Non-Equilibrium Continuum Physics **HW set #4**

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Finite elasticity

1. A Saint-Venant material is defined through the following elastic energy functional

$$u(\boldsymbol{E}) = \frac{\lambda}{2}\,\text{tr}^2\,\boldsymbol{E} + \mu\;\text{tr}\,\boldsymbol{E}^2\;, \tag{1}$$

where $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$ is the Green-Lagrange strain tensor. That is, the energy is the same as the linear-elastic energy, but the Green-Lagrange tensor $\boldsymbol{E}$ is used instead of the linearized Cauchy tensor $\boldsymbol{\varepsilon}$.

(a) Calculate the second Piola-Kirchhoff stress tensor $\boldsymbol{S} \equiv \frac{\partial u}{\partial \boldsymbol{E}}$ and the first Piola-Kirchhoff stress tensor $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$ in terms of the displacement gradient tensor $\boldsymbol{H}$.

(b) Plot the uniaxial stress-strain relation for both a Saint-Venant material and a linear elastic Hookean material setting the first Lamé constant to zero, $\lambda = 0$. That is, stretch the z-direction by λ_z while keeping $\lambda_x = \lambda_y = 1$ (do not confuse with Lamé's constant λ, sorry ☹) , and plot all the non zero components of $\boldsymbol{P}/\mu$ vs. the strain $\varepsilon \equiv \lambda_z - 1$. Plot both positive and negative values of the strain ε (say, between $-1 \le \varepsilon \le 1$), i.e. consider both tension and compression. Is there a problem with the Saint-Venant material? Hint: monotonicity.

(c) The last result shows that a Saint-Venant material is different from a linear elastic Hookean material, even though they are both defined through the Lamé coefficients λ and μ, and share the same functional structure. What is the origin of this difference? One way to look at this is to consider a dilation and a pure shear, i.e. the following deformation gradient:

$$\boldsymbol{F} = (1+\varepsilon)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\;. \tag{2}$$

The first term is pure volumetric dilation and the second term is pure shear (in the xy plane). Show that in the small deformation limit shear and dilation are decoupled (energy-wise), but not in the finite deformation case.

(d) In class we discussed the incompressible neo-Hookean material for which the uniaxial relation reads

$$\frac{P}{\mu} = \lambda_y - \lambda_y^{-2}\;. \tag{3}$$

Plot P/μ vs. $\varepsilon = \lambda_y - 1$ for this material, together with its linear elastic approximation, for both tension and compression. What is the qualitative difference as compared to question

Saint-Venant material (irrespective of the problem that you found in the previous question)? Think of pulling/compressing the materials beyond the linear regime.

2. As we discussed in class, linear elasticity is a first order perturbation theory in $\boldsymbol{H}$. A Saint-Venant constitutive law can be regarded as a first order perturbation theory in $\boldsymbol{E}$ (note, however, that it mixes up different orders of $\boldsymbol{H}$). Suppose now we want to go to the next (3^{rd}) order in $\boldsymbol{E}$.

   (a) Write down the most general *isotropic* energy functional for such a theory. It is called second order elasticity. How many new (second order) elastic constants one has to define? (you **don't have to** give a rigorous proof).

   (b) *QUALITATIVE QUESTION:* What is the physical origin of the second order elastic constants? Can you think of other physical quantities that have a related origin? How can one measure them in the experiment (when the fully nonlinear constitutive law in not known)?