MRI: Assignment #2 Solution

MRI Sensitivity

In the institute, the only human MRI machine, the Siemens 3 Tesla Tim Trio, is housed next to the neurobiology building. The highest field NMR spectrometer is the DRX800 (housed right in front of Ziskind). Complete the following table, intended to give you a rough feel for the orders of magnitude involved in typical magnetic resonance experiments:

	Siemens MRI Trio human MRI machine (approx.)	DRX800 NMR spectrometer	Earth's magnetic field
Precession frequency of proton ^1H nuclear magnetic moment	128 MHz	800 MHz	2.1 kHz
Precession frequency of Phosphorous ^{31}P nuclear magnetic moment	51 MHz	324 MHz	861 Hz
Precession frequency of Carbon ^{13}C nuclear magnetic moment	32 MHz	201 MHz	535 Hz
Precession frequency of electron's magnetic moment	84 GHz	~0.5 THz	1.4 MHz
Field strength (in Tesla)	3	18.8	0.00005
Field strength (in Gauss)	30,000	188,000	0.5

The gyromagnetic ratio of the electron is about 28 GHz/T. The gyromagnetic ratios of the nuclei are shown in the table in the previous question.

Current And Fields

Plugging in r=0.5 m, I=16A, we get a field that's on par with the Earth's magnetic field:

$$B = \frac{\mu_0 I}{2r} = \frac{4\pi \times 10^{-7} \times 16}{2 \times 0.5} \cdot T \approx 0.2 \text{ Gauss}$$

To create a 3 Tesla field we would need a current obtained by solving

$$B = \frac{\mu_0 I}{2r} = 3 \text{ T} \quad \text{or} \quad \text{about 2 million Amperes (!!!)}$$

For our r=0.5 meter radius wire, we have $l = 2\pi r \approx 3.14$ meters. For a "regular" power cable (כבל קומקום ...) we can estimate the internal radius as being about 0.5 cm, meaning

$$A = \pi a^2 = \pi \left(0.5\ cm\right)^2 = 0.785\ cm^2 = 0.785 \times 10^{-4}\ m^2 .$$

This wire's total resistance is

$$R = \frac{\rho l}{A} = \frac{1.7\times10^{-8}\times3.14}{0.785\times10^{-4}} \approx 0.7\times10^{-3}\,\Omega\,.$$

This means the power dissipated in the wire is

$$P = (\text{resistance of wire})\times(2\text{ million amps})^2 \approx 3.9\text{ GW}$$

This is equivalent to about half of the total electrical power requirements of Israel during the day!

The specific heat of water – the amount of heat energy required to raise the temperature of unit mass by 1°C – is

$$C = 4.186\,\frac{\text{Joule}}{\text{gram}\cdot\text{Celsius}}\,.$$

In 1 minute, 3.9 GW of power would create

$$E = (3.9\ GW)\times(60\text{ sec}) \approx 2\times10^{11}\text{ Joule}$$

which would raise the temperature of a liter of water (equivalent to 1000 grams) by

$$\frac{E}{C} = \frac{2\times10^{11}}{4186} \approx 4.8\times10^{7}\text{ deg. Celsius}\,.$$

Obviously a nonsensical result since by then the water will have evaporated, but this could also mean that we would raise the temperature of ~10^7 liters of water by 1 deg. Celsius.

Magnetic Moment of the Earth

The magnetic field created at a point **r** by a magnetic moment at the origin is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5}$$

For the Earth, **m** is parallel to the Earth's axis of rotation (assuming the geographic and magnetic north coincide). Looking at the equator, **r** and **m** are orthogonal, and r=R=Earth's radius=6371 km, so

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{R^3} = 0.5\text{ G} = 0.5\cdot10^{-4}\,T\,.$$

Using $\mu_0 = 4\pi\cdot10^{-7}\,T\cdot m/A$, we solve for $|\mathbf{m}|$:

$$m = \frac{0.5\cdot10^{-4}\cdot\left(6.37\cdot10^{6}\right)^3}{10^{-7}} \approx 1.3\cdot10^{23}\,\frac{J}{T}$$

This number is pretty close to more careful estimations.

Bloch Equations Induce Rotations

If **B** doesn't point along the z-axis we can simply re-orient our coordinate system so that it does (we're free to choose the orientation of our coordinate axes). A left handed rotation matrix about z is

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The vector expression for **m**(t) is

$$\mathbf{m}(t) = R_z(\omega t)\mathbf{m}(0) = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{x,0} \\ m_{y,0} \\ m_{z,0} \end{pmatrix} = \begin{pmatrix} \cos(\omega t) m_{x,0} + \sin(\omega t) m_{y,0} \\ -\sin(\omega t) m_{x,0} + \cos(\omega t) m_{y,0} \\ m_{z,0} \end{pmatrix}.$$

Differentiating, we obtain (note the last step!)

$$\frac{d\mathbf{m}(t)}{dt} = \omega \begin{pmatrix} -\sin(\omega t) m_{x,0} + \cos(\omega t) m_{y,0} \\ -\cos(\omega t) m_{x,0} - \sin(\omega t) m_{y,0} \\ 0 \end{pmatrix} = \omega \begin{pmatrix} m_y(t) \\ -m_x(t) \\ 0 \end{pmatrix}. \quad \text{(Expression for the left-hand side)}$$

On the other hand, calculating the right hand side of the Bloch equation for $\gamma\mathbf{B} = (0,0,\gamma B) = (0,0,\omega)$, we obtain, just by carrying out the cross product,

$$\mathbf{m}(t) \times \omega\hat{\mathbf{z}} = \begin{pmatrix} m_y(t)\omega \\ -m_x(t)\omega \\ 0 \end{pmatrix} \quad \text{(Expression for the right hand side)}$$

which agrees with our expression for the left hand side.