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Advanced mathematics from an elementary standpoint: functions, operations and modelling

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Example I

A very small child is asked to find the total number of building bricks in two small piles on the floor.

~~He~~ She pushes the piles together and counts. Picture

An older child asked to do the same thing counts each pile separately and adds. Picture

The younger child uses the scheme

$$A\ ,\ B \longrightarrow A \cup B \longrightarrow n(A \cup B)\ ,$$

whereas the older child uses the scheme

$$A\ ,\ B \longrightarrow n(A)\ ,\ n(B) \longrightarrow n(A) + n(B),$$

where A and B are the two piles and n(A) represents the number of bricks in A.

The two children get the same result (if they counted correctly) because

$$n(A \cup B) = n(A) + n(B)$$

We can describe the two methods by the following diagram.

older child

small child

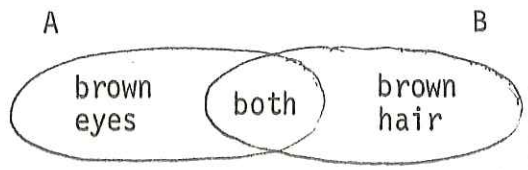
$$\begin{tikzcd}
A\ ,\ B \arrow[r, "n"] \arrow[d, "\cup"'] & n(A)\ ,\ n(B) \arrow[d, "+"] \\
A \cup B \arrow[r, "n"] & n(A \cup B) = n(A) + n(B)
\end{tikzcd}$$

Note that n is a function which maps some set of sets to the non-negative integers. Union is a binary operation on the domain of n and addition is a binary operation on the codomain.

The smaller child first performs the binary operation on the domain and then finds the image under the function, whereas the older child, first performs the function and then the binary operation on the images in the codomain.

Note also that $n(A \cup B) = n(A) + n(B)$ is not true in general.

Thus if the same two children were asked to count the number of children in the class who had brown eyes (A) and/or brown hair (B), then, assuming the children each followed his own procedure as before, only the younger child would get the right answer. The older child would count the children who had brown eyes <u>and</u> brown hair twice.



In terms of the diagram we would have

older child

$$\begin{tikzcd} & A , B \arrow[r, "n"] \arrow[d, "\cup"'] & n(A), n(B) \arrow[d, "(?)"] \\ \text{small child} & A \cup B \arrow[r, "n"'] & n(A \cup B) = \end{tikzcd}$$

Apparently, there is no binary operation $\square$, such that

$$n(A \cup B) = n(A) \square n(B).$$

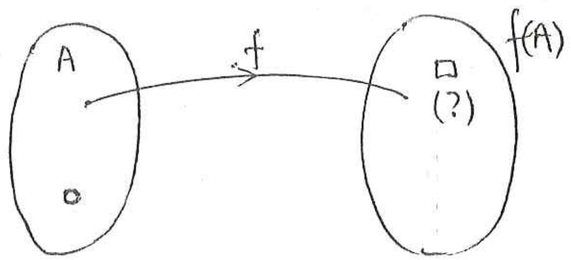
Note that although we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

the right-hand side does not define a binary operation $\square$ between $n(A)$ and $n(B)$.

Modelling I

The above is an example of a very simple but precise modelling situation, which occurs throughout mathematics, both pure and applied. We have a situation in which we are given a set A, a (closed) binary operation $\circ$ on A and a function with domain A. The problem then is to find a binary operation $\square$ on f(A) which models $\circ$ on A, where we mean by modelling that



performing $\square$ in f(A) corresponds to performing $\circ$ in A. Precisely, this means that if $a_1, a_2 \in A$

$$a_1 \circ a_2 \text{ in A corresponds to } f(a_1) \square f(a_2) \text{ in } f(A)$$

i.e. $$f(a_1 \circ a_2) = f(a_1) \square f(a_2).$$

or diagramatically

$$\begin{tikzcd} a_1 , a_2 \arrow[r, "f"] \arrow[d, "\circ"'] & f(a_1) , f(a_2) \arrow[d, "\square"] \\ a_1 \circ a_2 \arrow[r, "f"] & f(a_1 \circ a_2) = f(a_1) \square f(a_2) . \end{tikzcd}$$

We have deliberately stated the modelling situation as a problem:

given A, $\circ$ and f to find $\square$ (if it exists)
such that A, $\circ$ f, $\square$ is a modelling situation.

<u>Example II</u>

Consider the modulus function

$$f : x \longrightarrow |x| \qquad x \in R$$

with image set R_0^+, the non-negative real numbers.

There are all sorts of binary operations on R, let's look for the corresponding "modelling operations" on R_0^+ .

a) What is the model for (R, x)? We start going round our diagram and see how far we can get

$$\begin{tikzcd} x , y \arrow[r, "f"] \arrow[d, "\times"'] & {|x| , |y|} \arrow[d, phantom, "(?)"] \\ x \times y \arrow[r, "f"'] & {|x \times y|} \end{tikzcd}$$

Since $|x \times y| = |x| \times |y|$, we can complete the diagram

$$\begin{tikzcd} {|x| , |y|} \arrow[d, "\times"] \\ {|x \times y| = |x| \times |y|} \end{tikzcd}$$

and (R, x) is modelled by $(R_0^+ , \times)$

b) What is the model for (R, +)? As before

$$\begin{tikzcd} x , y \arrow[r, "f"] \arrow[d, "+"'] & {|x| , |y|} \arrow[d, "(?)"] \\ x + y \arrow[r, "f"'] & {|x + y|} \end{tikzcd}$$

In order to complete the model we need to find some binary combination of $|x|$ and $|y|$ which is equal to $|x + y|$.

(The common student error $|x + y| = |x| + |y|$ can be seen as a mistaken "solution" to our problem.). The problem is apparently solved by simply defining $|\square|$ by

$$|x| \; |\square| \; |y| = |x + y|$$

But, consider

$$\begin{tikzcd} -2,\ 3 \arrow[r, "f"] \arrow[d, "+"'] & 2,\ 3 \arrow[d, "\square"] \\ 1 \arrow[r, "f"'] & 1 = 2 \; |\square| \; 3 \end{tikzcd}$$

hence, apparently $2 \; |\square| \; 3 = 1$

But, consider further

$$\begin{tikzcd} 2,\ 3 \arrow[r, "f"] \arrow[d, "+"'] & 2,\ 3 \arrow[d] \\ 5 \arrow[r, "f"'] & 5 = 2 \; |\square| \; 3 \end{tikzcd}$$

hence, apparently $2 \; |\square| \; 3 = 5$

And we cannot have both. We are led to the conclusion that no $|\square|$ exists for the modulus function, which models addition.

Modelling II

Our general modelling problem was stated in the form

given A, o and f to find $|\square|$ (if it exists) such that A, o, f, $|\square|$ is a modelling situation

In view of the previous example we see that the problem of existence of $\square$ is a real one - and in the following we shall try to find under what conditions $\square$ exists. If we know it exists, then its definition follows from the modelling condition

$$f(a_1)\ \square\ f(a_2) = f(a_1 \circ a_2)$$

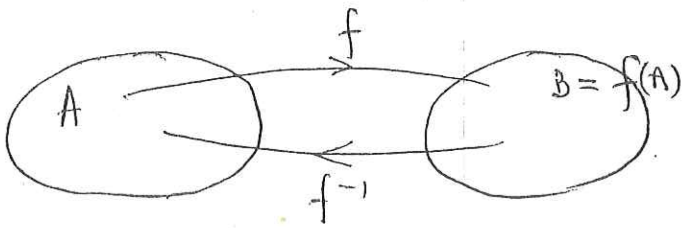
The only trouble with this definition is that when we want to perform $\square$, we are not given elements in the form $f(a_1)$ and $f(a_2)$, but in the form b_1, b_2, where $b_1, b_2 \in f(A)$. How do we define

$$b_1\ \square\ b_2 = \quad ?$$

If f is one-one, there is little problem. We simply find the a_1, and a_2 corresponding to b_1 and b_2; that is

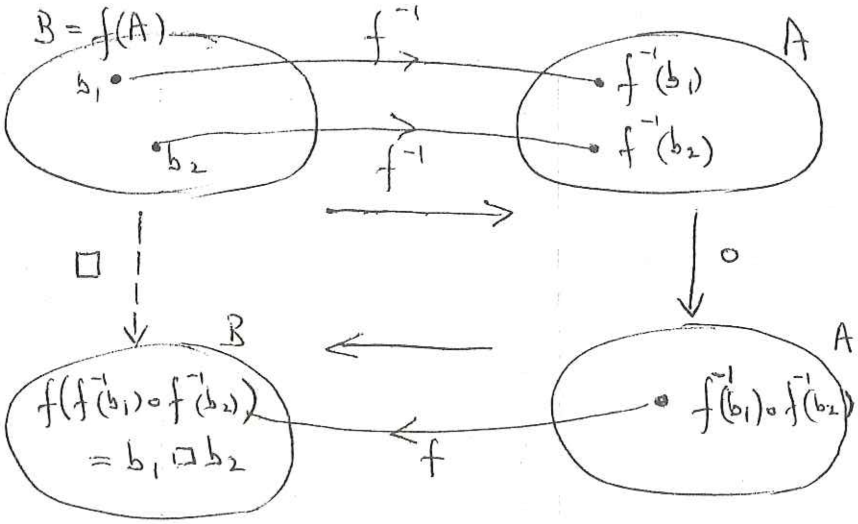
$$f^{-1}(b_1) = a_1 \text{ and } f^{-1}(b_2) = a_2 ,$$

where f^{-1} is the inverse function of f.



Then, in place of $f(a_1)\ \square\ f(a_2) = f(a_1 \circ a_2)$, we have

$$b_1\ \square\ b_2 = f(f^{-1}(b_1) \circ f^{-1}(b_2))$$



There is nothing missing here

Part of the problem is thus theoretically solved: whenever f is one-one, the modelling situation is satisfied, and we can define $|\square$ by using our modelling diagram in reverse order.

<u>Example III</u>

$$f : x \longrightarrow \frac{1}{x}$$

$$\text{i)}\ (R_{-o}, +) \longrightarrow (R_{-o}, ?)$$

$$\text{ii)}\ (R_{-o}, \times) \longrightarrow (R_{-o}, ?)$$

where R_{-o} is the set of real numbers without zero.

The function is one-one (and also $f^{-1} = f$ in this case), hence we know that the modelling operation exists and is defined by

$$\text{i)}\ x \,|\square\, y = f^{-1}\left(\frac{1}{x} + \frac{1}{y}\right) = f^{-1}\left(\frac{x+y}{xy}\right) = \frac{xy}{x+y}$$

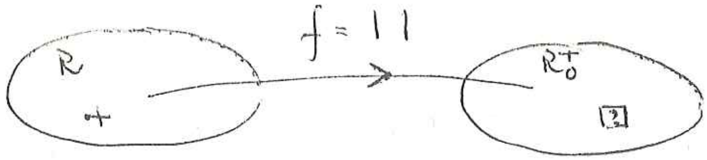
$$\text{ii)}\ x \,|\square\, y = \frac{1}{x} \times \frac{1}{y} = \frac{1}{xy} = xy$$

In the second case $|\square = \times$ and is well-known, whereas in the first case we have a new operation

$$x \,|\square\, y = \frac{xy}{x+y}\ .$$

Modelling III

We are left to discuss the case when f is many-one, and to determine a condition under which the modelling operation exists. Let's look again at Example II, where we failed to define a modelling operation corresponding to +.



Given two elements x, $y \in R_0^+$, we want to define $x \boxed{} y$ such that it models +. To do this we have to use f^{-1} to return to R_0^+, there perform addition , and then return by ~~memo~~ means of f.

$$f^{-1}(x) = \begin{array}{l} \nearrow x \\ \searrow -x \end{array} \qquad\qquad f^{-1}(y) = \begin{array}{l} \nearrow y \\ \searrow -y \end{array}$$

Hence we have four possibilities for $f^{-1}(x) + f^{-1}(y)$

$$x + y \,,\quad -x + y \,,\quad x - y \,,\quad -x - y$$

These four reduce to two when we return to R_0^+; i.e.

$$f\,(f^{-1}(x) + f^{-1}(y) = \begin{array}{l} \nearrow |x + y| \\ \searrow |x - y| \end{array}$$

In order to define $\boxed{}$ satisfactorily we should have had a unique answer at this stage.

In other words, in order to satisfactorily model the system (R, +) by the many-one function f, we must have that the sum of elements having the same images also has the same image.

In general if $f(a_1) = f(a_2)$ and $f(a_3) = f(a_4)$ then

$$f(a_1 \circ a_3) = f(a_2 \circ a_4) \qquad a_1, a_2, a_3, a_4 \in A$$

In our example, if we choose $a_1 = -a_2 = x$ and $a_3 = a_4 = y$, then we have

$f(x) = f(-x)$ and $f(y) = f(y)$

but $\qquad f(x + y) \neq f(-x + y)$

(Note that a_i do not have to be distinct: the result must be true for all a_i with the above property.)

Summary I

In order not to be too long-winded we introduce a little precise mathematical terminology.

Let f be a function with domain A and image set $f(A) = B$.

Further, let o be a (closed) binary operation on A, then f and o are said to be compatible if, for all $a_1, a_2, a_3, a_4 \in A$ such that

$$f(a_1) = f(a_2)\ , \quad f(a_3) = f(a_4)\ ,$$

we have $\qquad f(a_1 \circ a_3) = f(a_2 \circ a_4)$

Given compatibility we can define a binary operation $\square$ on B by

$$b_1 \ \square\ b_2 = f(a_1 \circ a_2)$$

where a_1 and a_2 are any two elements of A which satisfy

$$f(a_1) = b_1 \text{ and } f(a_2) = b_2\ .$$

If we have compatibility and define $\square$ in this way, then we have satisfied our modelling condition and

$$A,\ \circ,\ f,\ \square$$

is a modelling situation: $\square$ is called the induced binary operation.

In other words $(B\ ,\ \square)$ is a model of $(A, \circ)$ for the function f if

$$f(a_1 \circ a_2) = f(a_1)\ \square\ f(a_2)$$

The corresponding diagram

$$\begin{tikzcd}
a_1 , a_2 \arrow[r, "f"] \arrow[d, "o"'] & f(a_1) , f(a_2) \arrow[d, "\square"] \\
a_1 \circ a_2 \arrow[r, "f"] & f(a_1 \circ a_2) = f(a_1) \square f(a_2)
\end{tikzcd}$$

is called a commutative diagram, because we can go either way from top-left to bottom-right. The function f is called a morphism (or homomorphism) of $(A, \circ)$ onto $(B, \square)$.

Examples IV etc.

Mathematics from elementary school to advanced research is permeated with morphisms. Their existence or non-existence is central to many theories. In this section we shall give many examples with annotations as appropriate - we shall leave most of the examples as investigations to the reader, to discover whether the compatibility condition is satisfied - and if so what is the induced binary operation.

1. (a) Our first example at the beginning of this article was

   A = set of subsets of a (finite) set $\mathcal{U}$.

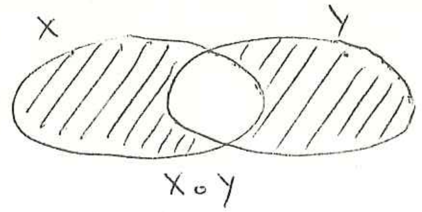
   f = n, where n maps a ~~good~~ given set onto the number of its elements

   $\circ = \cup$

   (In general, the compatibility condition is not satisfied, except in the event that the sets are disjoint, as in the counting problem, where we do use the morphism property.)

   (b) ditto with $\circ = \cap$.

(c) ditto with X o Y defined as X U Y - X ∩ Y . That is



This binary operation is called the symmetric difference of X and Y.

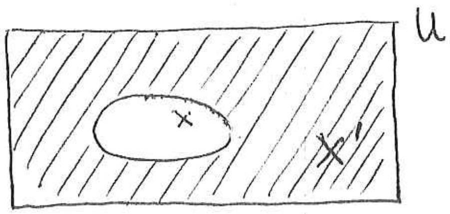
(d) ditto with X o Y defined as Z = $\{(x, y) \mid x \in X, y \in Y\}$.

(Note that in this case the compatibility condition is satisfied and $|\square|$ corresponds to multiplication.)

(The examples in (a) and (d) are basic to the definitions of addition and multiplication in the set of natural numbers, as given in texts on the foundation of mathematics.)

2. (a) A = set of subsets of a set U

f(X) = X', where X' is the ~~implement~~ complement of X in U, i.e.



o = U

(b) ditto with o = ∩.

(These two sections both give use to morphisms - f is a one-one function known as the <u>complement</u> - the corresponding modelling equations are known in set theory as <u>de Morgan's laws</u>.)

3. (a) A = set of subsets of a set $\mathcal{U}$.

$f(X) = K \cap X$, where K is a fixed subset of $\mathcal{U}$.

$o = \cup$.

(b) ditto with $o = \cap$.

(c) ditto with $f(X) = K \cup X$, where K is a fixed subset of $\mathcal{U}$

and $o = \cap$.

(In sections (a) and (c) the modelling equation can be interpreted as the appropriate distributive law. In fact, wherever the distributive law occurs, we can find a morphism. For example, in the set of real numbers

$$a \times (x + y) = a \cdot x + a \cdot y$$

can be reinterpreted as

$$f : x \longrightarrow ax$$

$$o = +$$

and the corresponding diagram

$$\begin{tikzcd}
x, y \arrow[r, "f"] \arrow[d, "+"'] & ax, ay \arrow[d, "\square"] \\
x + y \arrow[r, "f"] & a(x+y) = ax \,\square\, ay = ax + ay \text{ .)}
\end{tikzcd}$$

4. (a) $A = R$, set of real numbers

$f(x) = a + x$, for fixed $a \in R$

$o = +$.

(Note that since f is one-one, the compatibility condition is trivially satisfied. In such cases, the interest lies in "recognising" the induced binary operation. It is "defined" by

$$(a + x) \,\square\, (a + y) = a + (x + y),$$

but this definition is implicit, in the sense that the elements of the image set $f(R) = R$ are not given in the form $a + x$, but as elements of R. In other words, given $w, z \in R$

$$\omega \boxed{} z = ?$$

The solution to this problem can always be found by using the commutative diagram "backwards", i.e. from top-right to bottom-right.)

(b) ditto, but with $o = x$

(c) $A = R$

$f(x) = ax$, for fixed $a \in R$

$o = +$.

(See note after example 3(c).)

(d) ditto, but with $o = x$.

(e) $A = R$

$f(x) = -x$

$o = +$.

(This is a special case of (c) but corresponds to an important property in elementary arithmetic and algebra.)

5. (a) A = set of all real numbers, which have a terminating decimal representation.

$f(x)$ = number of digits after (to the right of) the decimal point

$o = +$.

(b) ditto, but with $o = x$.

(Note that the morphism in section (b) is a very important calculating aid.)

6. (a) $A = R$

$f(x) = a^x$, for some fixed $a \in R^+$, $(a \neq 1)$.

$o = +$

(b) ditto but with $o = x$.

(Note that the first morphism is fundamental to much mathematics from high school onwards. In fact, the equation $f(x + y) = f(x)\cdot f(y)$ is often used as part of an ~~axiometric~~ axiomatic definition of the exponential function. Because the function is one-one, section (b) also gives rise to a morphism, but it is not mathematically significant. The inverse function of a one-one morphism is always a morphism; in the case of section (a) it is, of course the logarithmic function.)

7. (a) $A = R_0^+$

$f(x) = x^2$

$o = +$.

(b) ditto but with $o = \times$

(Note that the fact that o and f are not compatible in (a), "corresponds" to the common student error $(x + y)^2 = x^2 + y^2$.)

(c) As in (a) and (b) but with $f(x) = \sqrt{x}$

(d) ditto, but with $f(x) = x^n$, for some fixed n.

(The morphisms in sections b and d are fundamental to factoring in arithmetic and algebra.)

8. (a) A is a set of data, or numbers representing approximations.

$f(x) = e_x$, the absolute error in x; i.e. $x = X + e_n$, where X is the true value of x.

$o = +$

(b) ditto, but with $o = \times$

(c) A as in (a)

$f(x) = r_x$, the relative error in x, i.e. $r_x = \frac{e_x}{x}$

$o = +$

(d) ditto, but with $o = x$.

(In section (a), f and o are compatible, and resulting morphism is basic to "error arithmetic": the absolute error in the sum is equal to the sum of the absolute errors.

Section (d) is also of some interest.

$$\begin{tikzcd} x\,,\ y \arrow[r, "f"] & r_x\,,\ r_y \\ \times & \\ x \circ y \arrow[r] & r_{xy} \end{tikzcd}$$

Now we can find r_{xy} from the calculations in (b). There we have

$$\begin{aligned} x \circ y &= (X + e_x)(Y + e_y) \\ &= XY + e_xY + e_yX + e_xe_y \end{aligned}$$

Hence
$$e_{xy} = e_xY + e_yX + e_xe_y$$

and
$$r_{xy} = \frac{e_x}{X} + \frac{e_y}{Y} + \frac{e_xe_y}{XY} = r_x + r_y + r_xr_y \ .$$

Thus, we have compatibility, and the induced binary operation on the image set as defined by

$$\omega \,\square\, z = \omega + z + \omega z \ .$$
$$\cancel{W \boxminus Z = W + Z + WZ}\ .$$

In practice, that is in "error arithmetic", this "exact" result is rarely used. If the (relative) errors are small, then their product is usually very much smaller, and can safely be ignored. So the usually used induced operation is simply addition: the relative error in the product is the sum of the relative errors.)

9. (a) $A = N$, the set of natural numbers

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is even} \\ -1, & \text{if } x \text{ is odd} \end{cases}$$

$o = +$

(b) ditto, but with $o = \times$.

(Suggest a more "natural" definition for f in section (b).)

10. (a) $A = N$

$f(x) = x(\text{mod } n)$, for some fixed $n \in N$.

$o = +$

(b) ditto, but with $o = \times$.

11. (a) $A = N$

$f(x)$ = the set of prime factors of x, recorded according to their multiplicity; e.g. $f(12) = \{2, 2, 3\}$.

o = highest common factor

(b) ditto, but o = least common multiple.

12. (a) A = open sentences with substitution set R

$f(x)$ = truth set of x.

o = "and"

(b) ditto, but with o = "or"

(In both cases we have compatibility. The induced binary operations are $\cap$ and $\cup$, respectively. These morphisms are, of course, extremely important in solving equations and inequations and graphing.)

13. (a) $A = R$

$f(x) = \sin x$

$o = +$

(b) ditto, but with $f(x) = \cos x$

(c) ditto, but with $f(x) = \tan x$.

(The tangent function is compatible with addition, but care is needed with domain R, because of problems at $\pm \pi/2$, etc. The induced binary operation is not trivial

$$x \boxed{} y = \frac{x + y}{1 - xy}.)$$

14. (a) $A = \mathcal{F}$, set of differentiable functions

$f = D$, the differentiation function

$o = +$, function addition .

(b) ditto , but $o = \times$.

(In the first case D can be replaced by any polynomial in D; e.g.

$$D^2 + 3D + 1$$

The fact that these functions are all morphisms, with induced binary operation also +, is fundamental to the theory of linear differential equations.

As in sections 13(a) and (b), so here in section 14(b), the absence of compatibility indicates that if a formula for the image of the combination exists, then it must be more complex than ~~same~~ some combination of the images. This, in itself, can be used to motivate the search for such a formula.)

Summary II

The concept of modelling is fundamental to much human intellectual activity. The reasons for modelling are various - the model is usually chosen to be simpler, or more familiar, than the original. The general concept of model takes many forms, in these notes we have given it a very specific meaning. A set with a binary operation is a structure and the function copies this structure onto the image set - In order that the copy should be a model, we have required that the result of the binary operation in the domain should be reflected by the result of performing some (induced) binary operation in the image set. Although this model is extremely simple, it has many uses - from the simple one of being a calculating aid or providing a formula useful in the solution of problems, to the extensive theory which we have not yet discussed here.

The number of examples of morphisms is endless - we have only touched on some of those to be found in elementary mathematics - the further we go, the more important the idea of morphism becomes. We have also not touched upon applied mathematics. The multiplicity of examples alone should convince one of the importance of the idea. Given that function and binary operation are now explicitly treated in the school curriculum, there is no reason why morphism and associated ideas should not be used implicitly to give the curriculum further unity and cohesion, as well as providing a basis for further work in mathematics for those who continue. The concept of modelling and the commutative diagram ~~is~~ as introduced in the section Modelling I, can easily be developed by their use in numerous examples. The precise definitions, especially of compatibility, as given in Summary I, belong to a second more sophisticated stage.

Although, rightly, the emphasis has been on morphisms, where they exist, the non-existence of morphisms (i.e. when the function and binary operation are not compatible), can be used didactically to motivate the search for a more complicated formula.

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