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WHAT DO ADULTS REMEMBER FROM THEIR HIGH SCHOOL MATHEMATICS? THE CASE OF LINEAR FUNCTIONS ⋆

ABSTRACT. A qualitative study was designed to investigate adults' long-term memory of mathematics learned in high school. Twenty-four men and women, aged 30 to 45, were requested to recall mathematical concepts and procedures during individual interviews. This article reports findings regarding the subjects' attempts to draw graphs of simple linear functions. In general, these findings support the idea that retaining high school mathematical content strongly depends on the number, level, and total length of mathematics courses taken by the student. Diverse responses to the task of drawing a graph of a linear function such as y=2x, were documented and categorized. In many of these responses, the basic mathematical communal notion of linear graphing was replaced with personal on-the-spot constructing of ideas. Detailed analysis of three cases is presented, based on recall theories that explain the mechanism of recalling in terms of reconstruction vs. reproduction.

KEY WORDS: adults, functions, graphs, long term memory, qualitative methods, recall theories

1. The long-term maintenance of knowledge learned in school: A road (almost) not taken

The educational community is a rather multifaceted one. Different sectors in this community – parents, teachers, researchers and other members – do not always see eye to eye, to say the least. However, if there is something that educators do share, I suppose that it is the basic aspiration that makes us send our children to twelve years of schooling: We all want our children to be 'educated', to know things about the world, to broaden their horizons. Bearing in mind this naive but probably most agreed-upon goal, it is quite surprising that of the immense existing body of school research, only a minute portion is dedicated to questions of knowledge maintenance after school. Although the scarcity of research in this field was referred to more

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than twenty years ago by memory researchers Bahrick (1979) and Neisser (1978), their call for innovative ecological memory studies was, in regard to this specific issue of knowledge learned in school, not widely answered. The following citation from Bahrick (1979) expresses his dissatisfaction with the state of affairs at the time, and it is still quite relevant today:

> It is disappointing that nearly 100 years of research have not yielded much progress toward specification of the conditions under which information, once acquired, can be maintained indefinitely. [. . .] Much of the information acquired in classrooms is lost after the final examinations are taken, but beyond the general advice to practice and rehearse frequently, we have little to offer those who wish to minimize or prevent such losses. (Bahrick, 1979: 297)

Nevertheless, some research on long-term memory for knowledge learned in school does exist. Semb and Ellis (1992, cited in Semb et al., 1993), who reviewed studies concerning this issue, pointed out that some of these studies were difficult to trace, due to the fact that they appeared in discipline-specific journals. Indeed, if the small aggregate of relevant research is to be sorted by disciplines, we might find that for certain school subjects there are few studies, at most, which address the issue of how much is remembered of those subjects in the after-school years.

In regard to mathematics, a unique work is that of Bahrick and Hall (1991). This large-scale quantitative study was designed to identify variables that affect losses in recall of high school algebra and geometry contents, throughout the life span. A major finding of this work was:

> When the acquisition period extends over several years, during which the original content is relearned and used in additional mathematics courses, the performance level at the end of the training is retained for more than 50 years, even for participants who report no significant additional rehearsal during this long period. In contrast, those whose acquisition period is limited to a single year perform at near-chance levels. . . (Bahrick and Hall, 1991: 30)

Bahrick and Hall also found that grades in courses, as well as standardized test scores, have minor influences on the rate of loss, although they reflect differences in the original degree of knowledge. The rate of loss is most affected, as emerges from the citation above, by variables concerned with the amount and distribution of practice. This conclusion is in agreement with findings from several memory studies in other disciplines of knowledge (Semb et al., 1993), and we will return to it later on. However, Bahrick and Hall's work does not tackle the following question directly: *what* do adults remember from their past mathematical studies? The performance of Bahrick and Hall's research participants (about 1700 in number) was measured by psychometric means (i.e., correct/incorrect answers). For obvious reasons, a statistical study of such a large scale does not usually involve cognitive analysis of answers to open-ended questions.

Since cognitive analysis could potentially reveal phenomena that remain unnoticed by psychometric analysis (Karsenty and Vinner, 1996; Cooper and Dunne, 2000), it seems that a qualitative study, scrutinizing adults' responses to mathematical tasks, could add to the general picture in the issue of maintenance of high school mathematical content. Thus, the research reported herein concentrated on a small group of adults, in order to allow an in-depth analysis. The analysis was based in part on theories of recall, which will be described in the next section.

2. THEORIES OF RECALL: A BRIEF REMINDER

The ways in which people recall different contents have long been an attractive subject for researchers. However, almost all of the traditional memory studies were held in research laboratories, and hence referred mostly to contents that could be acquired in laboratory settings (e.g., words, scripts, lists of various kinds). The well-acknowledged classic work of Bartlett (1932) laid the foundations of this field. In Bartlett's famous research, subjects were requested to recall a story they had read, within several time intervals. After meticulous examination of the discrepancies between different versions of the story, and based on other experiments, Bartlett suggested that recalling is a mechanism of *reconstruction* rather than *reproduction*. These terms will now be explained briefly. Reproduction means that specific details from the past are coded in memory, in such a way that enables the eliciting of 'copies', however pale or dim, in the present. Prior to Bartlett's publication, most theories of recall focused on the idea of reproduction, as can be recognized from various titles given to these theories: 'copy theories', 'trace theories' and 'reappearance hypothesis' (see Neisser, 1967). Bartlett rejected these theories, suggesting that recall is performed through a process of reconstruction: This means that attempts to elicit past experiences tend to yield an altered version, that might be different from the original one in a substantial manner. For example, when requested to repeat a story, people are likely to produce an interpretation, though they may be unaware of doing so. Thus, some details might be omitted, others emphasized or even added. According to Bartlett (1932), reconstruction is controlled by schemas, which he defined as active organizers of past experiences. More recent memory researchers, such as Neisser (1984), basically agree with Bartlett's ideas, although they suggest some refinements and modifications. Brewer and Nakamura (1984), for instance, talk about partial reconstructive recall. Yet, the context of many discussions about the nature of the recalling mechanism, is what Brewer (1986) called *personal memory*, or, to use Tulving's (1972) original distinction, *episodic memory*[1]. These terms refer to the recall of events and occurrences personally experienced by an individual. Theories of recall are less frequently applied to depersonalized knowledge, much less to knowledge learned in school. In this article I would like to offer a glimpse into adults' attempts to recall mathematical material, in a way that suggests that a mechanism of reconstruction might take place. Moreover, I intend to show that such a reconstruction, idiosyncratic as it might be, sometimes follows a certain inner logic that is used in the absence of accessible relevant details.

3. THE STUDY: CHARACTERISTICS AND METHODS

3.1. *The type of research conducted*

The research presented here is defined, according to Stake's classification (1994, 1995), as a collective case study. This category of qualitative research refers to studies in which a certain number of cases are thoroughly examined in order to highlight a particular issue. In contrast to an intrinsic case study, in which the focus of interest is on the specific case uniquely, a collective case study is usually defined as instrumental, that is, the analyses of cases are meant to serve as a vehicle for enhancing a more general understanding in regard to some phenomena or theory. However, as Stake (1994) notes, the primary purpose of case studies, even when they are collective and instrumental, should be to gain a better understanding of the cases themselves.

This study was therefore designed in light of two complementary objectives: to learn as much as possible from each single case, but also to obtain a more general picture in regard to all cases. This dual intention affected the case selection procedure, as will be described below.

3.2. *Target population and case selection*

The study took place in Israel, and was restricted to Israeli high school graduates who had passed the Israeli Matriculation Exam in mathematics, which is an external, nation-wide test, taken at the end of high school. The first decision that had to be made in regard to subjects' selection was the sector from which to choose candidates: Clearly, an attempt to represent the whole range of high school graduates would have been unfeasible, since the number of cases was to be relatively small. Therefore it was decided to focus on adults with post-secondary education. The implied assumption was that the voices of subjects holding college or university degrees might define ‘the upper bound’ of people’s recollections in regard to mathematics.

The second methodological decision concerned the issue of accessibility. The main research method was an extensive interview. Such a procedure is, on the subject’s part, not only time consuming, but also – and this might be even more crucial – emotionally demanding. Hence it could not be expected that randomly chosen adults would cooperate with a researcher unknown to them. It became clear that some degree of personal acquaintance between the researcher and the subjects was a prior condition to the study’s implementation. After examining different social circles in which I had some involvement, I decided to select subjects from the population of a small village near a city where I was living at the time. This village was founded twenty years ago, and the adult residents – approximately 200 in number – came from all over the country and had attended different high schools in their youth. Members of this community can be seen as representing the higher-educated sector of the society in Israel. Restricting the study to volunteers between the ages of 30–45[2] who were not mathematics graduates or mathematics teachers, led to a group of 105 candidates. Then, in light of the dual objective mentioned above, personal data about the candidates were gathered, in order to maximize diversity when choosing subjects. The goal was to achieve, within the limited sector I focused on, a wide spectrum of cases that would consequently enable a broader view, based on investigating potentially different past mathematics learners. Three factors were considered: gender, level of mathematics taken in high school[3] and current profession[4]. After mapping these characteristics in a sampling table, a subject was chosen randomly from each non-empty cell. The final number of subjects was 24. Of these, 12 were men and 12 women, 12 took mathematics in the low-level track and 12 in the intermediate or high level tracks (see note 3). All subjects had post-secondary education, mostly from colleges or universities, and they were all engaged in professional careers in areas such as law, medicine, psychology, art, business, high school teaching and other.

3.3. *Research methods*

Each subject participated in an individual session, which lasted between 2 to 3 hours. A lion’s share of this time was dedicated to a semi-structured interview, consisting of two parts. The first part was dedicated to affective aspects, later analyzed within a framework I called “the personal mathematical profile”. In this article I will not refer to this affective part of the research. Some results can be found in Karsenty and Vinner (2000). Detailed results will be published elsewhere.

In the second part of the interview, subjects were asked to solve mathematical tasks involving basic concepts and procedures. With a list of nine questions as a starting point, the conversation was constructed spontaneously according to the flow of ideas expressed by the subject. If interesting themes emerged, usually further unplanned questions were posed in order to follow the subject's line of thought. Moreover, it is important to emphasize that the interview was interactive, as the interviewer occasionally responded to the subject's answers. Responses were either in the form of feedback when necessary (e.g. if it became clear that the subject would not continue without it), or brief reminders, instruction, or hints in case the subject was 'stuck' in the process of recall. The intention was to investigate if and how a certain memory could be elicited through a certain trigger. The interviews were recorded and later transcribed.

Of the data collected in the research, I will report here on results concerning the subjects' attempts to draw graphs of simple linear functions, as documented during interviews. These results are presented in the next section.

4. Recalling Graphical Descriptions of Simple Linear Functions

4.1. *Two levels of reporting subjects' reactions to the task*

Subjects were requested to draw a graphical representation of a simple linear function. In most cases the function was y=x. However, in some cases the function was different, for instance in the case of Ilan[5], who mentioned the function y=2x during a previous answer, and thus was asked to draw it, in the natural flow of the interview. Usually, the discussion with each subject around the issue of drawing the given function did not end quickly. In fact, of all the tasks presented during interviews, this one produced some of the richest and most comprehensive data. In many cases, after the subject gave a spontaneous answer, and was asked to explain this answer, the discussion evolved in one (or a combination) of the following ways. Either the subject thought some more and changed or refined the original answer, or the subject justified the answer in a way that evoked further questions in regard to other functions. In addition, as the discussions involved mild interventions, in the manner explained above (see section 3.3), this process yielded some further interesting responses. In light of the extensive data, introducing results is a complicated and problematic issue, to be considered carefully. Since reporting different elements in each of the 24 discussions was unfeasible, it was decided to report the results in

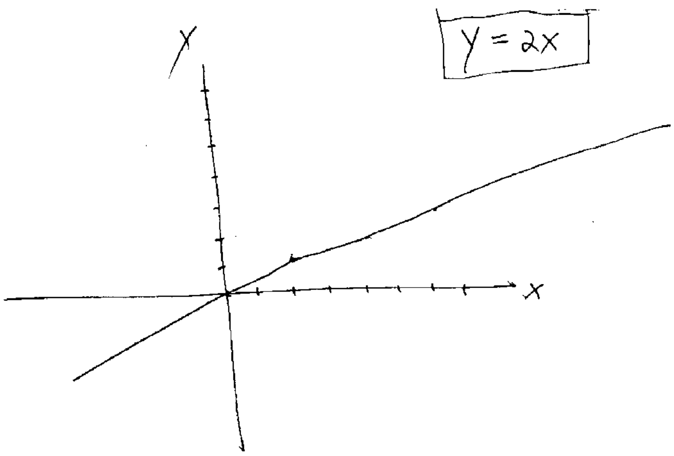

Figure 1. Shaul's sketch for the function y=2x.

two levels. First, the spontaneous answers will be introduced, grouped into categories. This level of report gives a general picture about what people remember on the surface. Then, three cases will be reported and analyzed in detail, including the evolution of ideas during discussions. This level of report is intended to exemplify processes of reconstructing memory.

4.2. *Research results (1): Subjects' spontaneous responses to the task of graphing a linear function*

The initial attempts of all subjects to draw the function y=x (or some other linear function) were grouped into six categories, described below.

Category I. Sketching a correct graph by marking two or three points in a coordinate system and connecting them with a straight line.

Category II. Sketching a straight line that reflects a misinterpretation of the relationship between x and y.

Example: Shaul (male), a 45-year-old sound technician who studied mathematics in the high level track, sketched the graph of y=2x shown in Figure 1. His reasoning was:

S: If y equals two x, so each segment in y equals two segments of x. [. . .] So it's about here, right? [Marks points at (2,1), (4,2) and draws a straight line through them].

Shaul translates y=2x as "having two x's for each y". This translation re-

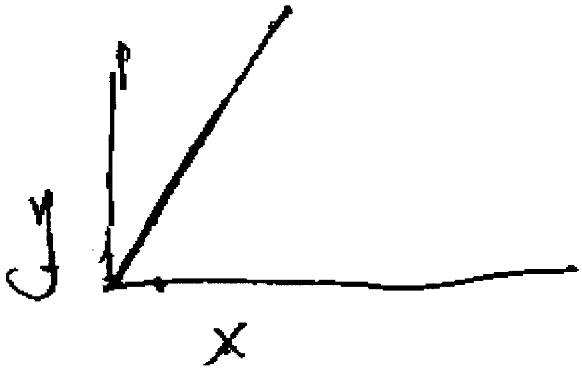

Figure 2. Gadi's sketch for the function y=2x+1.

flects a well-known proportional misconception, usually referred to in the literature from the opposite direction, i.e., mistranslating word sentences describing relations into symbols, as in the famous student-professor problem (Rosnick and Clement, 1980; Rosnick, 1981; Kaput and Sims-Knight, 1983; Philipp, 1992). However, if we are to look beyond this misconception, Shaul's answer communicates the general idea of using two points as means of drawing a straight graph in a Cartesian system.

Category III. Sketching an incorrect graph based on a holistic estimation of the behavior of the function.

Example: Gadi (male), a 41-year-old architect, who studied mathematics in the intermediate level track, sketched the graph of y=2x+1 shown in Figure 2. His reasoning was:

G: y is at least twice as big as x, so on every move of 1 here [on the x-axis], I need here [on the y-axis] 2 and more.

Gadi does not plug numbers in the given function, but rather analyzes the relationship between x and y, and draws the line accordingly. A further question posed to Gadi, requesting him to draw y=2x+3, yielded another straight line through the origin, with a steeper slope. Gadi explained: "So it's a little more. Not much. What counts is the ratio of 2". Two motives are revealed by Gadi's reasoning. First, Gadi appears to be confident that these two functions are represented by straight lines. It should be noted that later in the interview, when asked about the function $y=x^2$, Gadi sketched the correct parabola – again without plugging in numbers, but as a holistic picture preserved in memory. This link between the symbolic and the graphic representations can be regarded as an example of what Skemp (1976) called relational understanding. Indeed, Skemp suggested that such understanding of mathematical content has a positive impact on maintaining this content for long periods of time.

Second, Gadi recalls both straight graphs (y=2x+1, y=2x+3) as starting from the origin. He regards the ratio *m* as the dominant aspect in the linear expression y=mx+n, while *n* appears to have less effect, yet they both determine the slope. This idiosyncratic idea seems to have taken over the original learning.

Category IV. Marking only one point in a coordinate system.

Example: Amira (female), 31, a museum director who studied mathematics in the intermediate level track, was requested to draw the graph of the function y=x. In response, she sketched a Cartesian axes system and marked the point (1,1). Her reasoning was:

A: You said that x is equal to y, and if this is x and this is y, and these are the positive points, and this is 1 and this is 1, so let's say I did it in the middle.

Amira's response suggests that the request to draw the graph of y=x is interpreted as "solving", i.e. finding a point in the Cartesian plane where y is indeed equal to x. The point marked is an arbitrary representative of solutions to the equation y=x.

Category V. Drawing a graph by allocating segments on the x-axis and the y-axis and connecting the two endpoints.

Example: Tamar, (female), 42, a high school art history teacher, who studied mathematics in the low level track, sketched the graph of y=x shown in Figure 3. Her reasoning was:

T: Well, anyway, there's got to be something equal here.

Tamar's explanation is based on an attempt to make sense of the symbol phrase "y=x" in the context of an axes system, a context that she recalls. Tamar develops an idea that seems to be an improvisation created on the spot. However, we shall see later that this idea is rather persistent. The full discussion with Tamar will be presented in section 4.3.

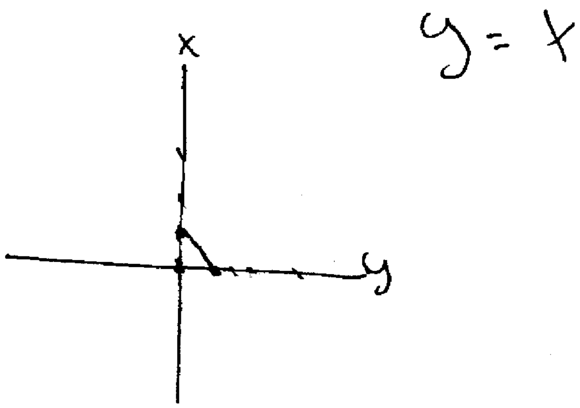

Figure 3. Tamar's sketch for the function y=x.

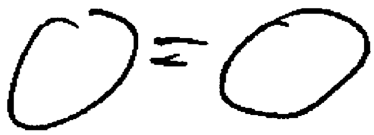

Figure 4. Yafa's sketch for the function y=x.

Category VI. Describing the function through equality between shapes or line segments.

Example (a): Yafa, (female), 38, a high school humanities teacher, who studied mathematics in the high level track, sketched the graph of y=x shown in Figure 4.

Example (b): Dov, (male), 37, a government official, who studied mathematics in the low level track, sketched the graph of y=x shown in Figure 5.

These two examples reflect a switch from ideas shared by the mathematical community to idiosyncratic reasoning.

The distribution of subjects within the categories I-VI (and a seventh 'no response' category) is shown in Table I. Observing this table, a connection can be noticed between high school mathematics level tracks and

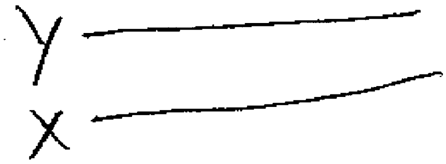

Figure 5. Dov’s sketch for the function y=x.

TABLE I

Distribution of subjects within the categories formed for drawing a graph of a linear function (N = 24)

Category	No. of subjects assigned to this category, distributed by level of math taken in high school			
	High level:	Medium level:	Low level:	Total in this category:
I. Correct graph.	5	1	1	7
II. Straight line reflecting a misinterpretation of the relationship between x and y.	1	–	4	5
III. Incorrect graph based on holistic estimations.	–	2	–	2
IV. Marking only one point on an axes system.	–	1	2	3
V. Allocating segments on the x-axis and the y-axis and connecting the two endpoints.	–	–	3	3
VI. Equality between shapes or line segments.	1	1	1	3
VII. No response.	–	–	1	1

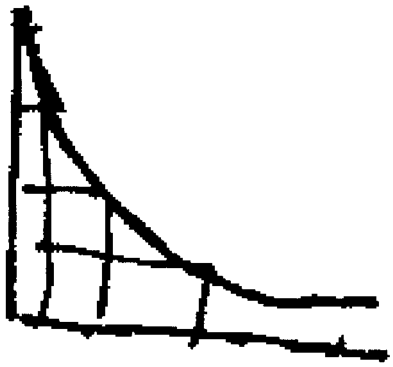

Figure 6. Tamar's sketch, drawn while explaining what is a function.

the category distribution: Most of the subjects classified within the three top categories (9 of 14), which in general can be characterized as expressing some degree of acquaintance with representations of linear functions in a Cartesian system, are subjects who participated in the high and intermediate tracks in high school. Most of the subjects classified in the other four categories, are low-level graduates (7 of 10). In these categories, the basic notion of linear graphing is replaced with personal on-the-spot constructing of ideas. In the next section, some of these ideas will be scrutinized.

4.3. *Research results (2): Three cases of reconstructing memory*

4.3.1. *The case of Tamar*

As mentioned above in section 4.2, Tamar is a 42-year-old high school teacher. She teaches art history, a subject in which she has a Bachelor's degree. Tamar studied mathematics in the low level track, and her final matriculation grade was 90. Earlier in the interview, when Tamar was asked what a function is, she said the following, while sketching the drawing presented in Figure 6.

T: There is a horizontal axis and a vertical axis, and there are points. You connect the points and you get a function.

Note that Tamar is using x and y coordinates in her drawing. However, right after that drawing was done, when Tamar was requested to draw a graph for the function y=x, she abandoned this use of coordinates and drew the sketch presented earlier in Figure 3. As explained in Category

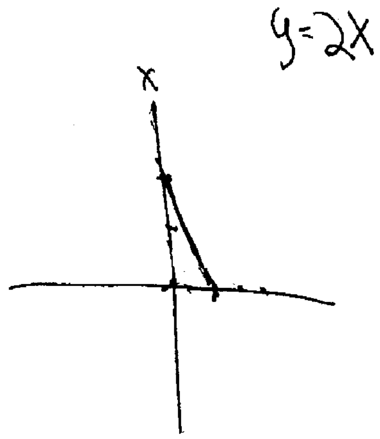

Figure 7. Tamar's sketch for the function y=2x.

V above, she allocated equal segments of one unit on both axes. She then joined the two endpoints by a line. In light of this description, Tamar was asked to draw the function y=2x. As can be seen, Tamar persisted with the same logic, drawing the graph of y=2x presented in Figure 7. This time the segment allocated on the x-axis (which is the vertical axis in the drawing) is twice as long as the one allocated on the y-axis, thus reflecting a proportional misconception held also by other subjects (see category II in section 4.2 above). After Tamar sketched these drawings for y=x and y=2x, the following conversation evolved:

T: It's something like that, I don't know.

Int: Have you any way to check this?

T: No, I don't know what this means.

Int: You said that there are points, you connect the points and you get a function.

T: Yes.

Int: [. . .] So, show me for instance a point here [referring to the graph of y=x] that keeps this rule, y=x.

Tamar responds by marking a point in the middle of the 'hypotenuse' created by the graph and the axes. The conversation continues:

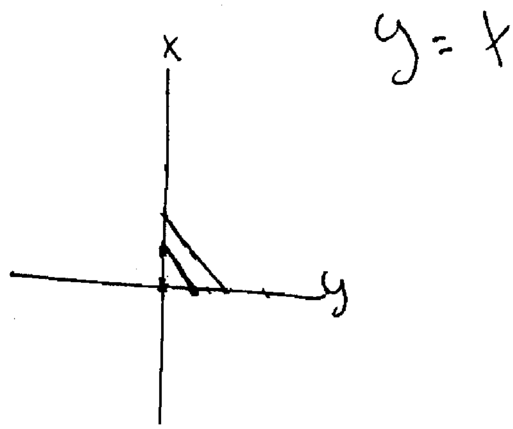

Figure 8. Tamar adds a parallel line to the original sketch of y=x.

Int: Why?

T: Because the distance here is equal to the distance here [refers to the two halves of the hypotenuse].

Int: I see. Can you give another one, another point on the graph that shows that its x is equal to its y?

At this point Tamar adds a second line, parallel to the first one, as shown in Figure 8. She explains:

T: What is a point? A place on these axes where this is equal to this? It could be anything.

Int: Must it be on the axes?

T: I don't know, yea, yea, sure. Along all the axes, this equals this, this equals this. . .

[Tamar adds more parallel lines, as shown in Figure 9].

Int: So the function is what, this collection of lines, one of these lines?

T: Maybe the function is this.

[Tamar bisects all the lines with one line, as shown in Figure 10]

Int: Ah. Why?

T: Because everywhere this distance equals this distance.

Int: I see. And what would you do here, with y=2x?

T: Ah, now I'm recalling.

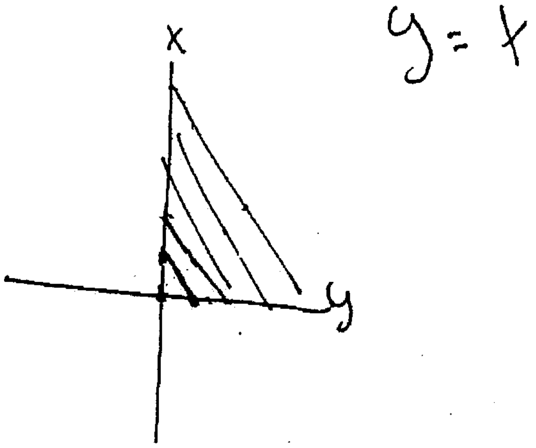

Figure 9. Tamar adds more lines to the sketch of y=x.

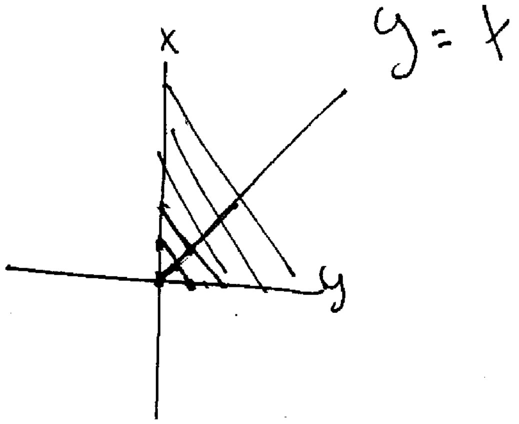

Figure 10. Tamar bisects the lines.

Now Tamar repeats the same process with y=2x, only this time the final line does not bisect the parallel lines, but divides them into thirds and two thirds (Figure 11). In order to examine this consistent line of thought, another question is posed to Tamar:

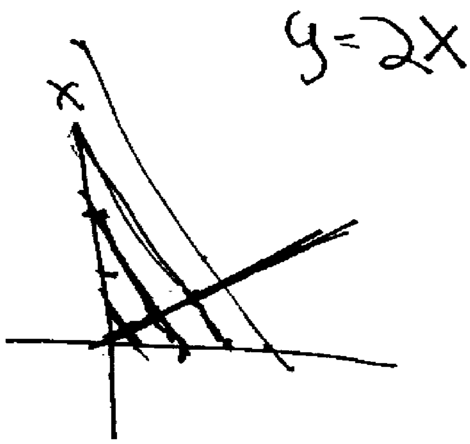

Figure 11. Tamar modifies the graph of y=2x.

Int: Ok, now I'd like to know about the point (2,3). Where is this point?

T: Here there's 1,2,3 [marks notches on a vertical axis] and here there's 1,2 [marks notches on a horizontal axis, see Figure 12].

Int: And where is the point (2,3)?

T: Here, in the middle? [refers to the midpoint of the line that joins the two points].

The conversation with Tamar brings forward two colliding ideas. On the one hand, the notion of function appears to be preserved in Tamar's memory, although as an image rather than a definition (Vinner, 1983). Her sketch, drawn while trying to recall this concept, clearly shows that the concept image includes Cartesian coordinates (see Figure 6). On the other hand, however, Tamar constructs a new schema for the process of graphing functions. This schema is embedded in some current common sense, but interestingly, Tamar refers to it using the phrase "now I'm recalling". In Bartlett's terms (1932, see section 2 above), this is an example of reconstruction of memory. The new reconstructed schema gains priority over the original recalled image. Tamar is remarkably persistent in following her own idea; at the end of the cited conversation, she abandons the coordinates even as descriptors of a single point, in favor of applying her new method to describe the point (2,3).

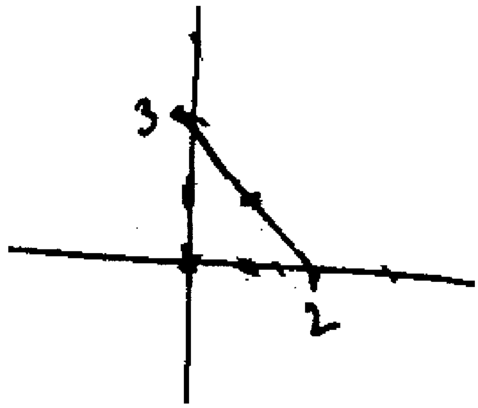

Figure 12. Tamar draws the point (2,3).

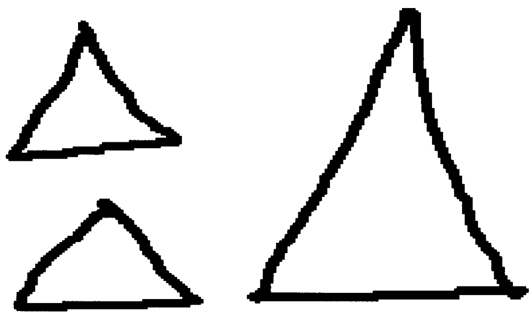

Figure 13. Nir's first attempt to draw the function y=2x.

4.3.2. *The case of Nir*

Nir, 43, studied professional photography in a college of technology, but is currently working as a furniture designer and a carpenter. In high school, he studied mathematics in the intermediate level track. Nir's first attempt to draw the function y=2x was classified within Category VI, and is shown in Figure 13. While drawing, Nir said:

N: Two triangles that, taken together, give one big triangle. This is how I see it today, and I know it's not correct in the mathematical way of thinking.

At this point, a small intervention was made in attempt to trigger Nir's recalling process, and Nir responded to it immediately, as can be seen in the following paragraph:

Int: So let's say it has to do with an axes system. Do these words ring a bell?

N: Yes.

Int: Can you sketch here an axes system?

[Nir sketches axes with notches from –10 to 10.]

Int: So now these axes... do you remember them having names, perhaps?

N: I think it was... x and y? [Writes x and y near the horizontal and the vertical axes, respectively].

Int: So now if I ask you, in this axes system, to describe the function y=2x, do you have an idea?

N: Yes. So all ah... that is if I move here one... the y is twice as large as x. So each step in y will give me half of a step in x. Two x's together will give me one y. So if one y will be 10, then two x's together will also be 10. So if in this plane [sic] I go up [on the y-axis] and I go up to 10, then one x here will give me 5 and two x's will give me 10... yea, yea. So the line will go in such a way that one y will give me two x's. So if it's one here and two here, this line won't be 45 degrees, but it will be like this [draws a sketch, see Figure 14] [...] If I'll go 45 degrees, they'll be equal to one another. The more I tilt the angle, the change will be, in this case of tilting towards this direction [points towards the x-axis] the y in relation to the x will be larger, and if the opposite then the y will be smaller in relation to x. The more I tilt it the more x's will be equal to one y, according to the angle.

Note that although Nir began with a correct analysis of the ratio between x and y, saying that "each step in y will give half of a step in x", nonetheless the analysis is later replaced with the proportional misconception pointed out earlier in Category II. Generally, as a response to the interviewer's mentioning of axes, Nir shifted from an idiosyncratic idea typical of Category VI, to the domain of mathematical conventions. He now treats the function y=2x holistically, estimating its behavior in a manner typical of Category III. The next question posed to Nir was meant to examine this holistic view in a case of a function with an additional free term.

Int: What about if I give you the function y=2x+3, what will you do?

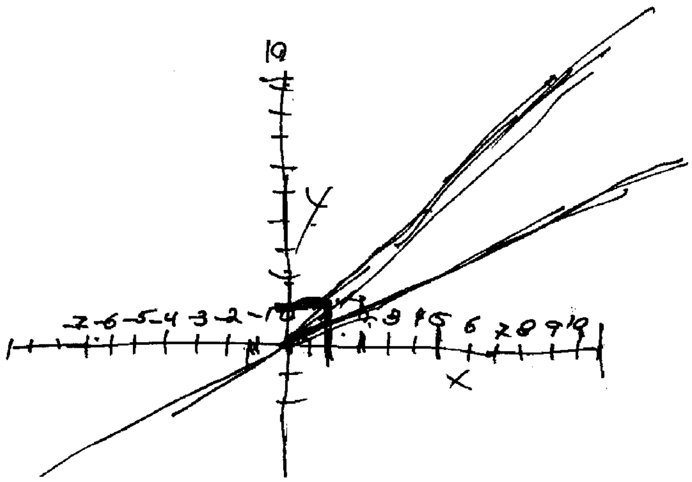

Figure 14. Nir's second attempt to draw the function y=2x.

N: One y we say is equal to two x's plus 3. Ah... how do I treat this 3... [pause]. Look, what's clear to me, without understanding what this means, what this 3 is saying, it's saying that in principle the y in relation to the x's is even more... [...] What this tells me is, that if we add 3 to these x's, then they will equal one y. This means, that these two x's are even less, even less than before. So if this was my line before, when y equals 2x, then the line should be even flatter, because to the 2x you have to add 3 for it to be equal to y. Now, how to combine between the number and the x's, this I can't figure out. I think I know that the line now is... is... its angle is smaller in relation to the x-axis by 3. Now, this 3, the question is how it relates to the x's. What I feel, this is unpleasant, like the tools are gone. I don't feel like the brain has weakened, but I feel as if I need to do a job but I haven't got what I need to do it.

At this stage Nir was stuck; according to his analysis he now had to reduce the angle between the line y=2x+3 and the x-axis, compared to the angle of y=2x, but he did not know how to perform this in a way that will reflect the number 3. Here the interviewer intervened again:

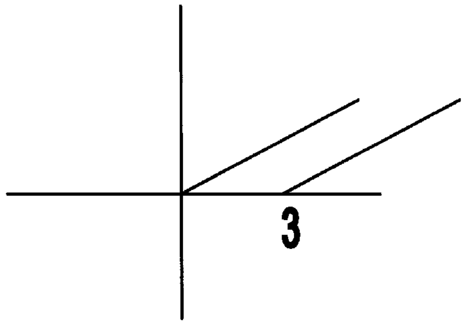

Figure 15. Illustration of Nir's sketch of the function y=2x+3. The original sketch appears as part of Figure 16.

Int: I see. So, according to your method, if we are to challenge you a bit, all the functions of this sort will go through the origin, that is (0,0).

N: Yes.

Int: And this is...

N: It's not right, no, I can't accept this. So what this is saying is that the reference begins from 3 [draws new axes and sketches the graph of y=2x again]. If we got the y=2x and started it from here, now we actually start it from here [draws a parallel line to y=2x through (3,0), see illustration in Figure 15], and this is our point of reference, yes.

Investigating Nir's new direction of thought, the interviewer now introduced another task. Two new linear graphs were drawn in the same coordinate system, and Nir was requested to state their equations. The first line was parallel to y=2x and y=2x+3 as drawn by Nir, but 'starting' from (0,3). To find its equation, Nir extended the line to its x-intercept, estimating that point as negative six. He then concluded that the equation was y=2x–6. The second line was through (0,4) and (4,0). Nir speculated that the equation of this line was y+4=x+4. Both lines are shown in Figure 16.

The discussion about linear graphs ended with Nir's expression of uncertainty about the last graph:

N: This is a bit confusing. I'm not sure, perhaps y+4=x+4? But I think I'm wrong.

Int: Why do you think you're wrong?

N: It seems logical, but... This is the y plane, and we started from plus 4, and this is the x plane, and we started it from 4 also... This is what I can figure out but I might be wrong.

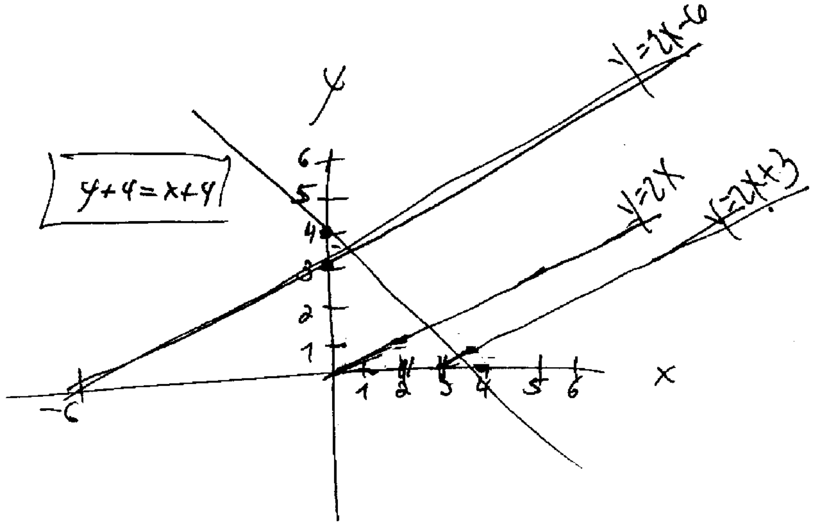

Figure 16. Nir's sketches of the functions y=2x and y=2x+3, and his equations for two other given graphs.

The conversation with Nir exemplifies a process of constructing a new schema for solving a problem, when the old schema, acquired many years ago, is no longer accessible. As in Tamar's case, Nir uses his common sense as a substitution for absent recollections. However, while Tamar continually suggested ideas that would agree with her new schema, in Nir's case we can notice that 'memory flashes', evoked by minor triggers, alter the ongoing schema construction. Thus, in the first stage Nir draws the function using geometrical shapes, knowing that this is not the expected answer. Eliciting the recollection of a coordinate system leads to dismissal of this direction and concentration on the relationship between x and y. In the second stage Nir perceives x and y as quantities whose ratio is expressed by the line's angle of inclination: As y equals more x's, the line gets flatter, and thus y=2x+3 is flatter than y=2x. Drawing Nir's attention to the consequence of this method, i.e., that all lines will pass through the origin, creates a contradiction with another 'memory flash', and Nir shifts to the third stage of the construction. In this stage Nir performs translations of y=2x according to the x-intercept, and constructs the idea that the x-value of the intercept is the number to be added on the right hand side of the equation. Nir extends this line of thought quite coherently, conjecturing about a line with a negative slope. Now the y-intercept is also taken into account, the y-value of this point being added to the left hand side of the equation. Yet, Nir is not satisfied with this idea and expresses his feeling of error. This is an example of the influence of a memory, existing even when it cannot be elicited, when one does not recall the right answer but can identify a wrong answer, sensing that ‘this is not it’.

4.3.3. *The case of Eli*

Eli is 40 years old, a principal of a post-secondary school for adults. In high school, he studied mathematics in the low-level track, but quit mathematics classes in twelfth grade, due to what he explained as ‘lack of interest’. After the army service he completed his mathematics matriculation exam and his final grade was 90.

Eli’s sketch of the function y=2x was classified within Category II, and was very similar to Shaul’s drawing, shown earlier in Figure 1. Eli obtained his drawing by marking the point (2,1) and tracing a straight line through the origin and this point. In his view, he explained, a line is “a point in motion”. Therefore, in order to draw a line, one has to find the direction of motion, which is set by the function. Eli said:

E: How can you define a line, you want to define its direction. I gave it some direction that was, like, determined by this function. That’s all. . . As far as I’m concerned, the first point has in fact determined the direction.

Eli was asked two more questions. First, he was requested to explain how he obtained the point (2,1). In response, he wrote y=2x and substituted 1 for y, forming the equation 1=2x. Eli solved this equation, not without difficulty, and obtained x=1/2. He then said:

E: So this contradicts my whole theory about this thing. It is something else. So, if I actually go like this. . . now, if I define this as the y-axis, I solved the problem [that is, Eli suggests switching the names of the axes]. [. . .] You see, because then I stay with the same. . . not that it matters. Actually it does matter. Because the minute this is y, then progress of one unit here means progress of half a unit here.

We see here a creative solution to the discrepancy that Eli discovered in regard to his first figure. Since the point x=1/2, y=1 did not fit the line already drawn, and Eli apparently knew that it should, he suggested renaming the x-axis ‘y’ [and vice versa]. This suggestion is, in my opinion, not only flexible but also indicative of a certain capability to look beyond mathematical conventions into the general idea, which in this case is the relationship between two variables. Eli used his common sense, and his solution can be considered no less rational than the expected and conventional one, i.e., drawing a new line through (1/2,1). We will now see that Eli continued to use his common sense when elaborating on his idea of a line as a point in motion. The second question posed to Eli was to draw the function y=x+3. Eli substituted 0 for y, and then substituted 1 for y. Again, solving the equations 0=x+3 and 1=x+3 turned out to be a complicated task for Eli, who at first did not seem to take into account the possibility of negative x's. After quite a long process of trial and error, and with the assistance of the interviewer, Eli finally arrived at the points (–3,0) and (–2,1). I will not go into details about this process here. The main point of interest at this stage is, what are Eli's actions once he marked these points on an axes system.

Eli started to connect these two points with a straight line, but changed his mind, and connected the origin with (–2,1), as shown in Figure 17. The following conversation evolved:

E: y equals 1 and x equals –2, so it goes here.

Int: What about the other point, when y equals 0 and x equals –3?

E: It was here.

Int: Yes. . .

E: What happens to the graph? This is the graph [connects the origin with (–3,0), see bold segment on the x-axis in Figure 17].

Int: You're making three suggestions.

E: Two.

Int: First you said it might go like this [refers to the beginning of the line connecting (–3,0) and (–2,1)], then you said it will go like this [the segment between the origin and (–2,1)], and now you're saying it goes like this [the segment between the origin and (–3,0)].

E: That's right.

Int: So I don't really understand if it's all three or one of the suggestions.

E: At first, with the first given point, it did go like this [the segment on the x-axis]. In my opinion, when the y equals zero we get one given figure. When the y equals 1 then the graph begins to rise, actually.

Int: I see. Do you mean that this [the first regretted line] is not. . .

E: No. This cannot be.

After marking down two points obeying the rule of the function, Eli's first and spontaneous intention was to connect these two points with a straight line. It seems as if a shred of recollection from the learned material has

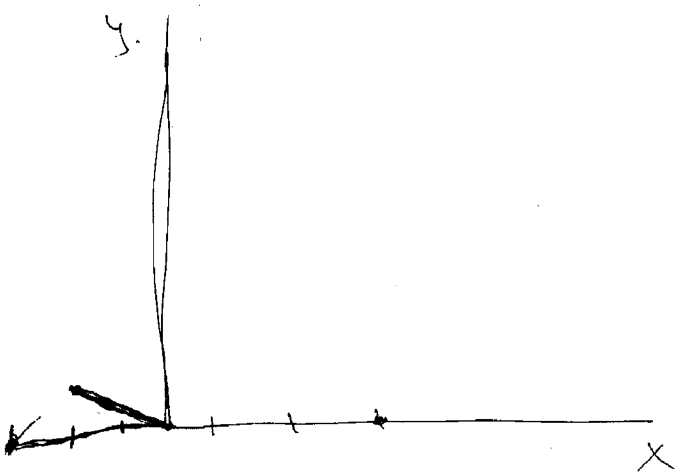

Figure 17. Eli's sketch of the function y=x+3.

emerged as a flashbulb, but Eli immediately rejected it, in light of his newly reconstructed idea of graphing straight lines according to a given direction. It is interesting to note that the result of this idea – that each given point creates a different graph to the same function – does not bother Eli and does not seem to cause any conflict with prior knowledge. On the contrary, it suits his view of a graph as a point in motion, a dynamic and alternating entity. This image is articulated in Eli's words "the graph begins to rise". Again, we see here an example of a reconstructed schema that attains dominance over an old learned one.

5. General summary

Investigating the long-term maintenance of knowledge learned in school is a challenging and fascinating pursuit. As Semb and Ellis (1993) noted, "theory development in this area is in its infancy" (p. 310), and therefore there is a potential significance in every effort to broaden the general picture emerging from the existing body of research. In this study I tried to exploit the power of qualitative methods, in order to gain some insight about how adults think when confronted with tasks related to mathematical material learned many years ago. Specifically, the task reported in this paper was to graph simple linear functions. The data, obtained through interviews, was rich and intriguing. As can be expected when it comes to human behavior, each case was unique in its own way, and demonstrated a different path of handling the task. Nevertheless, some shared characteristics made it feasible to form categories of recall. The findings reported in section 4 above, support, in general, fundamental assertions made by several memory researchers. Semb and Ellis (1993) claim, based on their review of studies, that the amount of original learning is a prime determinant of what is remembered. Bahrick and Hall (1991), who investigated specifically the maintenance of high school mathematics content, relate to this factor in detail, listing three elements in regard to the original conditions of learning: the number of mathematics courses taken by the student, the total length of the period in which mathematics was learned, and the highest level of the mathematics courses attained. In other words, the preservation of knowledge was found to be dependent on a certain threshold of exposure to the content. In the present study, it was found that graduates of the higher level tracks recalled graphical descriptions of linear functions better than graduates of the low level track. While the former group tended to produce linear graphs, however incorrectly placed, the latter group tended to create idiosyncratic solutions to the task. If we take yet a stricter view of the results, the fact is that most of the subjects were unable to draw a correct sketch of a simple linear function. Of the seven subjects who succeeded in this task, five were graduates of the highest level track.

However, the dichotomy of correct / incorrect was not the major theme of this study. Indeed, I believe that the term 'correct' is by itself problematic, especially in open settings where actions and responses are susceptible to different interpretations. The documentation of the interactions with the subjects around the issue of graphing functions revealed phenomena of recall that, in my opinion, are beyond the question of 'to remember or not to remember' what the function should look like. Such phenomena were demonstrated by presenting three cases, those of Tamar, Nir and Eli. All three have passed the matriculation exam in mathematics with high grades. It is reasonable to assume that in the past they knew how to create a graph of a linear function such as y=2x. Their present attempts to perform this task differed very much from each other. Yet, there was a salient resemblance in those efforts: All three recruited their common sense and tried to present actions that would adhere to some logical framework. They created schemas by which they could explain their actions. Using Bartlet's (1932) terms, they did not reproduce the learned material, but rather reconstructed it and in fact formed their own personal versions of representing a function in a graph. Acquainted with the basic notions of points and axes, each of them 'filled the gap' of the forgotten material with a new, but quite coherent, chain of ideas. As a concluding remark, I would like to offer an optimistic look upon these actions. In spite of the evident loss of basic mathematical principles, perhaps the flexibility and freedom to play with ideas in a rational way is the gain that some people receive from their mathematics lessons, rather than the information itself. However, the question remains if this optimistic view could still be held, if the whole range of high school graduates were to be investigated, not just the upper highly educated sector. This is yet to be explored.

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NOTES

1. The term episodic memory is used here according to Tulving's original definition, i.e., as referring to recalling personal experiences. In time, however, the use of this term expanded to include other phenomena of recall (see Brewer, 1986, p. 33).
2. These ages can be regarded as forming the 'settling down' phase in the adult life (see Levinson, 1978).
3. In Israel mathematics is a compulsory subject throughout high school, and can be studied at three levels, here referred to as high level, intermediate level and low level. A substantial difference exists between the low level and the other two, in terms of curriculum and number of hours alloted to mathematics at school. The difference between the intermediate level and the high level is smaller: In most cases, students at both of these levels study the same curriculum, with more complicated exercises given in the high level classes. The annual report of the Israeli Central Bureau of Statistics usually refers to the intermediate level and the high level together, in distinction from the low level.
4. The candidates' professions were categorized by The Standard Classification of Occupations, a scale of 10 categories published by the Israeli Central Bureau of Statistics. It is noteworthy that 83% of these 105 professions were classified within the top three categories.
5. This name is a pseudonym, as are all the other subjects' names.

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