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SOME ASPECTS OF THE FUNCTION CONCEPT IN COLLEGE STUDENTS AND JUNIOR HIGH SCHOOL TEACHERS.

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Some Aspects of the Function Concept in College Students and Junior High School Teachers(*)

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§ 1. Introduction

It is a common habit in Mathematics to introduce new notions by means of verbal definitions which characterize them definitely and ultimately. Some people even consider definitions crucial to the teaching of Mathematics. That is probably because definitions have a crucial role in the Formalist description of Mathematics. However, a cliam was made (Vinner, 1980; Vinner and Hershkowitz, 1980) that in mathematical behavior what mainly matters is the Concept Image and not the Concept Definition. The concept image consists of all mental pictures and all properties which are associated with a given notion in somebody's mind. The concept image is determined by specific examples and by the accumulated experience with the notion. Quite often there is a gap or even a conflict between the student's concept image and the concept definition as taught by the teacher.

This study examines some aspects of the notion of function in college students and junior high school teachers. A previous study (Vinner, 1980) examined similar aspects in high school students to whom functions were introduced in the Dirichlet-Bourbaki approach: a function is any correspondence between two sets (the domain and the range) which assigns to every element in the domain exactly one element in the range. It was found that only 57% of the students gave one version or another of the Dirichlet-Bourbaki definition when asked about the notion of function and only 34% of these really acted according to this definition when asked various questions about functions.

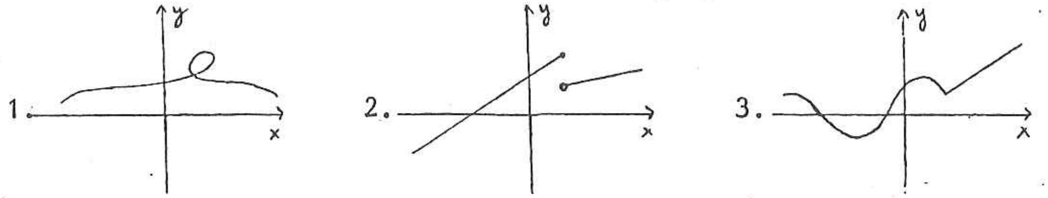
(*) This research was done while the authors were research fellows at the Weizmann Institute of Science, Rehovot, Israel.

It is not intended in this study to examine concept images versus concept definitions since it is impossible to find out how functions were taught to our students. What we try here is to reveal concept images of students a few years after they learned the notion of function in high school. We examined our students' concept of function before this concept was discussed in their present classes.

§ 2. The Notion of Function and the Questionnaire

The Dirichlet-Bourbaki definition of a function came to give recognition as functions to many correspondences which were not recognized as functions by previous generations of Mathematicians (see Malik, 1981). Among them are incontinuous functions, functions defined on a split domain, functions with a finite number of "exceptions" and functions defined by means of a graph. We designed a questionnaire to examine whether various correspondences are considered as functions by the students. In this paper, because of space problem, we will discuss only 5 out of its 7 items.

Does there exist a function the graph of which is:



4. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1.
5. What is a function in your opinion?

In each of the questions 1 to 4 the student had to choose between "yes", "no" and "I do not know" and to explain his choice. In question 5 we did not ask about the *definition* of function because we wanted also students who had forgotten the definition to come up with something that at least reflected their concept image.

§ 3. The Sample

The questionnaire was administered to several groups of first and second year college students whose main subjects of study were Mathematics, Physics, Chemistry, Biology, Economics, Agriculture, Technological Education and Industrial Design. It was also administered to junior high school Mathematics Teachers. When analysing the data we retained only those questionnaires which contained either a definition of the function concept (Question 5) plus at least one explanation to another question or, in case the definition was missing, at least two explanations to questions out of 1 to 4. Thus we were left with 271 students and 36 teachers. They were classified into five groups according to the level of their present Mathematics training: low level (33 students), intermediate (67), high (113), Math. major students (58) and teachers (36).

We also classified the students according to time and level of their previous mathematical training but no significant differences were found on the ground of these classifications.

§ 4. The Results

The first analysis we made is related to the concept definition of function. The various definitions (the answers to question 5) were categorized into 6 categories (which were a refinement of the categorization in Vinner, 1980):

I. The function is *any* correspondence between two sets that assigns to every element in the first set exactly one element in the second set (the Dirichlet-Bourbaki definition).

II. The function is a *dependence* relation between two variables (y depends on x).

III. The function is a *rule* of correspondence (this conception eliminates the possibility of *arbitrary* correspondences).

IV. The function is a manipulation or an operation (one acts on a given number, generally by means of algebraic operations, in order to get its image).

V. The function is a formula, an algebraic term or an equation.

VI. The function is identified, probably in a meaningless way, with one of its visual or symbolic representations (the graph, the symbols "$y = f(x)$", etc.).

Table I shows the distribution of the concept definition categories in the five groups mentioned at the beginning of this section.

Table I - Distribution of concept definitions in the subgroups (in percents)

Mathematical level / Categories	Low	Intermediate	High	Math-majors	Teachers
I	6%	18%	15%	45%	70%
II	37%	27%	32%	21%	8%
III	12%	13%	8%	12%	8%
IV	6%	2%	6%	5%	3%
V	18%	19%	7%	5%	0%
VI	12%	9%	10%	5%	3%
Others	9%	12%	22%	7%	8%
	100%	100%	100%	100%	100%

A χ^2-test shows that the differences between the subgroups are significant, $p < 0.0001$. The table shows that the percentage of the Dirichlet-Bourbaki definition increases with the level. It is specially high in the teachers. However, the difference between the 4 student groups are also significant, $p < 0.003$.

Table II shows the distribution of correct answers to questions 1, 2, 3 and 4 in the above 5 groups.

Table II - Distribution of correct answers in the sub-groups

Mathematical level / Question	Low	Intermediate	High	Math majors	Teachers
1	55%	66%	64%	74%	97%
2	27%	48%	67%	86%	94%
3	36%	40%	53%	72%	94%
4	9%	22%	50%	60%	75%

Again, the differences between the 5 subgroups are significant for each one of the question ($p < 0.0007$ for question 1 and $p < 0.0001$ for questions 2-4). When excluding the teachers from the sample the differences between the four student groups remain significant ($p < 0.0008$) for questions 2-4, but not for question 1.

An additional analysis we made concerned the explanations for accepting or rejecting the existence of functions as mentioned in the questionnaire. Here the concept image played very often a crucial role. In each question different aspects of the concept image were the ground for accepting or rejecting the existence of the function. Sometimes the same aspect was the ground for rejection for some students and the ground for acceptance for other students. For instance, in some explanations to *negative* answers to question 2 it was said that the graph is discontinuous and therefore it cannot be a graph of a function. In some explanations to *positive* answers to the same question it was said that discontinuous functions are legitimate members of the "function family".

We will list here some of the major aspects of the function notion that played a crucial role in the explanations. Table III will show their distribution in the answers to questions 1-4 for the entire sample.

1. One Value (OV): If a correspondence assigns exactly one value to every element in its domain then it is a function. If not - then it is not.

2. Discontinuity (D): The graph is discontinuous or changes its "character" (2 different straight lines).

3. Split Domain (SD): The domain of the function splits or the graph changes its "character" (a straight line and a curved line). (Also this aspect, as 2 above, was a ground for rejection and acceptance.)

4. Exceptional Point (EP): There is one point of exception (also a ground for rejection and acceptance).

Table III - Distribution of explanations to questions 1-4, in the entire sample

Question	Answer	Number of Answers	OV	D	SD	EP	Other	None
1	yes	46	0%	0%	4%	2%	66%	28%
	no	216	59%	0%	1%	4%	19%	17%
2	yes	203	33%	18%	16%	0%	18%	15%
	no	87	0%	40%	41%	0%	5%	14%
3	yes	176	23%	1%	28%	1%	13%	34%
	no	103	1%	0%	77%	3%	5%	14%
4	yes	138	10%	1%	35%	3%	27%	24%
	no	100	5%	1%	2%	38%	35%	19%

(The columns OV, D, SD, EP, Other and None appear under the spanning header "Aspect".)

Similar tables were constructed for each of the five subgroups. They will be discussed elsewhere.

It was also found that concept definitions remained very often inactive at decision making moments when (sometimes wrong) concept images took over. This resulted sometimes in an answer contradictory to the concept definition. This, however, happened less often than in the previous study (Vinner, 1980). This is probably because relatively long time had passed since the students learned the notion of function. During this time the concept image and the concept definition adjusted themselves to each other.

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