MRI Primer: Assignment #3

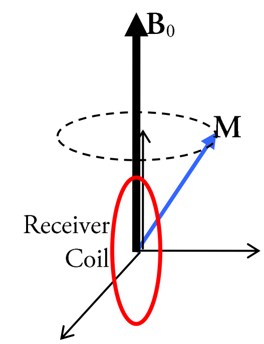
Due Tuesday, Nov. 30, 2021

Thermal Equilibrium Signal

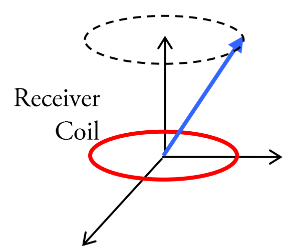
1. How many water molecules are in 1 mm^3 of "typical" brain tissue? This is the size of "typical" MRI voxel. A rough estimation would suffice. You can use scientific papers as long as you cite them properly, as you would in a paper.
2. Estimate the bulk nuclear thermal equilibrium (in J/T) in a 1 mm^3 "box" (i.e. a typical voxel) of "typical" human tissue in a 3T MRI magnet by using the expression we derived in class for the bulk magnetization.

Signal Induction

Imagine a point magnetic moment $\mathbf{M}(t)$ at the center of the coordinate system ($\mathbf{r}$=0) precessing around a constant field in the z-direction, $\mathbf{B}=(0,0,B_0)$. Assume that initially, at time t=0, $\mathbf{M}(0)=(M_{x0},0,0)$. Do not neglect T_1, T_2 relaxation.



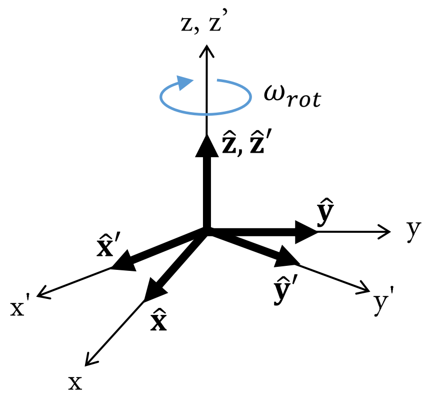
Part 2: coil is in xz plane



Part 4: coil is in xy plane

1. Using the results of the previous problem, and what you've seen in class, write down $M_x(t)$, $M_y(t)$ and $M_z(t)$.
2. Assume a circular coil in the x-z plane with a radius r and its center at $\mathbf{r}$=0. Write down an expression for the voltage v(t) picked up by the coil as a function of time, using the principle of reciprocity (the magnetic field of a circular loop of radius r and current I is perpendicular to the loop's plane and given in its center by $B=\mu_0 I/2r$).
3. Plot the signal as a function of time for M_{x0}=1 J/T, T_2=1 sec, $\omega=\gamma B_0=10\cdot 2\pi$ Hz·rad, r=20 cm (note: these are not realistic numbers for ω which should be $\omega=\gamma B_0=127\cdot 2\pi$ MHz·rad, but taking large values of ω would make the plotting very difficult and not contribute to the insights and overall shape of the plot). Hint: use the expressions we've derived in class for the solutions to the Bloch equations, $M_{xy}(t)$ and $M_z(t)$, in the presence of a field along the z-axis.
4. The coil is now placed in the xy plane. Show that the induced voltage is now approximately zero (i.e. much, much smaller than the voltage calculated in parts 2 & 3). Explain your answer, using the fact that the voltage is proportional to the time derivative of the magnetic flux.

Frame Transformations



In this exercise we will hone your frame-transformation skills. These will be important to understanding resonant excitations. Consider two frames, the static "lab" frame and a rotating frame which rotates about the z-axis with a constant angular velocity ω_{rot} (left-handed rotation). The unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ point along the x, y, and z-axes of the lab frame, and are static. The unit vectors $\hat{\mathbf{x}}'(\mathrm{t}), \hat{\mathbf{y}}'(\mathrm{t}), \hat{\mathbf{z}}'(\mathrm{t})$ point along the axes of the rotating frame, and are time-dependent.

1. Write down expressions for $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ as a function of time. That is, complete the following set of expressions (hint: it's a rotation around z):

$$\hat{\mathbf{x}}'(\mathrm{t}) = a_{11}\hat{\mathbf{x}} + a_{12}\hat{\mathbf{y}} + a_{13}\hat{\mathbf{z}}$$
$$\hat{\mathbf{y}}'(\mathrm{t}) = a_{21}\hat{\mathbf{x}} + a_{22}\hat{\mathbf{y}} + a_{23}\hat{\mathbf{z}}$$
$$\hat{\mathbf{z}}'(\mathrm{t}) = a_{31}\hat{\mathbf{x}} + a_{32}\hat{\mathbf{y}} + a_{33}\hat{\mathbf{z}}$$

   Similarly, describe how the basis vectors of the (xy) system depend on the (x'y') basis vectors (hint: you can either try to invert the equations above using algebra and trigonometry, or use the fact that, when viewed from the x'y' frame, the xy vectors seem to rotate in an opposite direction than the x'y' vectors when viewed in the xy frame):

$$\hat{\mathbf{x}}(\mathrm{t}) = b_{11}\hat{\mathbf{x}}'(t) + b_{12}\hat{\mathbf{y}}'(t) + b_{13}\hat{\mathbf{z}}'(t)$$
$$\hat{\mathbf{y}}(\mathrm{t}) = b_{21}\hat{\mathbf{x}}'(t) + b_{22}\hat{\mathbf{y}}'(t) + b_{23}\hat{\mathbf{z}}'(t)$$
$$\hat{\mathbf{z}}(\mathrm{t}) = b_{31}\hat{\mathbf{x}}'(t) + b_{32}\hat{\mathbf{y}}'(t) + b_{33}\hat{\mathbf{z}}'(t)$$

2. A vector $\mathbf{u}$ can be expressed in terms of either set of basis vectors: $\mathbf{u} = \mathrm{u_x}\hat{\mathbf{x}} + \mathrm{u_y}\hat{\mathbf{y}} + \mathrm{u_z}\hat{\mathbf{z}} = \mathrm{u_{x'}}\hat{\mathbf{x}}' + \mathrm{u_{y'}}\hat{\mathbf{y}}' + \mathrm{u_{z'}}\hat{\mathbf{z}}'$. Consider the RF field which has the following expansion in the lab frame:

$$\mathbf{B}_{\mathrm{RF}}(t) = B_1 \cos(\omega_{RF}t)\,\hat{\mathbf{x}} - B_1 \sin(\omega_{RF}t)\,\hat{\mathbf{y}}.$$

   Using (1), write down the expression for $\mathbf{B}_{\mathrm{RF}}$ in the rotating frame (i.e. in terms of the primed set of basis vectors). Use trigonometric identities to simplify as needed.

3. Show that, when $\omega_{RF} = \omega_{rot}$, the expansion coefficients of the RF field in the rotating frame become constant (what are they equal to?). Explain why this is so intuitively.
4. How does a static magnetic field along the z-axis appear in both frames of reference? That is, repeat (2) for a magnetic field of the form $\mathbf{B} = B\hat{\mathbf{z}}$. Again, provide an intuitive explanation for your result.

Time Derivatives in the Rotating Frame

In class we derived (or will derive) the form of the Bloch equation in the rotating frame:

$$\left(\frac{d\mathbf{M}}{dt}\right)_{rot} = \mathbf{M} \times \left(\gamma \mathbf{B} - \boldsymbol{\omega}_{rot}\right)$$

We remarked that the vector

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} \equiv \frac{dM_{x,rot}}{dt}\hat{\boldsymbol{x}}' + \frac{dM_{y,rot}}{dt}\hat{\boldsymbol{y}}' + \frac{dM_{z,rot}}{dt}\hat{\boldsymbol{z}}'$$

has the following meaning: it is the time derivative of the vector **M**, where **M** is viewed by an observer in the rotating frame. This is not the same as the time derivative of **M**! In this exercise, we are going to clarify this statement and show how it differs from d**M**/dt. Consider a vector **M** which rotates in the xy plane:

$$\boldsymbol{M}(t) = \cos(\omega t)\,\hat{\boldsymbol{x}} - \sin(\omega t)\,\hat{\boldsymbol{y}}$$

1. Differentiate **M**(t) to obtain d**M**/dt. Express its components in the (xy) frame.
2. Express the components of both vectors, **M** and d**M**/dt, in the (x'y'z') frame, where the (x'y'z') frame is the same as the one described in the previous problem (with a left-handed rotation ω around the z-axis). This tells you what **M** and d**M**/dt would look like to an observer in the x'y'z' frame. Hint: use the expressions for the unit vectors in the xyz frame in terms of the x'y'z' frame. In particular, show **M** and d**M**/dt are both **constant** and **non-zero**.
3. Now, consider the vector **M**(t) as it appears to an observer in the x'y'z' frame. If asked, what would an observer in the x'y'z' frame (who is unaware of the xyz frame) think the time derivative of **M** should be? In other words, what is $\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot}$? (Hint: it's not the same as d**M**/dt in the x'y'z' frame)

On and Off-Resonance Excitation

When one speaks of a "90°-pulse" or a "180°-pulse", it is implicitly assumed that they are referring to the rotation angle induced by the pulse when on resonance. Let us explore this point in greater detail, and also see what happens to spins that are "off resonance", meaning their frequency in the rotating frame does not match the frequency of the RF field. Assume we are dealing with protons.

1. What would be the magnitude, B_1 (in μT) of a rectangular 90°_x pulse - that is, a pulse that has a constant amplitude, and is applied along the x-axis on resonance in the rotating frame - given that its duration is 1 ms?
2. What would be the pulse's bandwidth (in kHz, approximately)?

3. Draw, in the rotating frame: (i) the effective field for an on-resonance spin; (ii) the trajectory of the magnetization vector, assuming it starts out from thermal equilibrium and is on resonance.
4. What would be the precession frequency of an on-resonance spin around this effective field, in kHz?
5. Now assume another spin, which has a non-zero offset in the rotating frame, $\Delta B = 1\ \mu T$, feel the same B_1 field. Write down B_{eff}. What is the precession frequency of the spin around the effective field in the rotating field?
6. What is the total angle by which the spin has rotated by the end of the pulse? (Hint: it is **not** 90° ...). Remember this: a 90° pulse is a 90° pulse only for on-resonance spins!
7. Repeat part (3) for the off-resonance spin. No need to calculate explicitly the magnetization's trajectory - a qualitative drawing would suffice.

Flip Angles Are Nucleus-Dependent

An experimentalist applies a 90°-pulse to a proton nucleus. Would this also be a 90° pulse when applied to a ^{13}C nucleus? If yes, explain why. If not, calculate its rotation angle, and explain how you would make it a 90° pulse. Assume you are on resonance for both nuclei.