Index Gymnastics: Gauss' Theorem, Isotropic Tensors, NS Equations

The purpose of today's TA session is to mess a bit with tensors and indices, which are a necessary tool for continuum theories and in particular for Solid Mechanics. We'll see some simple examples and try to become comfortable with these mathematical tools. If time permits we'll discuss the subject of dimensional analysis which, although very basic, is sometimes not understood well enough.

1 Isotropic tensors

A tensor is called *isotropic* if its coordinate representation is independent under coordinate rotation. Let's look at all the possible forms of isotropic tensors of low ranks.

1.0 0^{th} rank tensors

A 0^{th} rank tensor, a.k.a a scalar, does not change under rotations, therefore all scalars are isotropic (surprise!).

1.1 1^{st} rank tensors

A vector $\vec{v}$ is isotropic if for every rotation matrix R_{ij} we have

$$R_{ij}\, v_j = v_i \ . \tag{1}$$

You can easily show that this condition is satisfied for arbitrary $\boldsymbol{R}$ only if $\vec{v} = 0$. So the zero vector is the only isotropic vector (surprise #2!!).

1.2 2^{nd} rank tensors

Let's hope we're gonna get something a bit more interesting. A matrix $\boldsymbol{A}$ is isotropic if for every rotation matrix $\boldsymbol{R}$ we have $A_{ij} = R_{ik}R_{jl}A_{kl}$, or in matrix notation:

$$\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^T = \boldsymbol{A} \ . \tag{2}$$

Let's choose a specific rotation matrix, say a rotation of angle α around $\hat{z}$,

$$\boldsymbol{R}^z(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{3}$$

The invariance equation now takes the form

$$\boldsymbol{A}(0) = \boldsymbol{A}(\alpha) \equiv \boldsymbol{R}^z(\alpha)\boldsymbol{A}\,\boldsymbol{R}^z(\alpha)^T = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4}$$

This is a complicated equation, with cosines and sines all mixed up in a very unpleasant manner. Luckily, we can find an equivalent condition that is significantly simpler. Differentiating with respect to α and plugging $\alpha = 0$ gives

$$0 = \frac{\partial \boldsymbol{A}(0)}{\partial \alpha} = \left.\frac{\partial \boldsymbol{A}(\alpha)}{\partial \alpha}\right|_{\alpha=0} = \left.\frac{\partial \boldsymbol{R}^z(\alpha)}{\partial \alpha}\right|_{\alpha=0} \boldsymbol{A}\, \boldsymbol{R}^z(0) + \boldsymbol{R}^z(0)\, \boldsymbol{A}\ \left.\frac{\partial \boldsymbol{R}^z(\alpha)^T}{\partial \alpha}\right|_{\alpha=0} , \tag{5}$$

but since $\boldsymbol{R}^z(0)$ is the identity matrix, this reduces to the simple equation

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\equiv \boldsymbol{L}^z} \boldsymbol{A} + \boldsymbol{A} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \ . \tag{6}$$

$\boldsymbol{L}^z = \partial_\alpha \boldsymbol{R}|_{\alpha=0}$ is sometimes called "the generator of rotations around the z axis", because $\boldsymbol{R}^z(\alpha) = e^{\alpha \boldsymbol{L}^z}$. We see that equation $\boldsymbol{A}(0) = \boldsymbol{A}(\alpha)$ is equivalent to the much easier equation (notice the sign change)

$$\boldsymbol{A}(0) = \boldsymbol{A}(\alpha) \quad \iff \quad [\boldsymbol{A}, \boldsymbol{L}^z] = 0 \ . \tag{7}$$

Explicitly calculating $[\boldsymbol{A}, \boldsymbol{L}^z]$ gives

$$[\boldsymbol{A}, \boldsymbol{L}^z] = \begin{pmatrix} -A_{12} - A_{21} & A_{11} - A_{22} & -A_{23} \\ A_{11} - A_{22} & A_{12} + A_{21} & A_{13} \\ -A_{32} & A_{31} & 0 \end{pmatrix} \ . \tag{8}$$

We see that commutation with $\boldsymbol{L}^z$ requires (a) $A_{13} = A_{31} = A_{23} = A_{32} = 0$ and (b) $A_{11} = A_{22}$. Obviously, the choice of $\hat{z}$ is arbitrary and isotropy means that $\boldsymbol{A}$ should also commute with $\boldsymbol{L}^x$ and $\boldsymbol{L}^y$. If we repeat the above procedure for the other $\boldsymbol{L}$'s, the analog of (a) will be that all off-diagonal elements must vanish, and the analog of (b) will be that all diagonal elements must be equal. That is,

$$A_{ij} \propto \delta_{ij} \ . \tag{9}$$

I stress that this is true only in dimensions ≥ 3. In the HW you'll see that in 2D there are isotropic tensors that are not proportional to the identity (can you already see how the above argument fails in 2D?).

1.3 3$^{\text{rd}}$ rank tensors

Here we can use the same trick. A 3$^{\text{rd}}$ rank tensor $\boldsymbol{A}$ is isotropic iff for every rotation matrix R_{ij} we have

$$R_{i\alpha}\, R_{j\beta}\, R_{k\gamma} A_{\alpha\beta\gamma} = A_{ijk} \ . \tag{10}$$

You can imagine the mess that comes out of this if you plug in a real rotation matrix with sines and cosines and whatnot, and then start using trig identities. Phew, no thanks!

So like before, we choose $\boldsymbol{R} = \boldsymbol{R}^z(\alpha)$, differentiate, and set $\alpha = 0$. This gives

$$\begin{aligned} 0 &= \Big(L^z_{i\alpha}\, \delta_{j\beta}\, \delta_{k\gamma} + \delta_{i\alpha}\, L^z_{j\beta}\, \delta_{k\gamma} + \delta_{i\alpha}\, \delta_{j\beta}\, L^z_{k\gamma} \Big) A_{\alpha\beta\gamma} \\ &= L^z_{i\alpha}\, A_{\alpha jk} + L^z_{j\beta}\, A_{i\beta k} + L^z_{k\gamma}\, A_{ij\gamma} \ . \end{aligned} \tag{11}$$

To see what kind of equation we got, let's choose $i = 1,\ j = 3,\ k = 3$. Since the only non-zero elements of $\boldsymbol{L}^z$ are L^z_{12} and L^z_{21}, we get

$$0 = L^z_{1\alpha}\, A_{\alpha 33} + L^z_{3\beta}\, A_{1\beta 3} + L^z_{3\gamma}\, A_{13\gamma} = A_{233}\ . \tag{12}$$

Similarly, by choosing different combinations of i, j, k and/or different $\boldsymbol{L}$'s, you get that $A_{ijk} = 0$ whenever i, j, k are not all different, that is, if (ijk) is not a permutation of (123).

Using this knowledge, we can choose now $i = 1,\ j = 1,\ k = 3$, and we get

$$A_{113} = 0 = L^z_{1\alpha}\, A_{\alpha 13} + L^z_{1\beta}\, A_{1\beta 3} + L^z_{3\gamma}\, A_{11\gamma} = A_{213} + A_{123}\ ,$$

or put differently, $A_{213} = -A_{123}$. Similarly, we can show that every time we flip two indices we get a minus sign. Therefore, we conclude that the only isotropic 3rd rank tensor is equal, up to a multiplicative constant, to $\boldsymbol{\mathcal{E}}$,

$$\mathcal{E}_{ijk} = \begin{cases} 0 & (ijk) \text{ is not a permutation of } (123) \\ \text{sign of permutation} & \text{otherwise} \end{cases}\ . \tag{13}$$

As you probably know, $\boldsymbol{\mathcal{E}}$ is called the Levi-Civita completely anti-symmetric tensor[1].

1.4 4th rank tensors

Seriously? No. We're not going to redo the algebra. But can we guess the form of some isotropic 4th rank tensors? We can easily build them from lower rank isotropic tensors. Here are a few examples that come to mind:

$$A_{ijkl} = \delta_{ij}\, \delta_{kl}\ , \tag{14}$$
$$A_{ijkl} = \delta_{il}\, \delta_{jk}\ , \tag{15}$$
$$A_{ijkl} = \delta_{ik}\, \delta_{jl}\ , \tag{16}$$
$$A_{ijkl} = \mathcal{E}_{ij\alpha}\, \mathcal{E}_{\alpha kl}\ . \tag{17}$$

We did a really good job there, because it turns out that these are the only options. In fact, this list is even redundant, because each of the lines can be written as a linear combination of the other three (can you find it?). You may want to prove at home that there really are no other options - it's a nice exercise that can be easily automatized on `Mathematica`, and we're going to use this result in the course.

2 Navier-Stokes equation

We are now going to use the heavy arsenal developed above, and derive the Navier-Stokes (NS) equation solely from symmetry considerations. We want to find a dynamical

[1] This is true only in flat spaces. Those of you familiar with differential geometry might insist on calling it a "Tensor density". Since we are (thankfully) only considering flat space here, we'll disregard this subtelty.

equation for $\partial_t \vec{v}$ as a function of $\vec{v}$ and its spatial derivatives. We take a perturbative approach, and expand $\partial_t v$ to second order in $\vec{v}$ and in its gradients:

$$\partial_t v_i = A_{ij}\, v_j + B_{ijk}\, \partial_j v_k + D_{ijkl}\, v_j \partial_k v_l + E_{ijkl}\, \partial_j \partial_k v_l + F_{ijk}\, v_j\ v_k\ . \tag{18}$$

Since $\vec{v}$ is a physical quantity (specifically, a 1^{st} rank tensor), the dynamical equation for $\partial_t \vec{v}$ should be covariant under symmetries of the physical system in question. We'll see what these symmetries impose on the form of the various tensors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{F}$.

We begin with a Galilean transformation:

$$y_i = x_i - c_i t\ , \tag{19}$$

$$\tau = t\ . \tag{20}$$

Under this transformation, the velocity field now takes the form $\vec{w} = \vec{v} - \vec{c}$. Also, by the chain rule:

$$\partial_{x_i} = \frac{\partial y_j}{\partial x_i} \partial_{y_j} + \frac{\partial \tau}{\partial x_i} \partial_\tau = \partial_{y_i}\ , \tag{21}$$

$$\partial_t = \frac{\partial \tau}{\partial t} \partial_\tau + \frac{\partial y_j}{\partial t} \partial_{y_j} = \partial_\tau - c_j \partial_{y_j}\ . \tag{22}$$

Applying this to Eq. (18) gives

$$\begin{aligned} \partial_t w_i - c_j \partial_j w_i = A_{ij}(w_j + c_j) + B_{ijk} \partial_j w_k + D_{ijkl}(w_j + c_j) \partial_k w_l \\ + E_{ijkl} \partial_j \partial_k w_l + F_{ijk}(w_j + c_j)(w_k + c_k)\ . \end{aligned} \tag{23}$$

If we want the NS equation to be covariant, we need to impose that Eq. (23) will be equal, term by term, to Eq. (18), i.e.

$$A_{ij} c_j = 0\ , \tag{24}$$

$$-c_j \partial_j w_i = D_{ijkl}\, c_j \partial_k w_l, \tag{25}$$

$$F_{ijk}(w_j + c_j)(w_k + c_k) = F_{ijk} w_j w_k\ . \tag{26}$$

All these should hold for arbitrary $\vec{c}, \vec{w}$. The first constraint clearly means $\boldsymbol{A} = 0$. For the third one, choose for example $\vec{w} = -\vec{c}$, and get that $F_{ijk} w_j w_k = 0$ for arbitrary $\vec{w}$. Note that this is exactly the last term in Eq. (18), so we see that it vanishes identically. The constraint Eq. (25) may be written as

$$-\delta_{il} \delta_{jk}\ c_j \partial_k w_l = D_{ijkl}\ c_j \partial_k w_l\ .$$

Since $\vec{c}, \vec{w}$ are arbitrary, $D_{ijkl} = -\delta_{il} \delta_{kj}$, and Eq. (18) can be written as

$$\partial_t v_i + v_j \partial_j v_i = B_{ijk} \partial_j v_k + E_{ijkl} \partial_j \partial_k v_l\ . \tag{27}$$

You have to admit that this is a very big improvement...

Now let's look at rotations $y_j = R_{ij} x_j$. Demanding Eq. (27) to be invariant means that the tensors $\mathbf{B}, \mathbf{E}$ are *isotropic*.

We've just seen that the only 3rd rank isotropic tensor is the Levi-Civita tensor, so the $\boldsymbol{B}$ term is proportional to $\vec{\nabla} \times \vec{v}$ and thus is forbidden by reflection symmetry. It's too bad that we know already that the $\boldsymbol{A}$ and $\boldsymbol{F}$ terms are gone, because they would also be forbidden by rotational symmetry. For example, the $\boldsymbol{F}$ term must be proportional to $\vec{v} \times \vec{v}$ and therefore vanishes identically (note that we didn't show that $\boldsymbol{F} = 0$, but only that it gives zero when it acts on the same vector in its two slots).

As for $\mathbf{E}$, we know that we have exactly three choices, given in Eqs. (14),(15),(16). These give, respectively,

$$\delta_{ij}\,\delta_{kl}\,\partial_j\partial_k\,v_l = \partial_i\partial_j\,v_j = \vec{\nabla}\,(\nabla\cdot\vec{v}) = \text{grad}\,(\text{div}\,\vec{v}) \;\; , \tag{28}$$

$$\delta_{il}\,\delta_{jk}\,\partial_j\partial_k\,v_l = \partial_j\partial_j\,v_i = \nabla^2\vec{v} = \text{div}\,(\text{grad}\,\vec{v}) \;\; , \tag{29}$$

$$\delta_{ik}\,\delta_{jl}\,\partial_j\partial_k\,v_l = \partial_i\partial_j\,v_j = \text{same as Eq. (28)} \;\; , \tag{30}$$

so the third option is redundant. Note that if we wanted to use Eq. (17) we'd get

$$\mathcal{E}_{ij\alpha}\,\mathcal{E}_{\alpha kl}\partial_j\partial_k\,v_l = \mathcal{E}_{ij\alpha}\,\partial_j\left(\vec{\nabla}\times\vec{v}\right)_\alpha = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{v}\right) \;\; ,$$

which is also redundant because of the vector calculus identity which you all know by heart: $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$.

To sum up, we see that the only form of $\partial_t\vec{v}$ which is covariant under rotations, reflections and Galilean symmetries is

$$\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = \eta\nabla^2\vec{v} + \mu\vec{\nabla}\,(\nabla\cdot\vec{v}) \;\; , \tag{31}$$

where η and μ are two scalars. In incompressible flows, $(\nabla \cdot \vec{v})$, there's only one η, as the μ term vanishes.

Lastly, note that there's another term that clearly does not violate any symmetries: $\vec{\nabla}P$ where P is some scalar function.

2.1 An historical note about the power of symmetries in continuum theories

Euler's equation $(\partial_t + v_j\partial_j)v_i = \partial_i P$, regarding inviscid incompressible flows, was derived sometime around 1750. It took the scientific community almost 80 years (!!) to understand how to incorporate viscosity into the business. Mind you, some of the greatest minds of the time were devoted to the problem, including Cauchy, Poisson, d'Alembert, Bernoulli, and of course, Navier and Stokes. Not exactly Elitzur Ra'anana, if you see what I mean. So what took them so long?

The answer, very very roughly, is that they tried to model viscosity on a molecular level: to understand the dissipation mechanisms, stress-transfer mechanisms, and whatnot. One of the great strengths of continuum theory is that measly insignificant mortals like us were able to do here in 45 minutes a derivation that the primordial gods needed 80 years to do. Moreover, we did that *without caring even the slightest bit about the underlying physics*.

In fact, this is the crux of the matter – the use of symmetries allows us to state very powerful statements about the functional form of the viscosity term, without having to deal with the microscopic mechanisms. It allows us to develop a predictive theory, where all the "microscopics" are lumped into a small number of parameters (in our case - μ and η), which of course must be determined experimentally.

The down side is that we can not say anything quantitative about the parameters. From our theory we can not give even an order-of-magnitude estimation of η or μ, let alone their dependence on the fluid's properties (although thermodynamics easily tells us that they are positive).

3 Dimensional analysis and π theorem

This is a bit of a detour from the course, but surprisingly many students are not familiar enough with this subject. Like unhappy families, every unfortunate scientific idea is unfortunate in its own way. Many of those who have taught dimensional analysis (or have merely thought about how it should be taught) have realized that it has suffered an unfortunate fate. In fact, the idea on which dimensional analysis is based is very simple, and can be understood by everybody: physical laws do not depend on arbitrarily chosen basic units of measurement. An important conclusion can be drawn from this simple idea, using a simple argument: the functions that express physical laws must possess a certain fundamental property, which in mathematics is called generalized homogeneity, or symmetry. This property allows the number of arguments in these functions to be reduced, thereby making it simpler to obtain them (by calculating them or determining them experimentally). This is, in fact, the entire content of dimensional analysis - there is nothing more to it. Yet, dimensional analysis was cursed and reproached for being untrustworthy and unfounded, even mystical. Paradoxically, the reason for this lack of success was that only a few people understood the content and real abilities of dimensional analysis.

I'll state the crux of the argument again: *Every physical functional relation can be formulated in terms of dimensionless function of dimensionless variables.* This statement is also called the Buckingham π theorem. Let's look at the most simple example I know of. Consider an ideal pendulum of length ℓ, with a mass m hanging at the bottom, in a gravity field of constant acceleration g. Let's say someone, maybe Galileo Galilei, told you that he noticed that for small deflections of the mass from its equilibrium position[2], the period of oscillation τ was independent of the amplitude of deflection. How would you go about trying to figure out what this amplitude was? Dimensional analysis is the answer!

If you were to know nothing else about this system, the most general relationship you should consider is something like

$$\tau = f(m, \ell, g) \ , \tag{32}$$

but, recalling our theorem, we'll look at what dimensionless parameters we can construct from our physical quantities $\{\tau, m, \ell, g\}$. We find that only one such parameter can exist

[2] I'm not sure if these are the exact words Galileo would have used.

$\sqrt{\frac{g}{\ell}}\tau$. This means that this combination has to be constant, therefore

$$\tau = c\sqrt{\frac{\ell}{g}} \ , \tag{33}$$

which is a remarkable result, considering that we know almost nothing about the system and no equations were solved. Fully formulating and solving the problem would of course tell us that $c = 2\pi$, but consider that even if we would not know how to do so, we can measure just one pendulum, find the value of c, and know it happily ever after for any other pendelum.

What happens if we relax the requirement of small deflections? Then we introduce a new parameter, α, the initial angle. This means that we now have two dimensionless combinations, so we have one function of one parameter

$$\sqrt{\frac{g}{\ell}}\tau = f(\alpha) \qquad \Rightarrow \qquad \tau = \sqrt{\frac{\ell}{g}}f(\alpha) \ . \tag{34}$$

We actually know f, which is an elliptical integral, but it's beyond the point right now. Consider the power of this technique. All you have to do is to measure enough values of the function f for one pendulum, and again you know enough about all pendelums in the world.

As a final example I'll use an historical anecdote. During the early days of atomic testing, the American government published a series of photos from a nuclear test, but kept the details of the explosion classified. British physicist Taylor looked at the images and realised that it gave him enough data to calculate the amount of energy released in the blast!

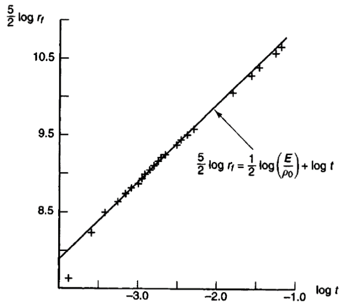

Figure 1: The experimental points determined by Taylor from the movie film lay on a single straight line with slope unity in the coordinates $\log t$, $(5/2)\log r_f$. Taylor was thus able to determine the energy of the explosion from the series of photographs.

Let's get to it. The quantities we need to consider are the radius at a specific time r, the time since detonation t, the energy released E, and the initial air density ρ. Looking at all the possible combinations $\Pi = r\rho^\gamma E^\beta t^\alpha$ we find that the only possible dimensionless combination is

$$\alpha = -\frac{2}{5}, \beta = -\frac{1}{5}, \gamma = \frac{1}{5} \Rightarrow \Pi = \frac{\rho^{1/5} r}{E^{1/5} t^{2/5}} \ , \tag{35}$$

which as before, being the only dimensionless quantity, has to be constant, so

$$r = \frac{cE^{1/5}}{\rho^{1/5}} t^{2/5} \ . \tag{36}$$

Thus, in a Log Log plot, we should expect to see a straight line, as can be seen in Fig. 1.

From the y axis intersection a value for $cE^{1/5}\rho^{-1/5}$ can be found, and knowing ρ and assuming c to be unity[3] Taylor got quite a good estimate for E. At the time, Taylor's publication of this value (which turned out to be approximately 1021 erg) caused, in his words, "much embarrassment" in American government circles: this figure was considered top secret, even though the film was not classified.

[3] A similar problem in gas dynamics has shown c to be about unity.