Non-Equilibrium Continuum Physics **HW set #3**

TA: Avraham Moriel and Yuri Lubomirsky Submit to: groupbouchbinder@gmail.com

Due 27.05.21 Submit as: NECP_HW3_⟨FirstName⟩_⟨LastName⟩.pdf

Elasticity and Thermo-Elasticity

1. Consider a 3D rectangular box, subject to uniaxial stress σ_0 in the z direction, as shown in Fig. 1. The faces in the x, y directions are traction-free. The rest-lengths of the boxes sides are a, b. Calculate the slope of the dashed line as a function of σ_0, E and ν. When $\sigma_0 \geq E$ something weird happens. Is linear elasticity wrong?

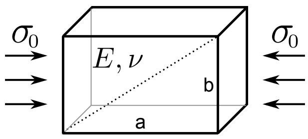

Figure 1: Box under uniaxial compression.

2. As a reminder, the Hertz Problem concerns two elastic bodies in contact. This problem was initially posed by Hertz when he considered Newton diffraction rings on the contact of two lenses. He solved the problem on his Xmas vacation in 1880, when he was 23 years old[1], and his treatment became canonical. In class, Eran "solved" this problem scaling-wise, and here you'll work out the solution a bit more carefully.

   The generic case is that of two paraboloids, and is justified by the assumption that near the contact point the surface of both bodies can be expanded to second order. The two bodies may have different radii of curvature and also different elastic properties. This case is completely tractable, and the procedure goes through reducing the problem to the case of contact between a rigid half-plane and an elastic sphere with some effective radius and elastic properties. We will not go through this reduction, and assume at the outset that this is the case - a sphere of radius R and elastic moduli E and ν is pressed against a rigid half plane with force F.

   The force induces a global displacement of δ. That is, the displacement of distant points in the body is $\boldsymbol{u} \approx \delta \hat{z}$. Cylindrical symmetry tells us that the area of contact will be a circle of radius a. The problem is to find the relationship between a, F, and δ, and also to find the pressure distribution on the contact. We will assume that $a \ll R$ (otherwise the strain is not small).

   (a) Assuming we have pressed the sphere down by an amount δ, creating an area of contact in a circle of radius a. What are the boundary conditions on the sphere now? Notice that we have a mixed BC - that is one BC in part of the system, and another BC in the other.

[1] What did you do by the time you were 23? I definitely did nothing as impressive.

(b) OK, but what next? Hertz pulled a dirty trick here. Since $a \ll R$, he assumed that the sphere can be treated as a half plane for the purposes of calculating the elastic responses. That is, he invoked the Green's function for a point load on a half surface (Eq. (5.26) in Eran's lecture notes). Following his path, use the Green's function formulation to write an integral equation for the normal stress field p_z, using the results of the previous question. Write down the results in polar coordinates, including the boundaries of integration.

(c) The form of p_z that solves the equation is

$$p_z(r) = p_0\sqrt{1-\frac{r^2}{a^2}} \qquad , \qquad r < a \ , \tag{1}$$

the proof of which is very cumbersome, so I won't give here. Of course, for $r > a$ $p_z = 0$. If any of you are interested, it can be found in K. L. Johnson's Contact mechanics, which is a great book with many solutions to all kind of contact problems. Setting Eq. (1) in Eq. (5.26) in Eran's lecture notes leads to

$$u_z(r) = \frac{\pi\left(1-\nu^2\right)p_0}{4aE}\left(2a^2 - r^2\right) \qquad , \qquad r < a \ . \tag{2}$$

Now finish it off and write down a, δ, and p_0 in terms of the material parameters, the loading force F, and the radius of the sphere R. Verify that it indeed agrees with Eran's scaling analysis.

(d) Small bonus - consider and tell us about your greatest achievement by the time you were 23. How does this compare with finding the solution to the elastic contact problem?

(e) Large bonus - Try to prove that the stress distribution given in Eq. (1) actually leads to the displacement given in Eq. (2). Keep in mind that it's rather long and complicated, so do it only if you really like this kind of stuff.

3. A rod of radius R and length L is pointed along the z direction, and is held between two rigid walls at $z = 0$ and $z = L$. It is free of constraints in the other two dimensions. The rod is then uniformly heated by some amount ΔT. Without solving the problem completely (although you may do it, if you have nothing better to do) estimate how the

   (a) stresses in the rod,

   (b) strains in the rod,

   (c) forces on the walls,

   (d) elastic energy stored in the rod

   scale with R, L and ΔT.

4. Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

5. Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x, y, z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

   Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

   Guidance: Begin with Navier-Lamé equation $(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.