ZDM Mathematics Education (2014) 46:109–122
DOI 10.1007/s11858-013-0551-1

ORIGINAL ARTICLE

# Exploring the potential role of visual reasoning tasks among inexperienced solvers

**Intisar Natsheh · Ronnie Karsenty**

Accepted: 4 October 2013 / Published online: 30 October 2013


**Abstract** The collective case study described herein explores solution approaches to a task requiring visual reasoning by students and teachers unfamiliar with such tasks. The context of this study is the teaching and learning of calculus in the Palestinian educational system. In the Palestinian mathematics curriculum the roles of visual displays rarely go beyond the illustrative and supplementary, while tasks which demand visual reasoning are absent. In the study, ten teachers and twelve secondary and first year university students were presented with a calculus problem, selected in an attempt to explore visual reasoning on the notions of function and its derivative and how it interrelates with conceptual reasoning. A construct named "visual inferential conceptual reasoning" was developed and implemented in order to analyze the responses. In addition, subjects' reflections on the task, as well as their attitudes about possible uses of visual reasoning tasks in general, were collected and analyzed. Most participants faced initial difficulties of different kinds while solving the problem; however, in their solution processes various approaches were developed. Reflecting on these processes, subjects tended to agree that such tasks can promote and enhance conceptual understanding, and thus their incorporation in the curriculum would be beneficial.

**Keywords** Visual reasoning · Teaching and learning calculus · Conceptual understanding

I. Natsheh
Abdallah Iben Al-Husein High School, Jerusalem, Israel

R. Karsenty (✉)
Weizmann Institute of Science, Rehovot, Israel
e-mail: ronnie.karsenty@weizmann.ac.il

## 1 Introduction

The study presented in this paper investigated how students and teachers, whose mathematical backgrounds do not include experiences with visual reasoning tasks, attempt to solve such a task in calculus. The motivation for this investigation emerged from previous findings (Natsheh 2012), indicating that within the Palestinian secondary school mathematics curriculum the role of visual displays is limited and rarely consists of more than illustrations or guiding tools for solutions. Visual reasoning is not used, nor necessitated, for purposes such as explaining, establishing and providing new information, or proving. This state of affairs created an opportunity to explore behaviors that might occur when adult solvers are challenged for the first time with tasks in which one is compelled to use images in order to engage meaning, infer, reason and finally produce a solution. Such an exploration has, in turn, enabled the development of a construct that links different types of visual reasoning with conceptual understanding of fundamental mathematical notions. We have termed this construct "visual inferential conceptual reasoning". The construct, presented in detail in Sect. 4 of this paper, essentially refers to the ability to infer visually, that is, to deduce implicit information from a given visual display, in order to produce correct justifications that reflect a conceptual understanding. Thus, the purpose of the investigation to be reported herein can be rearticulated as exploring the subjects' visual inferential conceptual reasoning on the notions of function and derivative.

In what follows, Sect. 2 provides the theoretical background for the study, concentrating on definitions of visualization and visual reasoning in the context of mathematical education, with some insights gained from neuroscience. This section also contains particular reference to the role that visual reasoning may play in the teaching and learning of calculus. In Sects. 3 through 5 we report on a collective case study, investigating the solutions of Palestinian students and teachers to a calculus task emphasizing visual reasoning. We analyze elements of visual inferential conceptual reasoning as manifested in the participants' performances. In addition, findings include affective aspects of using visual reasoning, by differentiating within the solvers' behaviors beliefs about potential roles of using visual reasoning tasks in teaching and learning mathematics. Section 6 presents the final discussion, including some implications for teaching.

## 2 Theoretical background

### 2.1 Visualization and visual reasoning in mathematics

The terms *visualization* and *visual reasoning* are often intermingled in the literature, thus sometimes both terms are used to describe the same cognitive process of mathematical thinking (e.g., Hershkowitz et al. 2001). In this paper, however, we refer to visualization as a broad term that incorporates a spectrum of cognitive processes, one of which is visual reasoning. We embrace Arcavi's definition of visualization:

> "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings." (Arcavi 2003, p. 217)

On the higher end of this wide spectrum of processes and thinking levels, we find visual reasoning, which can be defined as using pictures, images or diagrams effectively for solving tasks of higher-order thinking.

Visualization has always been an important part of mathematicians' ways of thinking (Cunningham 1994; Hadamard 1945; Presmeg 2006; Zarzycki 2004). Zarzycki (2004) points out that "We could not even imagine introducing many mathematical concepts without illustrating them by pictures, drawings, graphs, etc." (p. 108). However, within the realm of contemporary philosophy of mathematics, the place of visualization has been marginal throughout the first half of the twentieth century. Theories from the schools of Formalism, as represented by Hilbert, or Russell and Whitehead's Logicism, focused on the logical structure and consistency of mathematics. In the second half of the twentieth century, the focus of some mathematicians and mathematics educators has shifted to include considerations of what mathematicians are actually doing when 'producing' mathematics. It has been proposed that there is more to understanding mathematics than merely following its logical structure. Questions concerning concept formation, building of understandings, using heuristics, acknowledging the diversity of reasoning styles and the role of analogies became the subject of intense interest. In particular, the role that visual images and diagrams play in all mathematical activities was put forward, and is still being scrutinized (e.g., Mancosu et al. 2005).

The emergence of computerized technologies with their powerful graphical possibilities has broadened the horizons of mathematical research. These new powers have strengthened the perspective that the role of visualization goes beyond the traditional view which relegates it to a supportive role. Giaquinto (2007) argues that visual thinking in mathematics is rarely just a superfluous aid (e.g., for illustrating cases of a definition, or suggesting a proposition for investigation or an idea for a proof), and can have an epistemological value, often as a means of discovery, understanding, and even as a proof in itself. Furinghetti and her colleagues claim that "visualization allows one to control at the same time a larger number of hypotheses, it gives a sort of 'global' idea of the requirements, while in the symbolic approach you may control one requirement at a time" (Furinghetti et al. 2011, p. 226). Other scholars have described the role of visual displays as crucial to the work of both experts and students; such displays can condense information, suggest new results or propose potential approaches to proofs (Alcock and Simpson 2004; Harel and Sowder 1998; Presmeg 1986).

### 2.2 Visual reasoning in mathematics education

In the last two decades, mathematics education scholars have become increasingly interested in visualization and visual reasoning. It is claimed that visualization extends beyond the graphical, iconic and diagrammatic illustration of an idea to include many other roles associated with higher order visual thinking, thus it may serve as an alternative and powerful resource for mathematics teaching and learning (Giaquinto 2007; Hitt 2002; Mancosu et al. 2005; Presmeg 2006; Rivera 2013; van Garderen et al. 2014). Accumulating evidence links visual reasoning with deep understanding of concepts in various mathematical areas such as word problems (e.g., Abdullah et al. 2012), real analysis (Alcock and Simpson 2004; Furinghetti et al. 2011), linear algebra (e.g., Souto-Rubio 2012; Stylianou and Dubinsky 1999) and evidently calculus, as shall be detailed in Sect. 2.4. Scholars have emphasized the importance of integrating visual insights and tools into the curriculum (Arcavi 2003; Rivera 2011, 2013; Zazkis et al. 1996). This approach was manifested in the appearance of classroom materials such as *Math Made Visual: Creating Images for Understanding Mathematics* (Alsina and Nelsen 2006) and in the utilizing of a wealth of software to support the creation and manipulation of visual images (for instance Maple, Matlab, Cabri, Geometer Sketchpad, Geogebra and Derive).

Alongside the general agreement about the contribution of incorporating visual reasoning into mathematics instruction, limitations and difficulties in using such reasoning were raised and discussed by researchers (Alcock and Simpson 2004; Arcavi 2003; Eisenberg and Dreyfus 1991; Guzman 2002; Stylianou and Pitta-Pantazi 2002). Studies have shown that analytical approaches seem to dominate the way students learn mathematics (Eisenberg and Dreyfus 1994; Tall 1991; Vinner 1989), a phenomenon that may be attributed either to pervasive instructional models emphasizing analytic over visual reasoning, or to beliefs about the symbolic form as the most legitimate, accepted and valued mode of mathematical communication. Arcavi (2003), based on the work of Eisenberg and Dreyfus (1991), has classified difficulties in using visual reasoning into three categories: cultural, cognitive and sociological. In this paper we demonstrate cases in which participants' responses seem to reflect cognitive and cultural difficulties.

### 2.3 Visual reasoning from a neuroscientific perspective

The study of visual reasoning, similarly to studies of other cognitive-educational topics, can benefit from exploring the perspective of neuroscience. Following Van Nes (2011), it appears that neuroscientific research can help mathematics education researchers gain insight into brain–cognition relations, underlying cognitive processes that occur while people are engaged in mathematical tasks.

From the work of Dehaene and colleagues (Dehaene et al. 1999; 2003), we learn that activation of the two hemispheres of the brain and the interaction between them are essential for complex mathematical reasoning. More specifically, regarding the importance of visual reasoning, Terao et al. (2004) have found supporting evidence for the conjecture that mathematical thinking emerges from the interplay between symbolic and visuo-spatial brain systems. Lakoff and Núñez (2000) claim that "it makes neurological sense that structures in the visual system can be used for conceptual purposes" (p. 34). More recently, Thomas and his colleagues (Thomas et al. 2010b) have examined brain activity occurring whilst students translated between graphical and algebraic representations of functions, and suggested that instruction combining number sense and spatial cognition is critical for the acquisition of advanced mathematical concepts. It follows that the construction of mathematical knowledge may benefit from integrating visual reasoning into the logical-propositional and formal symbolic ways of thinking, as exemplified for instance in the study of Konyalioglu et al. (2003).

However, the endeavor to integrate visual reasoning and logical-symbolic thinking is far from being trivial. For some people, certain brain systems are dominant in determining a learning style in a manner that outweighs other routes of reasoning, thus hindering this integration. Sword (2000) describes the problems experienced by highly capable visual thinkers who may be placed at risk in the school system because their learning style is not recognized. She argues that traditional teaching techniques are designed for auditory-sequential learners, and hence disadvantage visual-spatial learners. Material introduced in a step-by-step manner, carefully graded from easy to difficult, with repetition to consolidate ideas, is not only unnecessary for the visual-spatial learner, but, by failing to create links to a holistic picture, may alienate these students. As a result, gifted visual-spatial learners may often exhibit lack of motivation, inattentiveness, weaknesses in basic calculations, and disorganization.

On the other hand, however, it has been found that instruction can develop and enhance spatial and visual ability (Hoffman 1998; Whiteley 2000). Students can be assisted to use visualization as an alternative or complementary resource to the formal symbolic ones, thus taking a step forward towards developing their visual reasoning.

### 2.4 Calculus and visual reasoning

The origins of calculus can be traced back to exploring geometrical and physical problems, thus it is certainly an area in which visual reasoning can play a major role. The first phase of modern calculus, in the seventeenth century, was characterized by a strong visual component, remaining so in the first centuries of its development, through continual interaction with geometry, physics and astronomy (Giaquinto 2007). However, during the second half of the nineteenth century, calculus underwent a process of arithmetization and formalization. This was in part due to the distrust that mathematicians began to develop towards intuitive, visual and informal arguments which may lead to contradictions, and resulted in a certain degree of delegitimization of visual thinking which permeated also into education. As intellectually satisfying as this process was, it meant that the study of calculus moved to methods that were rather complex (Gardiner 1982; Mancosu et al. 2005), particularly for introducing the ideas of this field to newcomers. The concept of limit, for example, provided a mathematical foundation for developing calculus and mathematical analysis, but its formal aspect became hard to grasp by high school and college students (Davis and

Vinner 1986; Sierpinska 1987; Tall and Vinner 1981; Williams 1991).

During the last two decades of the twentieth century, many scholars called for a reform in calculus instruction (e.g., Tall 1991; Vinner 1989; Zimmermann 1991), emphasizing the need to increase the role that visual reasoning should play in this instruction. Mathematics educators asserted that embedding visual thinking in calculus courses would contribute to the development of sound conceptual understanding:

> "The role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding, which is widely recognized to be lacking in many calculus courses as now taught. Symbol manipulation has been overemphasized and in the process the spirit of calculus has been lost." (Zimmermann 1991, p. 136)

More recently, Hoffkamp (2011) has used interactive activities for the teaching of calculus, based on dynamic geometry software. Her findings further support the link between visual thinking and the process of conceptualization.

Alongside this direction, however, research on the learning and understanding of calculus has shown that students' visualization skills are weak (Tall 1991), and that failure to employ both analytic and visual aspects of fundamental concepts in calculus can be an impediment to students' conceptual understanding (Haciomeroglu and Aspinwall 2007).

Thus, deficiencies in conceptual understanding of calculus were linked on the one hand to difficulties with the formalities of mathematical analysis, while, on the other hand, they alerted that visual reasoning should not be taken for granted. Visual thinking is an intellectual process which needs preparation; it takes time and intentional effort. When such preparation is absent, students may find visual thinking to be foreign and difficult. In addition, in many cases students may regard visual solutions as low-status, less valued mathematical means; hence they may not invoke it (Guzman 2002). It follows that a purposeful course of action should be taken to integrate visual ways of reasoning into the teaching and learning of calculus, in order to enhance conceptual understanding of this subject.

One possible way to incorporate visual reasoning in the teaching and learning of mathematics in general, and in calculus particularly, is to resort to tasks in which one is compelled to use images in order to engage meanings, to justify and to produce a solution. For example, the various relations between a function and its derivative may be studied by exploring and comparing their graphs, as was implemented in the study to be described below.

## 3 The study

### 3.1 The context: visualization in the Palestinian calculus curriculum

In the Palestinian mathematics curriculum, calculus is the main topic of study in 12th grade, for both the scientific and the humanities tracks. Students in the scientific track learn this topic at a higher level while the humanities track offers a more general overview of basic concepts. At the university level, an introductory course in calculus is part of the curriculum for 1st year mathematics, physics, computer science and engineering students.

The Palestinian curriculum is, on the whole, a traditional one. Natsheh (2012) has developed a framework for analyzing the different roles and functions that visualization plays in teaching and learning mathematics within the Palestinian educational system. By applying this framework, she analyzed the commonly used mathematics textbook for 12th grade (Youssef et al. 2006), and a textbook used in an introductory calculus course for 1st year university students (Thomas et al. 2010a). The analysis showed that, in general, the role of visualization in the Palestinian calculus curriculum is limited to illustrative purposes; visual reasoning is not required for objectives such as explaining, proving or discovering new information.

Given the limited role that visualization plays in the Palestinian curriculum, and considering the potential contribution of visual reasoning to the conceptual understanding of calculus, it seemed that a constructive path of investigation would be to explore Palestinian students' and teachers' cognitive and affective reactions to a calculus task demanding a high level of visual reasoning.

The topic chosen for this purpose was the relationships between a function and its derivative; this topic calls for visual reasoning about the basic concepts involved, yet, as said, such reasoning is usually absent from the textbooks in use. In addition, visual tasks are not included in the final exit exams (the Tawjihi), a fact that is likely to be connected with what the textbooks emphasize or leave out.

In the study, 22 students and teachers were presented with a task which can be solved only through visual-conceptual considerations (details follow).

### 3.2 The study's goal and research questions

The aim of the study was to document and analyze the subjects' solution approaches and reactions, in order to gain some insight about the role that visual reasoning might play in the teaching and learning of calculus within the Palestinian context. Specifically, we attempted to explore the following questions:

1. What are the processes that subjects undergo when confronted with a visual reasoning calculus task of a novel nature? What characterizes their solution approaches?
2. How do students and teachers view the role of visual reasoning tasks in the teaching and learning of calculus?

## 4 Methodology

The study is defined as a collective case study (Stake 1995), a category which is also found in the literature under the terms 'multiple case study' (Yin 2009) or 'multicase research' (Stake 2006). This type of qualitative research refers to studies in which a number of cases are examined in order to highlight a particular issue. In contrast to intrinsic case studies, where the focus of interest is on the specific case uniquely, a collective case study is usually defined as instrumental, that is, the analyses of cases are meant to serve as a vehicle for enhancing a more general understanding in regard to some phenomena or theory. Following the work of Guba and Lincoln (e.g., Guba and Lincoln 1994; Lincoln and Guba 1985) on qualitative research and the criteria by which it may be evaluated, the emphasis is not on how generalizable the findings are, but rather on how well they serve to establish new insights, through the presented data and its suggested interpretations. In the words of Arminio and Hultgren (2002), the study should endeavor to be "the art of meaning making: guided by the methodology and obtained by means of the data collection method, it is through the interpretation process and its presentation that new insight is gained" (p. 450).

In light of the study's aim and the questions introduced above, task-based individual interviews were used to investigate the solutions, views and reflections of a small sample of teachers and students. The collection of these interviews and their interpretation form the core of this qualitative study.

### 4.1 Subjects

The sample consisted of 22 subjects, of which 12 were students and 10 were teachers. High school students and teachers were from East Jerusalem, university students and teachers were from the West Bank. The selection procedure is described below.

*Students*: In East Jerusalem there are five governmental high schools; three for boys and two for girls. Subjects were selected from one girls' school and one boys' school chosen randomly. Administrators from each of these schools selected students learning in the scientific track who agreed to participate in the study. University students were selected from one of the six universities in the Palestinian Territories, within the group of students enrolled in an introductory course in calculus (unfortunately, since the data collection period overlapped with the final exams period, only three students agreed to take part in the study).

*Teachers:* University teachers were included in the study if they were teaching—or have taught in the past—a 1st year university mathematics course. High school teachers were included if they were teaching mathematics in the 12th grade scientific track. All teachers were selected on the basis of their willingness to participate in the study; this situation has resulted in a relatively small number of participants.

The final sample included:

1. Nine 12th grade scientific track students (STS), 4 boys and 5 girls. All these students study 5 weekly hours of mathematics. The first semester is dedicated to derivatives, followed by integrals in the second semester.
2. Three 1st year university students (US), 1 boy and 2 girls. These students study 5 weekly hours of calculus I (3 h of lectures, 2 h of exercise sessions).

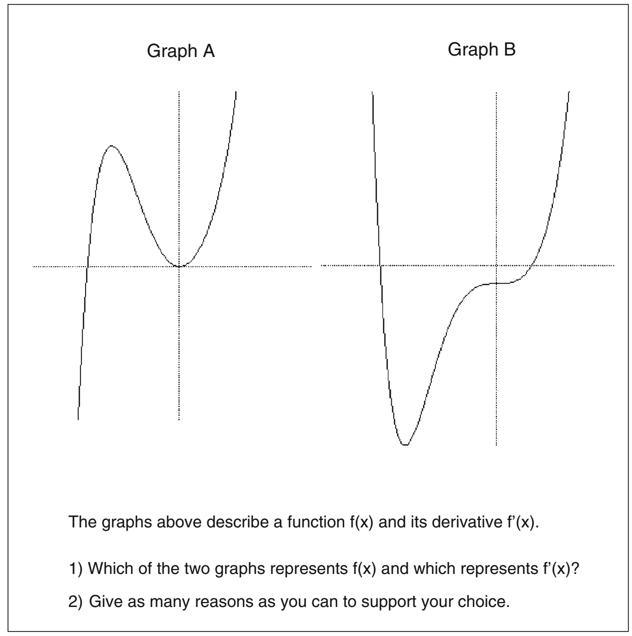


**Fig. 1** The task

3. Four 12th grade scientific track teachers (STT), one of which was the teacher of the boys included in the study; the others had no students among the subjects.
4. Six 1st year university teachers (UT). Their experience varied between 2 and 30 years (the average was about 15 years).

## 4.2 Interview procedure

Each of the 22 subjects was individually interviewed by the first author of this paper. Interviews were held in Arabic, and responses were recorded and transcribed.

The interviews consisted of a problem-solving part, completed by all participants, and a reflective part, completed by all students and three high school teachers (see below). The task that subjects were requested to solve is presented in Fig. 1. As can be seen, the graphs do not include any verbal, numerical or symbolical information, thus the only source of information is visual. Subjects were asked to verbalize their thinking processes while attempting to solve the task. The interviewer used prompts such as "why do you think so"; "can you give other reasons"; "can you show that on the graphs".

The reflective part of the interview was different for students and for teachers. Students were asked a general question, requesting them to express their opinions, as learners of calculus, regarding the use of such tasks both in the course of their learning and in exams.

As for teachers, a more elaborated reflective session was conducted individually with three high school teachers selected out of the ten participating teachers. This session can be described as a guided reflection, in the form of a semi-structured interview. The interviews took place in each teacher's work location, and lasted between 30 to 40 min. These interviews were held after all of the students' interviews were completed. The following guiding questions were posed to these three teachers:

1. Do you think that using such tasks is fruitful in explaining the relations between a function and its derivative?
2. Do you think that students are capable of developing, or consolidating, their understanding while solving this kind of problem?

After discussing these two questions, each teacher was presented with the solutions collected from students' interviews. Then, the following two questions were posed:

3. In your opinion, should such tasks be included in the mathematics curriculum? Why?
4. In your opinion, would the inclusion of such tasks in the Tawjihi (the Palestinian matriculation examinations) be a productive step? Why?

## 4.3 Data analysis

### 4.3.1 *Procedure*

The data generated through the problem-solving sessions, the students' reflective responses and the semi-structured reflective interviews with teachers were analyzed using the grounded theory approach (Glaser and Strauss 1967). The procedure of data analysis involved open, axial, and selective coding processes for qualitative data, following Strauss and Corbin (1990), to produce descriptive categories. As the process evolved, continuous comparisons were made between each category and the emerging new categories (Lincoln and Guba 1985).

Analysis of the task performance data involved multiple readings of both the transcripts and the corresponding written products, focusing on identification of solution procedures and classification of problem-solving behaviors, but also on affective aspects such as doubts about the task and curiosity. Analysis of the reflections obtained from students and teachers focused on the subjects' experiences with visual tasks and their perspectives on the role of visual reasoning in the mathematics curriculum.

Trustworthiness was enhanced through (a) ensuring that the collected data are accurate and complete, by way of administering the task in a written form and producing a verbatim transcription of each interview shortly after its recording; and (b) validating the process of coding and recoding of the different categories via discussions with several mathematics education specialists.

### 4.3.2 *Analysis framework: the construct of visual inferential conceptual reasoning*

The process of task performance analysis, described above, allowed us to develop a construct which we found helpful in characterizing the subjects' responses. This construct, termed "visual inferential conceptual reasoning", may be regarded as a specific type of visual reasoning, relating to one's ability to use visual considerations in order to elicit inferences that enhance conceptualization. It can also be seen as part of a more general framework, created by Natsheh (2012) for analyzing the different roles and functions that visualization may play in teaching and learning mathematics. Below is a brief outline of this general framework, and how the construct of visual inferential conceptual reasoning was derived from it.

The framework consists of three components: the visual display, the visual actions and the visual purposes:

- *Visual displays:* Objects (in either two, three or more dimensions) upon which one can exert certain visual actions. Images, charts, diagrams and graphs are examples of visual displays. (On certain occasions even a symbolic formula can be considered a visual display if a visual action is performed upon it.)

- *Visual actions:* The different processes and activities that a person can perform upon a visual display. For instance, a person can simply look at a visual display, and can also measure, read out information, make comparisons, add an auxiliary construction, decompose a visual display into parts or create a new visual display.
- *Visual purposes:* The goals of the visual actions performed on the visual displays, such as rephrasing information, describing, explaining and proving.

When the purpose of the visual action is to generate inferences from the visual display, in a way that reflects understanding of the underlying concepts, we see this as performing visual inferential conceptual reasoning. This construct is therefore useful in analyzing responses to tasks requiring a high level of visual thinking, such as the one used in this study. The visual displays in this case are the graphs presented in Fig. 1. The main visual actions called for by this display are describing and comparing the two graphs, in light of previous knowledge about the properties of a derivative. The purpose here is to establish as many connections between the graphs as possible, in order to infer about the requested correspondence, that is, which one is *f* and which is *f'*.

## 5 Findings

The analysis process yielded a categorization of the data around two major themes: (a) characterizing the subjects' visual inferential conceptual reasoning as reflected in their task performance; and (b) affective aspects regarding the use of visual reasoning. Each of these two themes was sub-categorized.

Theme (a), subjects' visual inferential conceptual reasoning, included the following:

(a) 1. Realization of the visual purpose (i.e., whether or not graphs B and A were correctly assigned to *f* and *f'*, respectively).

(a) 2. Type of visual actions.

Theme (b), affective aspects regarding the use of visual reasoning, included the following:

(b) 1. Doubts and curiosity about the given task.

(b) 2. Reflections on the use of visual reasoning tasks in teaching and learning mathematics.

In what follows, we present the findings according to each of these sub-categories, and illustrate them with examples where relevant.

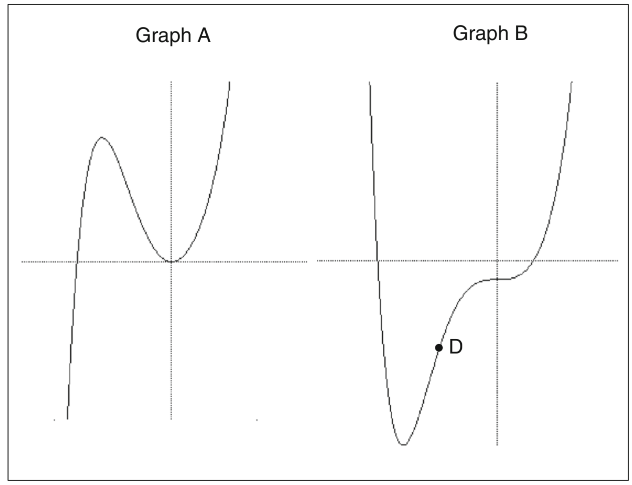


**Fig. 2** Point D added to the original graph

### 5.1 Theme (a): Subjects' visual inferential conceptual reasoning

#### *5.1.1 (a) 1. Realization of the visual purpose*

Twenty subjects correctly chose graph B to represent the function: all of the teachers and 10 out of the 12 students. One school student and one university student incorrectly chose graph A. Thus, in terms of the final visual purpose, which was to infer from the visual display and decide which graph represents *f* and which represents *f'*, the majority of responses were successful. However, in terms of the visual actions performed we found that the responses varied considerably in the number and nature of considerations employed, as we describe below.

#### *5.1.2 (a) 2. Type of visual actions*

In the second part of the task, subjects were explicitly prompted to elicit as many reasons as they could to support their answer to the first part. It was hoped that responding to this prompt would allow subjects to reflect their understanding of the derivative concept and its properties, as it urged them to expand their considerations beyond the first one that came to mind. The expected visual actions were to describe and compare the two graphs, utilizing previously acquired knowledge about functions and derivatives. Yet, we found that these actions were preformed differently among subjects, consequently generating a variety of justifications, some of which were mathematically valid while others reflected a lack of conceptual understanding.

The given justifications were classified into seven types. In the following we list these types and exemplify some of them with citations (translated from Arabic).

- Type 1. Matching intervals in which $f$ increases or decreases with the corresponding signs of $f'$.
- Type 2. Matching horizontal tangents of $f$ with zeros of $f'$.
- Type 3. Matching intervals in which $f'$ increases or decreases with the concavity of $f$.

Example (given by a university student, referring to the marked graph shown in Fig. 2):

*"Graph B is concaved upwards in $x > 0$ and graph A is increasing in $x > 0$. Also B is concaved downwards in [D, 0] and graph A is decreasing there."*

- Type 4. Matching inflection points of $f$ with horizontal tangents of $f'$.

Examples (the first given by a university teacher, the second by a high school teacher):

*"At $x = 0$ f has an inflection point $\rightarrow f''(0) = 0 \rightarrow f'$ at $x = 0$ has a horizontal tangent."*

*"At $x = 0$ we have an inflection point that has a horizontal tangent, and at $x = 0$ f' intersects the x-axis and has also a horizontal tangent."*

- Type 5. Matching the number of critical points and the degree of $f$ and $f'$.

Example (given by a university teacher; note that the justification is partly erroneous):

*"You can see that graph B has 3 critical points so its degree is 4, while graph A has 2 critical points so its degree is 3."*

- Type 6. Matching the number of "curves" with the degree of $f$ and $f'$.

Example (given by a high school teacher):

*"Graph B is the function graph since the number of its movements is more than the other, I can see that it has 3 curves so its degree is 4, while the other has 2 curves so its degree is 3."*

- Type 7. Inspecting the general shape of the graph and conjecturing about possible features of the symbolic representation of the function.

Examples (the first given by a university teacher, the second by a high school student):

*"Graph B is from degree 4 since its shape looks like w, and graph A is from degree 3 since the number of its curves is 2."*

*"Yes, I think graph B is f(x) because it looks like $f(x) = Ax^3 + Bx^2 + Cx + D$, and graph A looks like a square function."*

Table 1 summarizes the types and number of justifications given by the subjects. Note that the two subjects who erred in the first question are not included here. Two other subjects (STS2 and STS7) correctly matched the graphs with $f$ and $f'$, yet were unable to give a clear reason for their choice, thus their number of justifications is noted as zero.

As shown in Table 1, the two justification types most frequently used among all subjects, either students or teachers, were Type 1 (increasing and decreasing intervals of $f$ and the corresponding sign of $f'$, stated by 13 subjects), and Type 2 (horizontal tangents of $f$ and zeros of $f'$, stated by 16 subjects). All other justification types were used by 3–5 subjects. The predominant use of the visual features of increase, decrease and zero slopes may be attributed to their centrality in the curriculum; procedures of investigating algebraic representations of functions repeatedly relate to these features. In fact, even the simplest of inquiries, such as investigating a parabola, begins with finding extreme points and intervals of increase and decrease. Thus, it can be said that the visual actions applied by most subjects were strongly linked to concepts established through repeated algebraic practice. The ability of subjects to extract meaning from concepts that are usually presented in symbolic contexts and use that meaning in a context which is exclusively visual suggests a high degree of solidification of these concepts. In contrast, the concepts of inflection points and concavity are considered to be more advanced and are introduced only later in the curriculum. It seems that when confronted with an unfamiliar task demanding only visual considerations, the subjects tended to resort, or at least give priority, to visual actions reflecting more deeply established concepts. In other words, the infrequent use of inflection points and concavity might suggest a relatively lower degree of their conceptualization.

Taking a more general look at the findings described in Table 1, the emerging picture is that two major patterns of visual inferential conceptual reasoning may be characterized. The distinction between the two patterns is based on the difference in nature between visual actions reflected in justifications of types 1–4 and those reflected in types 5–7. Whereas the former types use only visual elements, the latter ones combine visual and symbolic aspects of function. Therefore, we can characterize one pattern of visual inferential conceptual reasoning as being "purely visual", while the other can be described as "partially visual", that is, a pattern in which the visual purpose was to construe, through the appearance of the graphs, possible features of the symbolic representation of the function. The "purely visual" pattern was demonstrated by 11 subjects. The "partially visual" pattern was demonstrated by 8 subjects (including STS7 who could not clarify his justification).

**Table 1** Types and number of justifications used by subjects who correctly solved the task ($n = 20$)

| Subjects | Types of justifications used | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1. Increasing and decreasing intervals of *f* and sign of *f*′ | 2. Horizontal tangents of *f* and zeros of *f*′ | 3. Increasing and decreasing intervals of *f* and concavity of *f*′ | 4. Inflection points of *f* and horizontal tangents of *f*′ | 5. No. of critical points and degree of *f* and *f*′ | 6. No. of curves and degree of *f* and *f*′ | 7. Shape of graphs as a basis for symbolic features of *f* and *f*′ | Total number of justifications presented by the subject |
| STS1 | √ | √ | | | | | | 2 |
| STS2 | | | | | | | | 0 |
| STS3 | | √ | | | | | | 1 |
| STS4 | √ | | | | | | | 1 |
| STS5 | | √ | | | | | √ | 2 |
| STS6 | √ | √ | | | | | √ | 3 |
| STS7 | | | | | | | | 0 |
| STS8 | | √ | | | | | | 1 |
| US1 | √ | √ | √ | | | | | 3 |
| US2 | √ | √ | √ | √ | | | | 4 |
| STT1 | √ | √ | | | | | | 2 |
| STT2 | √ | √ | | | | | | 2 |
| STT3 | √ | √ | √ | | √ | √ | | 5 |
| STT4 | √ | √ | √ | | | √ | | 4 |
| UT1 | √ | √ | | | | √ | √ | 4 |
| UT2 | | | √ | | √ | | | 2 |
| UT3 | √ | √ | | √ | √ | | | 4 |
| UT4 | | √ | | | | | | 1 |
| UT5 | √ | √ | | √ | | | | 3 |
| UT6 | √ | √ | | | | | | 2 |
| Number of subjects using this type | 13 | 16 | 5 | 3 | 3 | 3 | 3 | 46 |

*STS* scientific track student, *STT* scientific track teacher, *US* university student, *UT* university teacher

Interestingly, we further noted that these eight subjects, although sharing the same visual purpose, pursued it through different visual actions, as we shall now describe.

Three high school students attempted to fit their gestalt view of the graphs to a collection of prototypically known algebraic representations of functions:

> *"Yes, I think graph B is f(x) because it looks like* $f(x) = Ax^3 + Bx^2 + Cx + D$*, and graph A looks like a square function."* (STS5)

> *"This part of graph B is similar to* $f(x) = x^3$*, and that part of graph A looks like* $f(x) = x^2$*, so B is f(x) and A is for f'(x)."* (STS6)

> *"From the behavior of the function, we can see that this function (graph A) is similar to* $f(x) = x^2$ *added to or subtracted from something."* (STS7)

It seems that these students need an algebraic rule to hang on to. By fitting a prototype of a graph to a prototype of a symbolic form (though erroneously done, in these cases), they recruit what for them is a familiar and trustworthy resource, to assist them in tackling a novel experience.

Five of the teachers shared the "partially visual" pattern, with the same purpose of using the graphs to draw symbolic conclusions about the function; however, the visual action they employed was to infer the degree of the two functions by examining elements of the graphs and recalling relevant information about polynomials. For instance:

> *"I think graph B is f and graph A is the derivative graph. Graph B has two roots so its degree has to be at least three, and graph A has two roots, but from my*

*intuition B is the function graph, and its degree is 4."* (UT2)

*"The number of curves in graph B is more, they are 3 so its degree is 4 while in graph A there are two curves so its degree is 2."* (STT4)

Additionally, we noted that within the patterns of visual inferential conceptual reasoning portrayed above, two of the subjects demonstrated a slightly different pattern, or rather, a nuance of visual reasoning; they chose to prove by contradiction, using the visual action of reading information back and forth from the two graphs:

*"Yes, but if you take the point (0, 0) in graph A, the tangent is horizontal, but in graph B the value of the derivative is not zero, it is negative. We reach a contradiction to what the two graphs tell us, so graph B is f."* (STS5)

This nuance in visual reasoning, as captured in this citation, may be referred to as "visual deductive reasoning".

Lastly, Table 1 also enables a comparison between students' and teachers' performances. The data show that high school students gave, on average, fewer justifications than teachers (averages of 1.25 and 2.9 justifications for high school students and teachers, respectively). The two university students gave 3–4 justifications, higher than the teachers' average; however, their number is too small to be considered.

The number of justifications was not the only difference found between students and teachers. What Table 1 does not reveal is that they differed also with respect to the level, or quality, of the connections made between the graphs: students tried to visualize the exact location of corresponding points in the two graphs, but were unable to visualize more than one to two points. In contrast, teachers generally managed to visualize the location of more than two pairs of corresponding points in the two graphs. This finding is compatible with the expectation that teachers' experiences enable them to "extract" more information from given graphs.

To summarize theme (a) of the analysis: Two main patterns of visual inferential conceptual reasoning have emerged from the subjects' responses. The "purely visual" pattern, found among 11 of the 20 subjects, is characterized by actions that reflect visual considerations exclusively. Within this pattern, actions more frequently used were ones grounded in basic concepts such as the function's intervals of increase and decrease, while actions applying more advanced concepts such as concavity were less prevalent, suggesting a difference in the degree of their conceptualization. The "partially visual" pattern, found among 8 of the 20 subjects, is characterized by attempts to attach a symbolic representation to the graphs in order to base the decision on algebraic grounds.

In general, we can say that although subjects had initial difficulties while tackling the problem, and in spite of the doubts they expressed at first (see theme (b) below), their overall performance was quite impressive in terms of responsiveness, correctness, and the number and variety of reasons provided. This can be taken as an indication that both students and teachers had the necessary background knowledge to successfully cope with this kind of task, but given their lack of experience with visual tasks they had to overcome the initial surprise and think differently than what they were used to, in order to apply their knowledge. In addition, as expected, teachers were more successful than high school students in managing this problem.

## 5.2 Theme (b): Affective aspects regarding the use of visual reasoning

### 5.2.1 *(b) 1. Doubts and curiosity about the specific given task*

In all interviews, expressions of initial doubts were documented when the task was presented. All subjects commented on the novelty of such a task for them. For example:

*"They didn't teach us to solve such problems. Are you sure that one represents the graph of a function and the other is the derivative?"* (STS3)

*"I see, this is the first time I face a problem like this, I don't know how to solve it."* (STS2)

It took some time for all subjects to familiarize themselves with the task and start searching for a direction towards a solution. It is noteworthy that during the process of solving, some of the teachers were curious to know if they had exhausted all possible justifications, and inquired which other considerations could be elicited. Some of the students were curious about the correctness of their choice.

### 5.2.2 *(b) 2. Reflections on the use of visual reasoning tasks in teaching and learning mathematics*

As described above, students' and teachers' lack of experience with calculus tasks of this kind was evident throughout the problem-solving sessions. Not only was it manifested in their explicit expressions of unfamiliarity, but it was also apparent from subjects' initial difficulties and the amount of time it took them to develop their solutions. In spite of this shared situation, students' and teachers' reflections about possible uses of visual reasoning tasks were interestingly different. Once having completed the task, several students stressed the importance of exposing them to such tasks. For example:

*"I think if they teach us like this, it will be much better, I think it will not be an abstract way but more applied. I think I will be able to compare and see the different elements that ease the process of distinguishing between the two graphs."* (STS3)

*"Teaching us that way will make us understand more."* (SS5)

In contrast, when the three interviewed teachers were asked to assess students' capabilities to develop, or consolidate, their understanding while solving this kind of problem, two of them argued that students are not capable of solving such a task:

*"No, it's very difficult for our students."* (STT2)

*"I think it's difficult, since it needs a higher degree of understanding of the different relations between the graph of the function and its derivative."* (STT1)

The third teacher expressed a view that such tasks are appropriate for advanced students:

*"I think that good students are capable of dealing with such a question, I don't mean good in their marks in math, but those who have good mathematical thinking abilities."* (STT4)

However, once the researcher introduced some solutions produced by students, a certain change was noticed in the teachers' views. For instance:

*"That's good, maybe not just good students will be able to solve it, but it is hard for them to infer all the possible reasons to support their choice."* (STT1)

Inspecting the students' solutions, the teachers pointed out that the task may strengthen students' understanding of the concept of a derivative, and may give them different ways to comprehend the diverse relations between functions and their derivatives.

They also expressed their feeling that the Palestinian mathematics curriculum can benefit from the integration of visual tasks, yet that exposure to these tasks must be gradual, from simple to more complicated ones. Students should be trained to tackle such problems:

*"I still say it is hard to use, but if we start training students on similar tasks gradually from simple to more complicated ones, I think it will be fruitful."* (STT2)

*"If I'm going to use such a question I will use it in later stages, maybe after giving the students some tests, I see that this kind of question comes to deepen their understanding of the different concepts and the different relations between a function and its derivative. Or I may use some easier questions in which the behaviors of the two graphs are less complicated."* (STT4)

All three teachers said that this kind of question could be included in the Tawjihi's exams, because it provides an opportunity for excellent students to demonstrate their thinking.

## 6 Discussion and concluding comments

Developing visual reasoning is perceived as an important goal by mathematics educators (e.g., Arcavi 2003; Presmeg 2006; Rivera 2011); scholars advocate that helping students coordinate visual and analytic thinking can contribute to the consolidation of mathematical concepts (Zazkis et al. 1996; Mudaly 2010). This recommendation is supported also by findings from neuroscience (e.g., Thomas et al. 2010b), as we have discussed earlier in this paper. However, the acquisition of "visual language" and learning how to move back and forth between visual and symbolic mathematical modes demand training, and, as with other languages, may benefit from an early start. But what happens if an early start is barred, for instance by more traditional, pre-reformed curricula that have not yet integrated visual reasoning as a part of studies? Such is the case of the Palestinian curriculum, with which this study was concerned.

The subjects of the study were not familiar with visual reasoning tasks in mathematics in general, and in calculus in particular. As students and teachers of mathematics in either advanced high school tracks or university courses, their experiences were shaped by a curriculum in which visualization is seldom used as a tool for developing reasoning skills such as explanations and proof. Specifically, calculus tasks demand formal-symbolic and often procedural thinking and do not include tasks in which visual reasoning plays a central role. Thus, students may exercise drawing a schematic graph to a given symbolic representation of a function, but they do not encounter tasks in which the visual display is the major source of information from which inferences can be elicited. It can be said that the subjects in this collective case study lacked *visual literacy*, a term which describes learners' proficiency in "visualization combined with logical thought" (Mudaly 2010, p. 67). Therefore, the visual task presented to these subjects was nearly a "terra incognita" for them. This can explain the initial reactions to the task: doubt, surprise and difficulties in understanding what was required. However, the analysis of subjects' reactions has indicated that even without prior experience, most subjects were eventually able to establish connections between the visual data and the calculus concepts they have previously acquired through a formal-symbolic practice. The analysis was performed using the construct developed in the study, namely *visual inferential conceptual reasoning*, which relates to problem-solving situations in which visual actions are implemented upon a visual display, with the explicit purpose of generating inferences from it in a way that reflects understanding of the underlying concepts. Using this construct, we found that more than half of the subjects have followed what we called a "purely visual" pattern, that is, their visual inferential conceptual reasoning was based solely on visual considerations. It seemed that while tackling the problem, these subjects have managed to recruit a kind of "ad hoc" visual reasoning for the specific task in front of them. Thus, it is suggested that they have increased their visual literacy through this experience, which is a propitious finding in light of the fact that all subjects were adults or on the verge of adulthood.

Other responses were characterized by the "partially visual" pattern of reasoning, that is, an attempt to attach a symbolic interpretation to the graphs in order to bridge over the supposedly lack of information. Either way, the vast majority of the subjects have reached the visual purpose of the given task, and, moreover, found the task to be appealing and shared a view that such tasks have the potential to promote and enhance conceptualization. This is apparent from the reflections cited in the previous section, in which the visual actions performed are associated with a deep understanding of the concepts involved. We therefore support the assumption, implied by many scholars (e.g., Presmeg 2006; Zazkis et al. 1996) that solvers who have gained solid mathematical knowledge through symbolic-algebraic practice may still benefit from visual reasoning problems with which they are inexperienced. The introduced construct of visual inferential conceptual reasoning enables an analysis that discerns different levels of visual reasoning based on the various visual actions used by solvers.

In addition, we claim that the time may be ripe for a change in the Palestinian curriculum, as well as in similarly typified curricula, towards more use of "the visual culture" in mathematics education. This can be considered an optimistic reading of the findings; however, the issues at stake may be more complicated. Following the discussion in the study of Souto-Rubio (2012), who explored visualization in the context of a linear algebra university course, some further points are worth paying attention to. Souto-Rubio advocates the use of visualization in mathematics at the university level, claiming that visualization is essential to achieve deep understanding that enables advanced mathematical thinking. She argues that the following steps should be taken in order for visualization to become the valuable tool for understanding that it could be: (a) students must be guided to explicitly practice the languages and characteristics of visualization; (b) teacher knowledge about visualization should be improved; and (c) visualization must undergo a process of legitimization, a step which involves the institutional dimension, in introducing changes in both the curriculum and the assessment tools. We believe that similar steps apply when considering the higher end of the visualization spectrum, that is, visual reasoning. The findings of this study suggest that currently there may be several cognitive and socio-cultural obstacles for using visual reasoning in teaching and learning mathematics in Palestinian schools and universities. First, teachers are unaware of the utility of visual reasoning and its different roles in mathematics education. Second, teachers may be inclined to prejudge their students as incapable of solving high level visual tasks. Last and most important, teachers are not trained to use visual reasoning tasks in their classes.

Therefore, within this context, any attempt to shift the teaching and learning of mathematics towards including more use of visual reasoning must take into account the systemic aspects of such a shift. In other words, teachers should be provided with opportunities to get deeply acquainted with visual reasoning tasks, appreciate their educational potential and develop tools to use them timely and appropriately in their classrooms. This would imply adding a new focus to teacher education programs. Moreover, both the curriculum and the high-stake assessments procedures need to support this move by including visual tasks in all mathematical topics, promoting higher order visual actions with advanced visual purposes. Thus, the "visual road" ahead may be long, and further research is needed to support and guide future development. Nevertheless, we propose that the results of this study are encouraging. Students' recognition of a novel task which they initially were struggling to solve, as an activity that can enhance their understanding and enable them to relate to the different concepts in a meaningful way, is noteworthy. The finding that school teachers, who had never used visual reasoning tasks before, were positively affected by their students' success further reinforces this observation. This case study therefore adds another brick to the accumulating evidence on the contribution of visual reasoning to processes of conceptualization and meaning construction, by providing an "existence proof" relating specifically to adults who are newcomers to the idea of visualization in mathematics.

**Acknowledgments** We deeply thank Abraham Arcavi for sharing with us his wisdom and for his helpful and instructive comments on earlier versions of this manuscript.

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