MRI: Assignment #1 Solution

Rotations

The mathematical preliminaries lecture contains the expression for a right handed rotation matrix about an arbitrary axis and angle (you can also find it on Wikipedia if you look under “Rotation Matrix”):

$$R(\hat{\mathbf{n}},\theta)=\begin{pmatrix} c\theta+n_x^2(1-c\theta) & n_xn_y(1-c\theta)-n_zs\theta & n_xn_z(1-c\theta)+n_ys\theta \\ n_yn_x(1-c\theta)+n_zs\theta & c\theta+n_y^2(1-c\theta) & n_yn_z(1-c\theta)-n_xs\theta \\ n_xn_z(1-c\theta)-n_ys\theta & n_zn_y(1-c\theta)+n_xs\theta & c\theta+n_z^2(1-c\theta) \end{pmatrix}$$

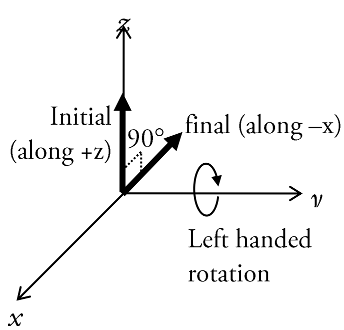
To turn it into a left handed rotation by 90° about the y-axis we set $\theta=-\frac{\pi}{2}$ and $\mathbf{n}=(0,1,0)$, so

$$R(\hat{\mathbf{y}},-\tfrac{\pi}{2})=\begin{pmatrix} \cos(-\frac{\pi}{2}) & 0 & \sin(-\frac{\pi}{2}) \\ 0 & 1 & 0 \\ -\sin(-\frac{\pi}{2}) & 0 & \cos(-\frac{\pi}{2}) \end{pmatrix}=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Applying this to a vector along z, we get

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

which is a unit vector pointing along the (-x) axis. This is indeed what one would get by rotating the z-vector by 90° about the y-axis according to the left-hand rule.



Rotations Do Not Commute

This is a straightforward matter of writing down the matrices and multiplying them in both possible orders. I'll go with left handed rotations, although this holds equally for right handed ones.

$$R\left(\hat{\mathbf{y}},-\tfrac{\pi}{2}\right)=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}$$

$$R\left(\hat{\mathbf{z}},-\tfrac{\pi}{2}\right)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}$$

Now, we multiply:

$$R\left(\hat{\mathbf{y}},-\tfrac{\pi}{2}\right)R\left(\hat{\mathbf{z}},-\tfrac{\pi}{2}\right)=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&0&-1\\-1&0&0\\0&1&0\end{pmatrix}$$

$$R\left(\hat{\mathbf{z}},-\tfrac{\pi}{2}\right)R\left(\hat{\mathbf{y}},-\tfrac{\pi}{2}\right)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

The two results are clearly not the same, proving rotations are not commutative.

Complex Numbers

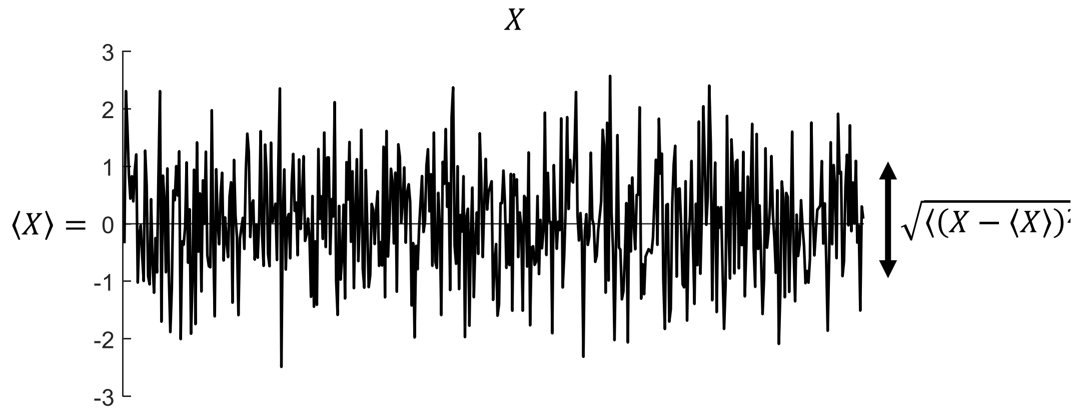
The general rule about exponential functions is that $e^a e^b = e^{a+b}$, so $e^{ix}e^{-ix} = e^{ix-ix} = e^0 = 1$. On the other hand, using Euler's identity,

$$\begin{aligned}e^{ix}e^{-ix} &= \left(\cos x + i\sin x\right)\left(\cos x - i\sin x\right)\\ &= \cos^2 x + i\sin x\cos x - i\sin x\cos x - i^2\sin^2 x\\ &= \cos^2 x + \sin^2 x\end{aligned}$$

where we have used i^2=-1. Equating both expressions, we immediately obtain the desired identity.

SNR And Averaging

Let's write the average of a random number using $\langle\ \ \rangle$. For example, if X is the result of rolling a die, then the average of rolling the die many times should be $\langle X\rangle = \frac{1+2+3+4+5+6}{6} = 3.5$. Means are additive, meaning that $\langle X+Y\rangle = \langle X\rangle + \langle Y\rangle$ (e.g., the mean sum of two dice equals the sum of the mean of each die). The standard deviation is defined as $\sqrt{\langle (X-\langle X\rangle)^2\rangle}$, and is the width of the distribution of X around its mean. For example, if you measure a noisy signal with zero mean, then



We calculate:

$$\begin{aligned}\sqrt{\left\langle (n_1+n_2)^2 \right\rangle} &= \sqrt{\left\langle n_1^2 + 2n_1n_2 + n_2^2 \right\rangle} \\ &= \sqrt{\left\langle n_1^2 \right\rangle + 2\left\langle n_1n_2 \right\rangle + \left\langle n_2^2 \right\rangle} \\ &= \sqrt{\sigma^2 + 0 + \sigma^2} = \sqrt{2}\sigma\end{aligned}$$

Here, we used the fact that because n_1 and n_2 are random, uncorrelated and have zero means, their product should be sometimes positive and sometimes negative, but on average zero: $\langle n_1 n_2 \rangle = 0$ (if you don't "believe" me, try using MATLAB or Python to generate two vectors of normally distributed random numbers, multiply them, and then look visually at the result).

If we repeated this exercise adding up M uncorrelated random numbers with zero mean, the SD of the sum would be $\sqrt{M}\sigma$.
For an image, the SNR is:

$$SNR = \frac{\text{signal}}{\text{SD of noise}} = \frac{\text{s}}{\sigma}$$

If we add two images with the same signal and same SD of the noise, the signal would double by the SD would only increase by $\sqrt{2}$, leading to an increase in SNR of only $\sqrt{2}$:

$$SNR_{\text{two images}} = \frac{2\text{s}}{\sqrt{2}\sigma} = \sqrt{2}\frac{s}{\sigma} = \sqrt{2}SNR_{\text{one image}}.$$

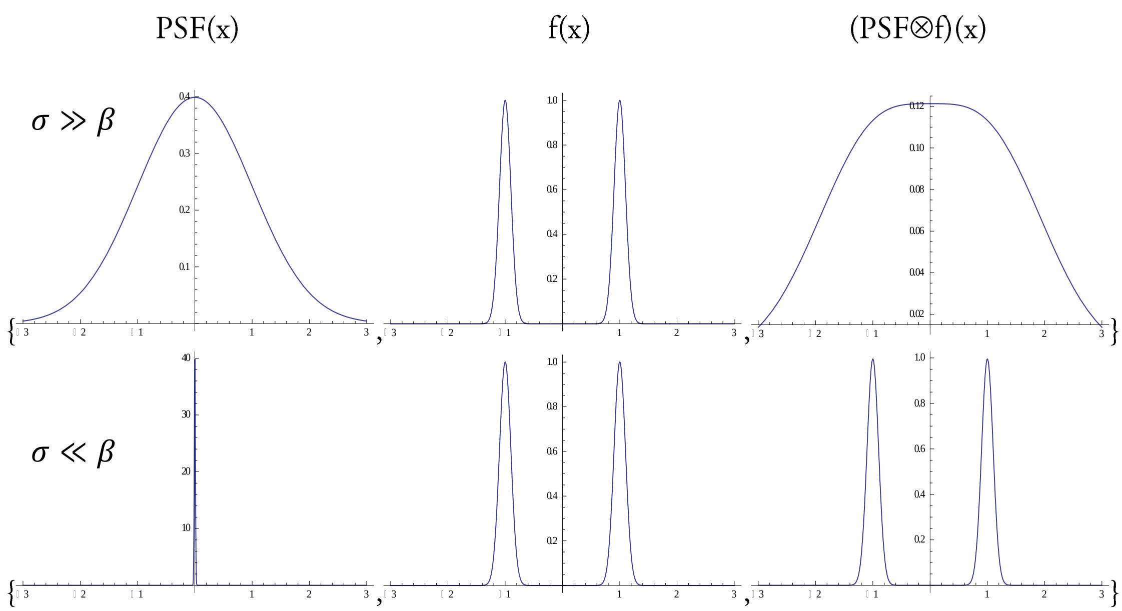
Now we can directly see that if we have double the time we can acquire two images and add them up, increasing the SNR by a factor of $\sqrt{2}$. This generalizes to an increase of the SNR by a factor of $\sqrt{\alpha}$ for an increase of scan time by a factor of α .

Convolution and The Point Spread Function (PSF)

Merely apply the definition.

$$\begin{aligned}(f\otimes PSF)(x) &= \int_{-\infty}^{\infty} f(x')\,PSF(x-x')\,dx' \\ &= \frac{1}{\sqrt{1+\frac{\sigma^2}{\beta^2}}}\left\{e^{-\frac{(y-x_0)^2}{2(\beta^2+\sigma^2)}}+e^{-\frac{(y+x_0)^2}{2(\beta^2+\sigma^2)}}\right\}\end{aligned}$$

Plots of PSF(x), f(x) and their convolution are shown below for the two cases outlined in the assignment:



One can say that $\sigma << \beta$ is preferable since it does not widen the profile f(x). The opposite ($\sigma >> \beta$) causes the two peaks to widen to the point of merging, making it impossible for anyone viewing the final "image" to know it originated from two distinct distributions.