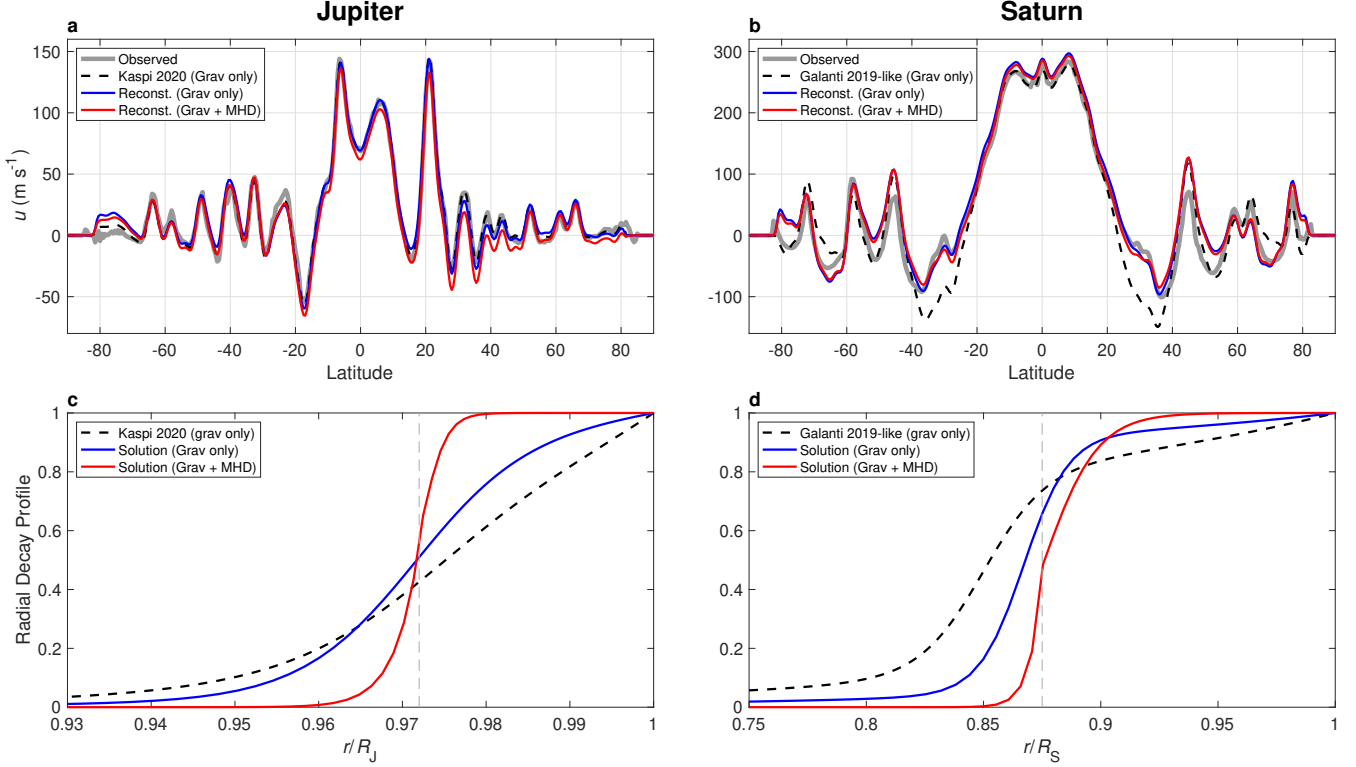


Figure 8. Wind profiles (top) for Jupiter (left) and Saturn (right) using the constraints from observed gravity data only (blue) or also from MHD (red). The black dashed lines for Jupiter are taken from [Kaspi et al. \(2018\)](#) (grav only), while for Saturn, they are adjusted from [Galanti & Kaspi \(2021\)](#) (grav only). Solutions for grav+MHD from that previous work are not shown since the solutions are close to the ones from this work. The grey lines in the top panels show the observed profiles ([Tollefson et al. 2017](#)).

is enough freedom in these solutions that other wind profiles with small shifts to the wind profiles can give fitting solutions as well ([Galanti et al. 2021](#)). For these wind profiles, grav and grav+MHD yield similar solutions. Jupiter’s wind profile is slightly less well matched (red and blue lines more strongly deviate from the observed profile (grey) than does the black-dashed line), while for Saturn, the shoulders at 20-40 degrees latitude are somewhat better matched than in previous work ([Iess et al. 2019](#); [Galanti et al. 2019](#)). The wind decay depths for grav only are slightly steeper than for previous interior models and thus closer to the grav+MHD solutions.

For each of Jupiter and Saturn, we picked only one specific interior model to calculate the wind contribution and optimize for the agreement with the observed wind velocities and gravitational harmonics. The fact that these two interior models allowed for solutions within the observed values and the ToF7/MOGROP uncertainty strongly suggests that there are further interior models for which such a fit can be obtained. This means that the joint interior and wind solutions are not unique, given the uncertainties we allowed for. In addition, alternative interior models which fit all the J_n when combined with a wind model may be possible for different equations of state and wind models, such as the MH13 H/He EOS and wind models that account for the oblate shape ([Cao & Stevenson 2017](#)) or solve for the gravo-thermal wind equation **that accounts for the dynamic self-gravity of the flow (TGWE)** ([Kong et al. 2018](#); [Wicht et al. 2020](#)). **We note that [Galanti et al. \(2017\)](#) find**

that these modified wind models introduce corrections that are an order of magnitude smaller for most J_n , while [Dietrich et al. \(2021\)](#) obtain corrections of respectively $\sim 60\%$ and $\sim 20\%$ for J_3 and J_5 when including the dynamic self-gravity for polytropic models and an additional correction of $\sim 40\%$ and $\sim 10\%$ when accounting for Jupiter-model specific background density and gravity profiles.

Internal flow structures, that are decoupled from the observed cloud-level winds, can also be found to fit the J_n ([Kaspi et al. 2018](#); [Kong et al. 2018](#)) and thus lead to non-uniqueness of the solutions ([Kong et al. 2018](#)). Here, we conclude for non-uniqueness because of uncertainties in the interior models, the high-order J_n to be fitted, and the wind profile.

7. DISCUSSION

Our Jupiter and Saturn models exhibit a strong trend toward low envelope metallicities which extend deep into Saturn’s interior to $\sim 0.4 R_S$ (Fig. 7) or are negative in Jupiter (Fig. 3). We have attributed these model properties to possible uncertainties in the H/He EOS, however, one may think of further processes.

7.1. Z and the adiabatic P - T profile

We did not include Z in the computation of the adiabatic P - T profile. This leads to a slight overestimation of the temperatures along the adiabat. Mixing first the equations of state H/He-REOS and H₂O-REOS linearly and then computing the adiabats as a function of Z using thermodynamic integration described in [Nettelmann et al. \(2012\)](#) shows that 10x solar water would lower the temperatures by only -100 K in the 10-100 GPa region relevant for J_2 and J_4 . Conversely, an adiabat more rich in atomic helium would be warmer. Considering molecular volatiles in the entropy calculation would tend to make the adiabat slightly cooler and denser, and therefore lead to even lower envelope metallicities, but our estimate shows this effect should be small.

7.2. H/He demixing?

Inspired by the recent experimentally derived H/He demixing boundary that extends over a large region from ~ 0.9 Mbar to ~ 10000 K (8–10 Mbar) in Jupiter ([Brygoo et al. 2021](#)) one could consider helium abundances which increase over a wide region, allowing for more heavy elements to replace helium. However, our variation of the H/He adiabat showed that reduced densities are needed near the top and beyond (~ 20 GPa) the demixing region. Exploration of the helium abundance profile deep inside may thus have too little influence to solve the low-atmospheric metallicity problem in Jupiter.

7.3. Deep internal flows in Jupiter?

[Guillot et al. \(2018\)](#) constrained the maximum amplitude of a deep wind that would extend along cylinders all the way to the center and be consistent with the even J_n to < 10 m/s. [Kong et al. \(2018\)](#) found that a flow with 1 m/s down to $0.8 R_J$ can explain the odd J_n ; but including the influence of the induced magnetic field through Ohmic heat dissipation bounded by the total convective power ([Wicht et al. 2019](#)), [Li et al. \(2020\)](#) are able to limit this depth to only $0.96 R_{Jup}$ results. [Moore et al. \(2019\)](#) even constrain the flow velocity to a few mm/s at depths of 0.93 – $0.95 R_J$ by explaining the observed

681 secular variation of the magnetic field with advection by the flow.

682 In contrast, in order to lift Jupiter’s atmospheric metallicities substantially, a much
 683 stronger and retrograde deep wind in the interior where J_2 and J_4 are sensitive would
 684 be needed. This deep wind must not be seen in the high-order gravity data, in the
 685 secular variation of the magnetic field, nor in the System III rotation period derived
 686 from magnetic field observations. It would be seen in the moment of inertia, the static
 687 Love numbers, and the shape. At present, there is no indication of a strong (> 10 m/s)
 688 wind in Jupiter’s deep interior.

689 7.4. *Uncertainty in the shape due to dynamical effects?*

690 With both CMS and ToF-method the interior models are derived from a self-consistent,
 691 static solution between the gravity field and the shape, however, the shape and the
 692 gravity field of the planet can be influenced by various dynamic effects.

693 For instance, [Kong & Zhang \(2020\)](#) propose that the winds are shallow while con-
 694 vective motions could induce a zonal flow disjunct from the surface winds. They find
 695 a dynamic influence of 1×10^6 in J_2 and 0.2×10^6 for J_4 . While small, this effect on J_4
 696 could be noticeable in the interior models. However, this estimate of the dynamic
 697 contribution due to convective motions is based on an Ekman number 5×10^{-5} , about 10
 698 orders of magnitude larger than in the real Jupiter and Saturn. It is therefore possible,
 699 that the dynamic contribution from convective motions on the low-order J_{2n} is smaller
 700 in the real planets.
 701

702 For Saturn, the uncertainty in its deep rotation rate maps on an uncertainty in
 703 equipotential shape of about 120 km ([Helled & Guillot 2013](#)), far outweighing other
 704 influences like from the winds, which lift the dynamical height above a reference iso-bar
 705 to no more than ~ 20 km ([Buccino et al. 2020](#)). Moreover, the zonal flows on Saturn
 706 are symmetric enough to be described by rotation along cylinders up to mid-latitudes
 707 ([Militzer et al. 2019](#)). In that case, equipotential theory still applies. We do not suggest
 708 that dynamic effects play a major role for the uncertainty in Saturn’s shape and gravity
 709 field.
 710

711 For Jupiter, the uncertainty in rotation rate is tiny, so that it is the influence of the
 712 winds of 2–4 km ([Buccino et al. 2020](#)) against which further effects must be compared.
 713 Such are the tidal bulges from the Galilean satellites. [Nettelmann \(2019\)](#) estimated
 714 a maximum elongation of 28 km in the direction of Io from static tidal response. The
 715 tidal flows around Jupiter are a dynamic perturbation and subject to Coriolis force
 716 ([Idini & Stevenson 2021; Lai 2021](#)). The flow and the Coriolis force acting upon it lead
 717 to dynamic contributions to the Love numbers k_{nm} . JUNO measurements revealed a
 718 deviation by 1–7% from the static k_2 -value ([Idini & Stevenson 2021](#)). Approximating the
 719 corresponding shape deformation h_2 by $h_2 = 1 + k_2$ yields a tentative estimate of a (1–7)%
 720 $\times 28$ km $\sim (3–21)$ km additional shape deformation due to dynamic tidal response, which
 721 exceeds the wind effect. Possible importance of (periodic) perturbations on Jupiter’s
 722 interior structure inference remains to be investigated.
 723

7.5. *A cold hot spot?*

Juno MWR data revealed that the ammonia abundance below the cloud level shows strong vertical and latitudinal variation (Guillot & Fletcher 2020). It is therefore possible that Jupiter’s atmosphere is not everywhere well mixed where observations were taken. Consequently, the abundances and temperatures measured by the Galileo entry probe in a hot spot may not be representative of Jupiter’s global atmosphere. On the other hand, analysis of Voyager 1 and 2 radio occultation data spanning a broad range of latitudes between 70 degrees South and the equator, yielded a 1-bar temperature of 165 K +/- 5 K (Lindal et al. 1981), consistent with the Galileo measurement of 166.1 K in the hot spot. Present data therefore do not indicate that hot spots, in which deeper layers are exposed that appear brighter than surrounding regions at higher altitudes, were particularly cool regions allowing us to suppose warmer global average temperatures. Rather, it is possible that the hot-spot temperature-gradient is steeper than the global one since it is close to a dry adiabat (Seiff et al. 1998), whereas moist regions above the water cloud level may follow a less steep P - T profile (Kurosaki & Ikoma 2017), implying an even cooler interior below the cloud base. A colder and thus denser interior would strain the low-metallicity models even more.

8. CONCLUSIONS

We present the expansion of the Theory of Figures (Zharkov & Trubitsyn 1978) from formerly 5th order (Zharkov & Trubitsyn 1975) to the 7th order. The coefficients are available in form of five read-in online tables and allow the computation of the even gravitational harmonics J_2 - J_{14} and the shape of a rotating fluid body in hydrostatic equilibrium.

We estimate the numerical accuracy of the ToF method carried out to 4th (Nettelmann 2017), 5th (Zharkov & Trubitsyn 1975), and 7th (this work) order by comparing to the analytic Bessel solution of Wisdom & Hubbard (2016) for the rotating $n = 1$ polytrope and by using three different codes. We find that the CEPAM code (Guillot & Morel 1995) with ToF5 (Ni 2020) has a superior performance in regard to the accuracy in J_2 , J_4 , and J_6 , while for J_8 and J_{10} , the MOGROP code with ToF7 reaches similar degree of accuracy for a practical number of radial grid points of a few thousand, although in J_{10} the error changes sign between both variants. The accuracy in J_8 , J_{10} , J_{12} falls by, respectively, 1, 2, 3 orders of magnitude below the current $3 \times 1\sigma$ formal uncertainty of the observational gravity data ”halfway through the Juno mission” (Durante et al. 2020). We also apply the CMS-2019 H/He EoS of Chabrier et al. (2019) to interior models of Jupiter and Saturn.

For Jupiter, the high-order J_n of the Jupiter models fall along the same line in J_n - J_{n+2} space as in previous work, regardless of detailed model assumptions and the H/He EOS used. We find that J_6 stands out in that it is neither adjusted, as J_2 and J_4 are, nor insensitive to model assumptions, as the J_n for $n \geq 8$ are. We match Jupiter’s observed J_6 value by placing the transition pressure between an outer, He-depleted envelope and an inner, He-enriched envelope at $P_{12} = 2$ -2.5 Mbar. Transition pressures farther out lead to lower J_6 values, while deeper transitions result in higher J_6 values. The same behavior but with a weaker amplitude is seen for the transition pressure of heavy elements, which we place between ~ 5 and 20 Mbar. Gaussian- Z profiles underneath can lead to high metallicities of up to $Z=0.5$ at the compact core-mantle boundary. However, the atmospheric heavy element abundance, represented by an EOS of water, always stays negative ($\sim -1 \times$ solar) if

766 the adiabat is defined by the 1-bar temperature of 166.1 K as measured by Galileo and extended
 767 downward. Alternatively, we set Z_1 to $1 \times$ solar, the 1σ -lower limit of the equatorial water abundance
 768 measured by Juno (Li et al. 2020) and perturb the adiabat to fit J_2 and J_4 . Such an optimized adiabat
 769 was found for a perturbation between 26 and 150 GPa and has a maximum density decrease of 12.6%
 770 at a midpoint of 63 GPa. Higher internal temperatures help to decrease the internal density as well.
 771 For $T_{1bar} = 180$ K and deep transition pressure $P_{trans,He} = 6$ Mbar we obtain $1x$ solar metallicity
 772 without H/He-EOS modification.

773 Our Saturn models with CMS-19 H/He EOS are characterized by a few-times solar envelope that
 774 extends deep down to $< 0.4 R_{Sat}$ and requires a compact core. Its density of $\sim 20 \text{ gcm}^{-3}$ is higher
 775 than the most likely central densities of Saturn that match the gravity field (Movshovitz et al. 2020),
 776 which in turn agree with density distributions of a largely stably stratified deep interior with a
 777 dilute core (Mankovich & Fuller 2021). By applying the Jupiter-optimized perturbation along the
 778 adiabat to Saturn, we are able to obtain density distributions with a dilute core of 30-40% H/He that
 779 reaches out to $\sim 0.4 R_{Sat}$ in the core. This moves the solution in the direction of density distribution
 780 constrained by seismic data. Our models suggest that an inhomogeneous central region out to ~ 0.6
 781 R_{Sat} (Mankovich & Fuller 2021) is due to both a dilute core and rained down helium.

782 **Overall, our Jupiter and Saturn models exhibit a strong trend toward low envelope**
 783 **metallicities which extend deep into Saturn’s interior to $\sim 0.4 R_S$ (Fig. 7) or are negative**
 784 **in Jupiter (Fig. 3). We have attributed these model properties to possible uncertainties**
 785 **in the H/He EOS. However, further processes one may think of and which certainly are**
 786 **at play are estimated to be too minor to solve that issue.**

787 This work demonstrates that our understanding of the internal heavy-element distribution of Jupiter
 788 and Saturn strongly depends on the properties of H and He. We conclude that part of the difficulties
 789 of obtaining Jupiter and Saturn models that are consistent with all observational constraints still
 790 lies in our imperfect understanding of the material properties. We therefore suggest that further
 791 measurements and calculations of the behavior of materials at planetary conditions could improve
 792 our understanding of the gas giants. This, in return, will also reflect on the characterization of
 793 gaseous planets orbiting other stars.

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797 APPENDIX

798 A. TECHNICAL NOTES ON TOF COEFFICIENT COMPUTATIONS

799 In the following, we abbreviate the term in brackets in Eq. (4) as $(1 + \Sigma)$ and write $\mu = \cos \vartheta$.
 800 Since any point $\mathbf{r} = (r, \varphi, \vartheta)$ in and near the planet can be associated with an equipotential surface,
 801 we can replace any dependence $f(r, \vartheta)$ by $f(l, \vartheta)$ using Eq. (4).

802 A.1. Centrifugal Potential Q

803 The first step of the Legendre polynomial expansion of the centrifugal potential $Q = 1/2 \omega^2 r^2 \sin^2 \vartheta$
 804 is to write $Q = -1/3 \omega^2 r^2 (P_0 - P_2(\mu))$. Its full expansion is obtained by replacing r with $r_l(\vartheta)$. $Q(l, \vartheta)$

can then be written in the form

$$Q(l, \vartheta) = -\frac{GM}{R_m} \left(\frac{l}{R_m}\right)^2 \sum_{k=0}^O A_{2k}^{(Q)} P_{2k}(\mu) \quad \text{with} \quad A_k^{(Q)} = \frac{m_{\text{rot}}}{3} \sum_i c_{i0k}. \quad (\text{A1})$$

A.2. Gravitational Potential V

The gravitational potential $V(\mathbf{r}) = -G \int d^3r' \rho(\mathbf{r}')/|\mathbf{r}' - \mathbf{r}|$ can be separated into an external contribution D from the mass density interior to a sphere of radius r , i.e. to which r is exterior ($r > r'$), and an internal contribution D' from the mass density exterior to a sphere of radius r , i.e. to which r is interior ($r < r'$). V then reads

$$V(r, \vartheta) = -G \sum_{n=0}^{\infty} (r^{-(n+1)} D_{2n}(r) + r^n D'_{2n}(r)) P_{2n}(\mu) \quad (\text{A2})$$

with

$$D_n = \int_{r' < r} d^3r' \rho r'^n P_n(\mu') \quad , \quad D'_n = \int_{r' > r} d^3r' \rho r'^{-(n+1)} P_n(\mu'). \quad (\text{A3})$$

Although this multipole expansion is valid only for spheres, in ToF, the radial coordinate r in Eqs. (A2,A3) is simply replaced by the non-spherical equipotential surface $r_l(\vartheta)$ and the interior/exterior criterion is transferred to l . In the CMS method, this expansion is also used but the expression for the external potential is only applied to spheroids of level surface $r_i(\mu)$ at or interior ($i \geq j$) to a point B of radial distance r_B that resides on a level surface $r_j(\mu)$. Since all spheroids share the same center but extend outward to different level surfaces $r_i(\mu)$ where $i = 0$ denotes the surface of the planet and $i = N$ the center, a point B on $r_j(\mu)$ is also located exterior to the mass of the spheroids of index $i < j$ but only as far as the radius r_B . This is taken care of in the CMS method by adding the external gravitational potential of the spheres of densities $\delta_{i, i < j}$ interior to B from the spheroids $i < j$. This improvement of the CMS method over ToF method is still limited by the deformation and spacing of the spheroids. Rapid rotation, or dense spacing, could lead to an overlap of the sphere of radius r_j with the spheroid $r_{j-1}(\mu)$. Kong et al. (2013) developed the full solution to the Poisson equation and demonstrated that the CMS method converges as long as the flattening ($R_{eq}/R_{pol} - 1$) remains sufficiently small. Similarly, Hubbard et al. (2014) showed that ToF converges for sufficiently small flattening and toward the correct solution.

By replacing all powers of r by $r_l(\vartheta)$ in Eq. (A2), one obtains

$$V(l, \vartheta) = -\frac{GM}{R_m} \left(\frac{l}{R_m}\right)^2 \sum_{n=0}^O ((1 + \Sigma)^{-(n+1)} S_{2n}(z) + (1 + \Sigma)^n S'_{2n}(z)) P_{2n}(\mu) \quad (\text{A4})$$

with $z = l/R_m$, $z \in [0, 1]$ and, with the help of the transformation

$$r^n dr = dl r^n \frac{dr}{dl} = \frac{1}{n+1} dl \frac{d}{dl} r^{n+1} \quad ,$$

$$S_n(z) = \frac{3}{2(n+3)} \frac{1}{z^{n+3}} \int_0^z dz' \frac{\rho(z')}{\bar{\rho}} \frac{d}{dz'} \left[z'^{n+3} \int_{-1}^1 d\mu' (1 + \Sigma)^{n+3} P_n(\mu') \right] \quad , \quad (\text{A5})$$

$$S'_n(z) = \frac{3}{2(2-n)} \frac{1}{z^{2-n}} \int_z^1 dz' \frac{\rho(z')}{\bar{\rho}} \frac{d}{dz'} \left[z'^{2-n} \int_{-1}^1 d\mu' (1 + \Sigma)^{2-n} P_n(\mu') \right] \quad . \quad (\text{A6})$$

837 This can further be written as

$$838 \quad S_n(z) = \frac{1}{z^{n+3}} \int_0^z dz' \frac{\rho(z')}{\bar{\rho}} \frac{d}{dz'} [z'^{n+3} f_n(z)] \quad , \quad (\text{A7})$$

$$839 \quad S'_n(z) = \frac{1}{z^{2-n}} \int_z^1 dz' \frac{\rho(z')}{\bar{\rho}} \frac{d}{dz'} [z'^{2-n} f'_n(z)] \quad , \quad (\text{A8})$$

840 with

$$841 \quad f_n(z) = \frac{3}{2(n+3)} \int_{-1}^1 d\mu P_n(\mu) (1+\Sigma)^{n+3} \quad ,$$

$$842 \quad f'_n(z) = \frac{3}{2(2-n)} \int_{-1}^1 d\mu P_n(\mu) (1+\Sigma)^{2-n} \quad (n \neq 2) \quad ,$$

$$843 \quad f'_2(z) = \frac{3}{2} \int_{-1}^1 d\mu P_n(\mu) \ln(1+\Sigma) \quad . \quad (\text{A9})$$

844 A.3. ToF7 Tables for public usage

845 The ToF coefficients c_{ink} are of the form

$$846 \quad c_{ink} = q_{ink} \prod_{j=1}^O s_{2j}^{p_{2j, ink}} \quad ,$$

847 where the q_{ink} are rational numbers and the exponents p_j are small natural numbers including 0.

848 Since the number of coefficients rises with the order of expansion faster than quadratically, it
849 becomes impractical to write down all the coefficients. We present them in the form of five online
850 tables. Tables `tab_Sn` and `tab_Snp` contain the coefficients c_{ink} and c'_{ink} in front of the S_n and S'_n in
851 the $A_k^{(V)}$, respectively, so that

$$852 \quad A_k^{(V)} = \sum_{i=1}^{N_{0k}} c_{i0k} S_0 + \sum_{i=1}^{N_{2k}} c_{i2k} S_2 + \dots + \sum_{i=1}^{N_{14k}} c_{i14k} S_{14} + \sum_{i=1}^{N'_{0k}} c'_{i0k} S'_0 + \sum_{i=1}^{N'_{2k}} c'_{i2k} S'_2 + \dots + \sum_{i=1}^{N'_{14k}} c'_{i14k} S'_{14}$$

853 Table `tab_m` contains the summands in the $A_k^{(Q)}$ so that $A_k^{(Q)} = m_{\text{rot}}/3 \sum_{i=1} c_{i0k}$ with $m_{\text{rot}} =$
854 $\omega^2 R_m^3/GM$. Tables `tab_fn` and `tab_fnp` contain the summands of the functions f_n and f'_n so that
855 $f_n = \sum_i c_{in}$, $f'_n = \sum_i c'_{in}$, respectively. In Table 2, we give an example of the read-in ascii-table
856 `tab_Sn`. All five tables have the same format. The A_{2k} in Eq. (3) are of the form $0 = A_{2k}(l) =$
858 $-s_{2k}(l)S_0(l) + B$ if $k \neq 0$. We rewrite this as an expression for direct, iterative computation of
859 the figure functions in the form $s_{2k} = B/S_0$, where the functions B depend on the $\{s_{2k}\}$ from the
860 previous iteration step. We omit the summand $-s_{2k}S_0$ from Table `tab_Sn`.

861 **To facilitate the application of our ToF7 tables by external users for their own plane-**
862 **tary models, we share routines for read-in of the tables in Matlab, Python, and C++.**
863 **For the latter variant, we also provide functions that can be used to easily access**
864 **the coefficient values. The archived service routines and descriptions can be found**
865 **at <https://doi.org/10.6084/m9.figshare.16822252>.**

n	k	N_{nk}							$q_{ink}, i = 1-N_{nk}$	comment
Order	p_2	p_4	p_6	p_8	p_{10}	p_{12}	p_{14}			
0	0	24								$n = 0, k = 0, \text{ next 24 rows}$
0	0	0	0	0	0	0	0	1.0000000000000000e+00	Order= 0, $c_{100} = 1$	
2	2	0	0	0	0	0	0	4.0000000000000000e-01	Order= 2, $c_{200} = 0.4 s_2^2$	
⋮										
7	2	1	1	0	0	0	0	2.157842157842158e-01	Order= 7, $c_{2400} = q_{2400} s_2^2 s_4^1 s_6^1$	
							⋯			
4	8	17								$n = 4, k = 8, \text{ next 17 rows}$
4	2	0	0	0	0	0	0	2.937062937062937e+00	Order= 4, $c_{148} = q_{148} s_2^2$	
⋮										

Table 2. Example of one of the five online-only ascii-tables. This example is for Table `tab_Sn` that contains the coefficients in front of the S_n in the A_k . Since the $A_{2k}^{(Q)}$ have no dependence on index n , index n is set to 0 in Table `tab_m`. Since the f_n, f'_n have no dependence on index k , k is set to 0 in Tables `tab_fn` and `tab_fn'`.

A.4. Powers of the radius

In the binomial expansion of $(1 + x)^{-m}$ for $m > 0$,

$$(1 + x)^{-m} = \sum_{i=0}^{\infty} \binom{-m}{i} x^i \quad , \tag{A10}$$

it is sufficient to expand to $i = 7$ because $x = \Sigma$ and the minimum order of Σ^i is i . The binomial expansion of $(1 + x)^m$ for $m > 0$,

$$(1 + x)^m = \sum_{i=0}^m \binom{m}{i} x^i \quad , \tag{A11}$$

is carried out to $m \leq 7$. Products $P_n P_m$ occurring in Σ^i and in $(1 + \Sigma)^i P_j$ are expanded as $\sum_{k=0}^{n+m} b_k P_k$. It becomes evident that all terms can linearly be expanded in Legendre polynomials and that numbers in the expansion coefficients are rational numbers $q = n_e/n_d$. The natural numbers n_e, n_d can be represented exactly on a computer, although size limitations may apply. For ToF7, the vast majority of numbers could be decomposed into prime numbers that individually do not exceed 3 million. However in rare cases this was not possible and larger prime numbers would have been required, perhaps indicating an error in the code used. In such a case, the given number is not decomposed into prime numbers. In any case, the enumerators and denominators are computed as exact numbers in all coefficients. They are cast to real numbers of 15 digits only for the purpose of printing the tables.

A.5. Figure function s_0

We calculate s_0 to the 7th order with the help of the defining integral Eq. (2), which, with $z = l/l_1$, can be written

$$\frac{4\pi}{3} l_1^3 = 2\pi l_1^3 \int_{-1}^1 d\mu \int_0^1 dz z^2 (1 + \Sigma(z, \mu))^3 \quad . \tag{A12}$$

886 Because of hemispheric symmetry and $1/3 = \int_0^1 dz z^2$, the comparison of integrands yields

$$887 \quad 1 = \int_0^1 d\mu (1 + \Sigma)^3 \quad , \quad (\text{A13})$$

888 where $\Sigma(z, \mu) = \sum_0^7 s_{2i}(z) P_{2i}(\mu)$ and $(1 + \Sigma)^3 = 1 + 3\Sigma + 3\Sigma^2 + \Sigma^3$. We calculate Σ^3 and safely
889 remove all terms containing $P_0 P_n P_m$ with $n \neq m$ or $P_0^2 P_n$, since they would contribute nothing to
890 the evaluated integral in Eq. (A13),

$$891 \quad \begin{aligned} \Sigma^3 = & s_0^3 P_0^3 + s_2^3 P_2^3 + s_4^3 P_4^3 \\ & + 3s_0 s_2^2 P_0 P_2^2 + 3s_2^2 s_4 P_2^2 P_4 + 3s_2^2 s_6 P_2^2 P_6 + 3s_2^2 s_8 P_2^2 P_8 + 3s_2^2 s_{10} P_2^2 P_{10} \\ & + 3s_0 s_4^2 P_0 P_4^2 + 3s_2 s_4^2 P_2 P_4^2 + 3s_4^2 s_6 P_4^2 P_6 + 3s_2 s_6^2 P_2 P_6^2 \\ & + 6s_2 s_4 s_6 P_2 P_4 P_6 + 6s_2 s_4 s_8 P_2 P_4 P_8 \quad . \end{aligned} \quad (\text{A14})$$

895 Terms containing $P_0 P_n^2$ yield a factor $1/(2n + 1)$ for the integral in (A13). By $(P_n P_m)_k$ we denote
896 the summand $b_k P_k$ in the expansion of $P_n \times P_m$. The other terms contribute

$$897 \quad \begin{array}{lll} P_0^3 & : & \rightarrow 1 \\ P_2^3 & : (P_2 P_2)_2 P_2 = \frac{2}{7} P_2^2 & \rightarrow \frac{2}{5 \cdot 7} \\ P_4^3 & : (P_4 P_4)_4 P_4 = \frac{2 \cdot 3^4}{7 \cdot 11 \cdot 13} P_4^2 & \rightarrow \frac{2 \cdot 3^2}{7 \cdot 11 \cdot 13} \\ P_2^2 P_4 & : (P_2 P_2)_4 P_4 = \frac{18}{35} P_4^2 & \rightarrow \frac{2}{5 \cdot 7} \\ P_2^2 P_6 & : (P_2 P_2)_6 P_6 & \rightarrow 0 \\ P_2^2 P_8 & : (P_2 P_2)_8 P_8 & \rightarrow 0 \\ P_2^2 P_{10} & : (P_2 P_2)_{10} P_{10} & \rightarrow 0 \\ P_2 P_4^2 & : P_2 (P_4 P_4)_2 = \frac{2^2 \cdot 5^2}{7 \cdot 11} P_2^2 & \rightarrow \frac{2^2 \cdot 5}{3^2 \cdot 7 \cdot 11} \\ P_4^2 P_6 & : (P_4 P_4)_6 P_6 = \frac{2^2 \cdot 5}{3^2 \cdot 11} P_6^2 & \rightarrow \frac{2^2 \cdot 5}{3^2 \cdot 11 \cdot 13} \\ P_2 P_6^2 & : P_2 (P_6 P_6)_2 = \frac{2 \cdot 7}{11 \cdot 13} P_2^2 & \rightarrow \frac{2 \cdot 7}{5 \cdot 11 \cdot 13} \\ P_2 P_4 P_6 & : (P_2 P_4)_6 P_6 = \frac{5}{11} P_6^2 & \rightarrow \frac{5}{11 \cdot 13} \\ P_2 P_4 P_8 & : (P_2 P_4)_8 P_8 & \rightarrow 0 \quad . \end{array} \quad (\text{A15})$$

898 Similarly in Σ^2 , terms $P_n P_m$ yield a factor $1/(2n + 1)$ if $n = m$ and 0 otherwise, so that

$$899 \quad \Sigma^2 = s_0^2 + \frac{1}{5} s_2^2 + \frac{1}{3^2} s_4^2 + \frac{1}{13} s_6^2 \quad .$$

900 Out of Σ^1 in the integral in Eq. (A13), only $s_0 P_0$ survives. Now, s_0 can be expressed in terms of the
901 $s_{n, n \geq 2}$, so that

$$902 \quad s_0 := s_0^{(1)} + s_0^{(2)} + \dots + s_0^{(7)},$$

903 where O denotes the order of expansion. Sorting terms according to their order, we obtain (note the
904 –sign)

$$905 \quad \begin{aligned} -s_0^{(1)} &= 0 \\ -s_0^{(2)} &= \frac{1}{5} s_2^2 \end{aligned}$$

906

$$\begin{aligned}
907 \quad -s_0^{(3)} &= \frac{2}{3 \cdot 5 \cdot 7} s_2^3 \\
908 \quad -s_0^{(4)} &= \frac{1}{3^2} s_4^2 + \frac{2}{5 \cdot 7} s_2^2 s_4 \\
909 \quad -s_0^{(5)} &= \frac{2}{3 \cdot 5^2 \cdot 7} s_2^5 + \frac{2^2 \cdot 5}{3^2 \cdot 7 \cdot 11} s_2 s_4^2 \\
910 \quad -s_0^{(6)} &= \frac{-127}{3^2 \cdot 5^3 \cdot 7^2} s_2^6 + \frac{2}{5^2 \cdot 7} s_2^4 s_4 + \frac{2 \cdot 3}{7 \cdot 11 \cdot 13} s_4^3 + \frac{2 \cdot 5}{11 \cdot 13} s_2 s_4 s_6 + \frac{1}{13} s_6^2 \\
911 \quad -s_0^{(7)} &= \frac{2}{3 \cdot 5^3 \cdot 7} s_2^7 + \frac{2 \cdot 41}{3^3 \cdot 5 \cdot 7 \cdot 11} s_2^3 s_4^2 + \frac{2^3}{3 \cdot 5^2 \cdot 7} s_2^5 s_4 + \frac{2 \cdot 7}{5 \cdot 11 \cdot 13} s_2 s_6^2 + \frac{2^2 \cdot 5}{3^2 \cdot 11 \cdot 13} s_4^2 s_6. \\
912 & \tag{A16}
\end{aligned}$$

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