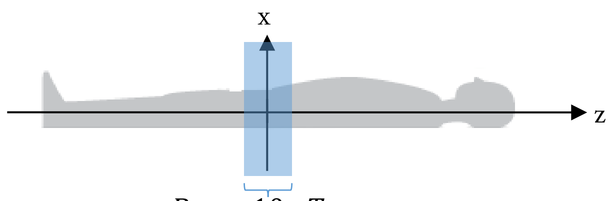
MRI Primer: Assignment #5 Solution

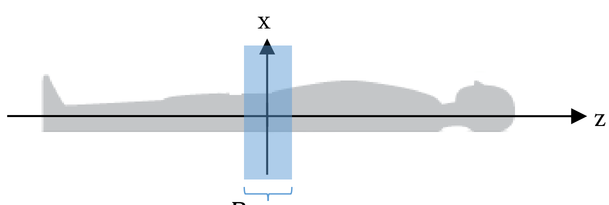
Selective Excitation & B_0-Field Inhomogeneity

1. The slice will be centered at 0, perpendicular to the z-axis, and have a 1 cm width:



$$\Delta z = \frac{B_1}{G} = \frac{10\ \mu T}{1mT/m} = 1\ cm$$

2. The pulse will excite frequencies between $\pm BW/2=\gamma B_1/2$. This corresponds to positions between $\omega(z_\pm) = \gamma G z_\pm + \gamma \eta_1 z_\pm = \pm\frac{2\pi\cdot BW}{2}$, where $\omega(z_\pm) = \pm\frac{2\pi BW}{2}$. Solving, we get $z_\pm = \pm\frac{\pi BW}{\gamma(G+\eta_1)} = \pm\frac{B_1}{2(G+\eta_1)}$. This means that the width of the slice is now $z_+ - z_- = \frac{B_1}{G+\eta_1}$. The center of the slice is at $\frac{z_+ + z_-}{2} = 0$. It is perpendicular to the z-axis.

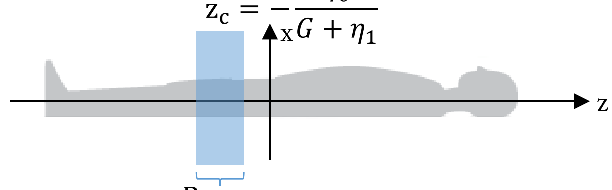


$$\Delta z = \frac{B_1}{\eta_1 + G} = 0.5\ cm$$

3. The effect of eta becomes negligible when it is small compared to the other term in the denominator, which is G. This translates to: $\eta_1 \ll G$.

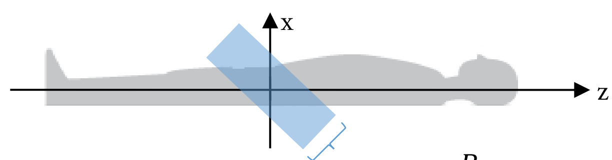
4. The constant term will shift the slice. To see this, repeat our reasoning from section 2: The pulse will excite frequencies between $\pm BW/2=\gamma B_1/2$. This corresponds to positions between $\omega(z_\pm) = \gamma G z_\pm + \gamma \eta_1 z_\pm + \gamma \eta_0 = \pm\frac{2\pi BW}{2}$. Solving, we get $z_\pm = \pm\frac{\pi BW}{\gamma(G+\eta_1)} - \frac{\eta_0}{G+\eta_1} = \pm\frac{B_1}{2(G+\eta_1)} - \frac{\eta_0}{G+\eta_1}$. The slice width is now $z_+ - z_- = \frac{B_1}{G+\eta_1}$ (same as in part 2), but the center shifts: $\frac{z_+ + z_-}{2} = -\frac{\eta_0}{G+\eta_1}$.

$$z_c = -\frac{\eta_0}{G + \eta_1}$$



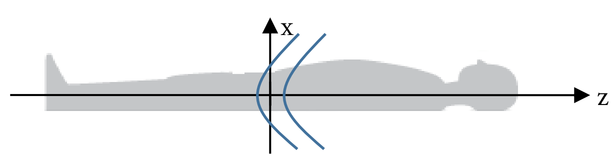
$$\Delta z = \frac{B_1}{\eta_1 + G} = 0.5\ cm$$

5. Conceptually, there is no difference between a linear inhomogeneity and a gradient! With the homogeneity along the x-axis, the offset as a function of position is $\omega(z,x) = \gamma G z + \gamma \eta_1 x$, which is what you would get with **no** inhomogeneity and a gradient $\boldsymbol{G} = (\eta_1, 0, G)$, such that $\omega(z,x) = \gamma \boldsymbol{G} \cdot \boldsymbol{r}$. If $\eta_1 = G = 1\frac{mT}{m}$ then this gradient points at a 45° angle to the x-axis. If we look along the axis that is parallel to **G**, then the angle between **G** and **r** is 0° and we can write $\omega(r) = \gamma|\boldsymbol{G}|r$, where r is the distance from the origin along the axis defined by **G**, and $|\boldsymbol{G}| = \sqrt{G^2 + \eta_1^2}$. The solution now becomes analogous to finding the thickness of a slice in 1D. The solution is $\Delta z = \frac{B_1}{|\boldsymbol{G}|}$:



$$\Delta r = \frac{B_1}{\sqrt{\eta_1^2 + G^2}} = 0.71\ cm$$

6. Pulses "talk" to different frequencies, not positions. The connection between frequency and position is made by the gradient and possible inhomogeneity. In our case, the total inhomogeneity+gradient mean that $\omega(x,z) = \gamma G z + \gamma \eta_2 x^2$. Lines of constant ω define the boundaries of the slice; these are parabolas. An exact calculation of slice profile and "thickness" is possible, although this thickness might even change as a function of position if the inhomogeneity is very nonlinear!



There Is More Than One Way of Exciting a Given Slice Thickness

B_1 is calculated out of $\gamma B_1 T = \alpha$, and $SAR = B_1^2 T$ for a constant amplitude pulse, where T is the pulse's duration.

Slice thickness (mm)	*Flip angle (Deg.)*	*B_1 (μT)*	*Bandwidth (Hz)*	*G (mT/m)*	*Duration (ms)*	*SAR ($\mu T^2 \cdot ms$)*
10	90	5.6	250	0.58	1	34.48
10	90	1.17	50	0.12	5	6.9
10	90	0.56	25	0.06	10	3.45

Possible reasons for using shorter pulses:

1. Minimize relaxation effects and signal decay during excitation.
2. Maximize G and (based on the previous problem) minimize the effects of B_0 inhomogeneity.

Possible reasons for using longer pulses:

1. Hardware might not be good enough to supply the high RF (B_1) and gradient amplitudes.
2. Less SAR.

1D Phase Encoding

1. At thermal equilibrium, our magnetization in the rotating frame starts from some initial equilibrium values along the z-axis:

$$M_z(z) = \begin{cases} M_A & z = 0 \\ M_B & z = \Delta z \\ M_C & z = 2\Delta z \\ M_D & z = 3\Delta z \\ 0 & \text{otherwise} \end{cases}$$

$$M_{xy}(z) = 0$$

Following a 90_{-y} pulse, the magnetization gets transferred to the xy-plane:

$$M_z(z)=0$$

$$M_{xy}(z)=\begin{cases} M_A & z=0 \\ M_B & z=\Delta z \\ M_C & z=2\Delta z \\ M_D & z=3\Delta z \\ 0 & \text{otherwise} \end{cases}$$

Once we turn on a gradient during the phase encoding part we create a spatially dependent frequency $\omega(z)=\gamma Gz$ in the rotating frame, and so the magnetization at point z will precess in the xy-plane and accumulate a phase $\phi(z)=\omega(z)t=\gamma Gtz$. In the four experiments, the four points will accumulate the following phases by the end of the phase encoding period (just before acquisition):

Exp	γGt	$\phi(z=0)$	$\phi(z=\Delta z)$	$\phi(z=2\Delta z)$	$\phi(z=3\Delta z)$
1	0	0°	0°	0°	0°
2	$\frac{\pi}{2\Delta z}$	0°	90°	180°	270°
3	$\frac{2\pi}{2\Delta z}$	0°	180°	360°	540°
4	$\frac{3\pi}{2\Delta z}$	0°	270°	540°	810°

Ergo, the transverse magnetization at each of these points at the end of the phase encoding period, $M_{xy}(z,t=0)e^{i\phi}$, will be (just substitute the phases and use $e^{i\pi/2}=i$, $e^{i\pi}=-1$, $e^{3\pi i/2}=-i$, etc …):

Exp	γGt	$M_{xy}(z=0)$	$M_{xy}(z=\Delta z)$	$M_{xy}(z=2\Delta z)$	$M_{xy}(z=3\Delta z)$
1	0	M_A	M_B	M_C	M_D
2	$\frac{\pi}{2\Delta z}$	M_A	iM_B	$-M_C$	$-iM_D$
3	$\frac{2\pi}{2\Delta z}$	M_A	$-M_B$	M_C	$-M_D$
4	$\frac{3\pi}{2\Delta z}$	M_A	$-iM_B$	$-M_C$	iM_D

The signal from an experiment is given by the signal equation:

$$s(t)\propto\omega_0\int\overline{B_{xy}^{(rec)}(\mathbf{r})}M_{xy}^{(rot)}(\mathbf{r},t)\,d\mathbf{r}$$

Here, $B_{xy}^{(rec)} = 1$ everywhere, ω_0 is just a proportionality constant and, once the phase encoding gradient is turned off and $\omega = 0$ at all points in the rotating frame, $M_{xy}(\mathbf{r}, t)$ is actually time independent and equal to its value at the beginning of acquisition (again, we're neglecting relaxation; had we not, we would've had to incorporate T_2 decay). Therefore, the signals from the four experiments are

$$\begin{aligned}
s_1 &= M_A + M_B + M_C + M_D \\
s_2 &= M_A + iM_B - M_C - iM_D \\
s_3 &= M_A - M_B + M_C - M_D \\
s_4 &= M_A - iM_B - M_C + iM_D
\end{aligned}$$

2. In matrix notation,

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} M_A \\ M_B \\ M_C \\ M_D \end{pmatrix}.$$

3. Using MATLAB we can define a matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

and invert it using the command inv(A), which yields

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

4. From Wikipedia (google "DFT Matrix"), this matrix is

$$W = \frac{1}{\sqrt{4}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}, \qquad \omega = e^{-\frac{2\pi i}{4}} = -i$$

so

$$W = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

We see that the reconstruction matrix (A^{-1}) is, up to a constant factor, precisely the 4^{th} order DFT matrix, while the encoding matrix is its inverse. It is not surprising therefore that the Discrete Fourier Transform (DFT) pops up repeatedly in MRI image reconstruction, as we will see throughout the course.
Incidentally, the 2^{nd} order DFT matrix is simply

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

If you recall, this is consistent with the phase encoding example shown in class in which we only used two spatial points A and B.

5. To understand how the **reconstruction matrix** changes we must first ask ourselves: how does the **encoding matrix** change when the spins are shifted? Going back to the first part we commented that the phase accumulated by spins at point z will be $\phi(z) = \gamma G t z$. Adding a constant shift δz would correspond to adding an additional phase as shown in the following table:

Exp	$\gamma G t$	$\phi(z=\delta z)$	$\phi(z=\delta z+\Delta z)$	$\phi(z=\delta z+2\Delta z)$	$\phi(z=\delta z+3\Delta z)$
1	0	$0°$	$0°$	$0°$	$0°$
2	$\frac{\pi}{2\Delta z}$	$0° + \frac{\delta z \cdot \pi}{2\Delta z}$	$90° + \frac{\delta z \cdot \pi}{2\Delta z}$	$180° + \frac{\delta z \cdot \pi}{2\Delta z}$	$270° + \frac{\delta z \cdot \pi}{2\Delta z}$
3	$\frac{2\pi}{2\Delta z}$	$0° + \frac{2\delta z \cdot \pi}{2\Delta z}$	$180° + \frac{2\delta z \cdot \pi}{2\Delta z}$	$360° + \frac{2\delta z \cdot \pi}{2\Delta z}$	$540° + \frac{2\delta z \cdot \pi}{2\Delta z}$
4	$\frac{3\pi}{2\Delta z}$	$0° + \frac{3\delta z \cdot \pi}{2\Delta z}$	$270° + \frac{3\delta z \cdot \pi}{2\Delta z}$	$540° + \frac{3\delta z \cdot \pi}{2\Delta z}$	$810° + \frac{3\delta z \cdot \pi}{2\Delta z}$

This would lead to the following transverse magnetizations at the end of the phase encoding block, just before acquisition:

Exp	γGt	$M_{xy}(z=0)$	$M_{xy}(z=\Delta z)$	$M_{xy}(z=2\Delta z)$	$M_{xy}(z=3\Delta z)$
1	0	M_A	M_B	M_C	M_D
2	$\dfrac{\pi}{2\Delta z}$	$e^{\frac{i\pi\delta z}{2\Delta z}} M_A$	$e^{\frac{i\pi\delta z}{2\Delta z}} iM_B$	$-e^{\frac{i\pi\delta z}{2\Delta z}} M_C$	$-e^{\frac{i\pi\delta z}{2\Delta z}} iM_D$
3	$\dfrac{2\pi}{2\Delta z}$	$e^{\frac{i2\pi\delta z}{2\Delta z}} M_A$	$-e^{\frac{i2\pi\delta z}{2\Delta z}} M_B$	$e^{\frac{i2\pi\delta z}{2\Delta z}} M_C$	$-e^{\frac{i2\pi\delta z}{2\Delta z}} M_D$
4	$\dfrac{3\pi}{2\Delta z}$	$e^{\frac{i3\pi\delta z}{2\Delta z}} M_A$	$-e^{\frac{i3\pi\delta z}{2\Delta z}} iM_B$	$-e^{\frac{i3\pi\delta z}{2\Delta z}} M_C$	$ie^{\frac{i3\pi\delta z}{2\Delta z}} M_D$

The corresponding signals in the four experiments will be

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{\frac{i\pi\delta z}{2\Delta z}} & ie^{\frac{i\pi\delta z}{2\Delta z}} & -e^{\frac{i\pi\delta z}{2\Delta z}} & -ie^{\frac{i\pi\delta z}{2\Delta z}} \\ e^{\frac{i2\pi\delta z}{2\Delta z}} & -e^{\frac{i2\pi\delta z}{2\Delta z}} & e^{\frac{i2\pi\delta z}{2\Delta z}} & -e^{\frac{i2\pi\delta z}{2\Delta z}} \\ e^{\frac{i3\pi\delta z}{2\Delta z}} & -ie^{\frac{i3\pi\delta z}{2\Delta z}} & -e^{\frac{i3\pi\delta z}{2\Delta z}} & ie^{\frac{i3\pi\delta z}{2\Delta z}} \end{pmatrix} \begin{pmatrix} M_A \\ M_B \\ M_C \\ M_D \end{pmatrix}$$

Now, you can invert this matrix by brute force, even analytically, but there's a trick here: the encoding matrix can be decomposed into the product of two sub-matrices:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} \eta^0 & 0 & 0 & 0 \\ 0 & \eta^1 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 0 & \eta^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} M_A \\ M_B \\ M_C \\ M_D \end{pmatrix}$$

where

$$\eta^0$$

$$\eta^1 = e^{\frac{i\pi\delta z}{2\Delta z}}$$

$$\eta^2 = \left(e^{\frac{i\pi\delta z}{2\Delta z}} \right)^2 = e^{\frac{i2\pi\delta z}{2\Delta z}}$$

$$\eta^3 = \left(e^{\frac{i\pi\delta z}{2\Delta z}} \right)^3 = e^{\frac{i3\pi\delta z}{2\Delta z}}$$

Inverting this becomes easy because matrix algebra tells us $(AB)^{-1} = B^{-1}A^{-1}$, we've already inverted the encoding matrix (and got the DFT matrix), and inverting a diagonal matrix is particularly easy (just invert each element along the diagonal):

$$\begin{pmatrix} M_A \\ M_B \\ M_C \\ M_D \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} \eta^0 & 0 & 0 & 0 \\ 0 & \eta^{-1} & 0 & 0 \\ 0 & 0 & \eta^{-2} & 0 \\ 0 & 0 & 0 & \eta^{-3} \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$$

Bottom line: all you need to do is apply a "phase correction" term to the reconstructed signal at each position (no correction at z=A, $\eta^{-1} = e^{-\frac{i(\pi\delta z)}{2\Delta z}}$ at z=B, η^{-2} at position C, etc ...)