Non-Equilibrium Continuum Physics **HW set #2**

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Due 13.05.2021 Submit as: NECP_HW2_⟨FirstName⟩_⟨LastName⟩.pdf

Kinematics - Solution

1. Consider the following 2D deformation:

$$x_1(t) = \cosh(t)X_1 + \sinh(t)X_2 \ , \qquad x_2(t) = \sinh(t)X_1 + \cosh(t)X_2 \ .$$

(a) Find the material velocity and the acceleration $\boldsymbol{V}, \boldsymbol{A}$ and express their spatial forms $\boldsymbol{v}, \boldsymbol{a}$. Remember to represent each field in the proper coordinates (i.e. $\boldsymbol{V}, \boldsymbol{A}$ in terms of $\boldsymbol{X}$ and $\boldsymbol{v}, \boldsymbol{a}$ in terms of $\boldsymbol{x}$). Plot schematically $\boldsymbol{V}$ and $\boldsymbol{v}$ at $t = -10, 0, 10$. Note how vastly different $\boldsymbol{V}$ and $\boldsymbol{v}$ are!

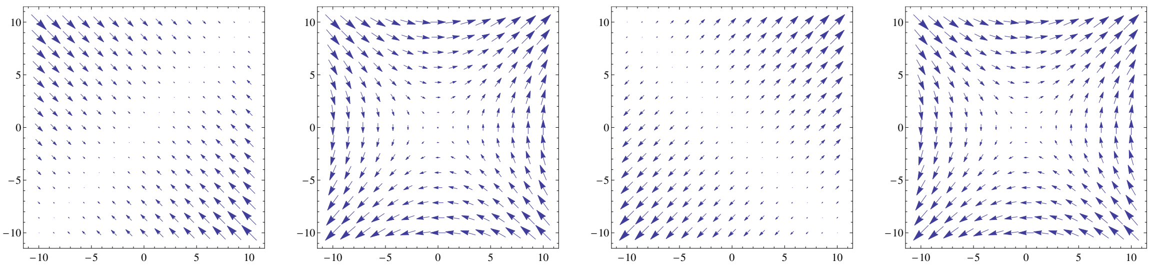
Solution

$$\boldsymbol{V} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sinh(t)X_1 + \cosh(t)X_2 \\ \cosh(t)X_1 + \sinh(t)X_2 \end{pmatrix} \ .$$

Note that this can be simply expressed as $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$, so we also found $\boldsymbol{v} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$.

Similarly,

$$\boldsymbol{A} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \cosh(t)X_1 + \sinh(t)X_2 \\ \sinh(t)X_1 + \cosh(t)X_2 \end{pmatrix} \ , \qquad \text{and } \boldsymbol{a} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$V(t = -10)$ $V(t = 0)$ $V(t = 10)$ v (for all t)

Note that $\boldsymbol{V}$ changes exponentially in time while $\boldsymbol{v}$ is constant (!!). This goes to show how different things may look like if they're presented as a function of $\boldsymbol{X}$ or $\boldsymbol{x}$.

(b) The acceleration $\boldsymbol{a}$ can also be calculated as a material derivative of the velocity:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \ .$$

Calculate $\boldsymbol{a}$ using this expression, and show that the results coincide.

Solution

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} = \vec{0} + (v_1 \partial_{x_1} + v_2 \partial_{x_2})\, \boldsymbol{v} = (x_2 \partial_{x_1} + x_1 \partial_{x_2}) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(c) Calculate $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ and $J = \det \boldsymbol{F}$ (we will use it in Q4).

Solution

$$\boldsymbol{F} = \begin{pmatrix} \partial_{X_1} x_1 & \partial_{X_2} x_1 \\ \partial_{X_1} x_2 & \partial_{X_2} x_2 \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} ,$$

and clearly $J = \det \boldsymbol{F} = 1$.

2. Solve these apparent contradictions:

(a) One may claim that $\nabla_{\boldsymbol{x}} \boldsymbol{v} \equiv 0$ because

$$\nabla_{\boldsymbol{x}} \boldsymbol{v} = \nabla_{\boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial}{\partial x_j} \frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x_i}{\partial x_j} = \frac{\partial \delta_{ij}}{\partial t} = 0 ,$$

is this true (hint: no)? What is wrong with this reasoning?

Solution

$\partial_t(\cdot)$ is defined to be $\left.\frac{\partial(\cdot)}{\partial t}\right|_{\boldsymbol{X}}$. Thus, ∂_t and ∂_x do not commute , but ∂_t and ∂_X do. To see this more explicitly, note that the expression $\nabla_{\boldsymbol{x}} \boldsymbol{v}$ is actually shorthand for

$$\nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x}, t) = \nabla_{\boldsymbol{x}} \partial_t \boldsymbol{\varphi}(\boldsymbol{X}(\boldsymbol{x}, t), t)$$

so you see that $\boldsymbol{x}$ is also time dependent.

(b) In Eq. (46) of TA session #3 we used the fact that $D_t \boldsymbol{x} = \boldsymbol{v}$ (there we denoted $\boldsymbol{x}$ by $\boldsymbol{r}$). One may claim that there's a factor of 2 missing, since

$$D_t \boldsymbol{x} \equiv \partial_t \boldsymbol{x} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{x} = \boldsymbol{v} + \boldsymbol{v} \boldsymbol{I} = 2\boldsymbol{v} .$$

Is this true (hint: no)? What is wrong with this reasoning?

Solution

Remind yourselves the derivation of the equation for the material derivative, Eqs. (3.7-8) in Eran's notes:

$$\begin{aligned}\frac{Df(\boldsymbol{x},t)}{Dt} &= \left(\frac{\partial f(\boldsymbol{\varphi}(\boldsymbol{X},t),t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} \\ &= \left(\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right)_{t}\left(\frac{\partial \boldsymbol{\varphi}(\boldsymbol{X},t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} . \end{aligned} \tag{1}$$

That is, in the above we should interpret $\partial_t \boldsymbol{x}$ as the time derivative of $\boldsymbol{x}$ when $\boldsymbol{x}$ is kept constant. In other words, it is strictly zero.

3. We use quite freely in class $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-T}$ and so on. What is the physical meaning of the assumption that $\boldsymbol{F}$ is always an invertible matrix?

Solution

$\det \boldsymbol{F}$ is the ratio of an infinitesimal volume element in the material coordinates to its volume in the deformed configuration. If $\boldsymbol{F}$ is non invertible, i.e. $\det \boldsymbol{F} = 0$, then the motion takes an infinitesimal volume and "squishes" it to a plane (or a line, or a point). That is, if $\boldsymbol{F}$ is non-invertible the motion maps a triad of basis vectors in the material coordinates $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}$ to a linearly dependent set $\{\boldsymbol{F}\boldsymbol{X}_1, \boldsymbol{F}\boldsymbol{X}_2, \boldsymbol{F}\boldsymbol{X}_3\}$ and the images of the basis vectors are co-planar and do not span a volume. Since we do not allow such a situation (what would you do with mass conservation then?), we assume that $\boldsymbol{F}$ is invertible.

Note that demanding that $\boldsymbol{F}$ is invertible is a stronger assumption than assuming that $\boldsymbol{\varphi}$ is invertible. Consider the motion

$$x_1 = X_1^3 , \qquad x_2 = X_2 , \qquad x_3 = X_3 .$$

This is clearly an invertible motion but $\det \boldsymbol{F}$ vanishes at $\boldsymbol{X} = 0$.

A side note for the rigorous-mathematics-oriented students: We just saw that the fact that $\boldsymbol{\varphi}$ is invertible does not imply that $\boldsymbol{F}$ is invertible. However, the other direction kind of works: the inverse-function theorem says that if $\det \boldsymbol{F} \neq 0$ then $\boldsymbol{\varphi}$ is *locally* invertible (i.e. that if $\det \boldsymbol{F} \neq 0$ at a point then there's a small environment around this point where $\boldsymbol{\varphi}$ is invertible).

4. The purpose of this exercise is to prove the relation $\partial_t J = J \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}$, and in the meanwhile to get a better intuition about how tensorial derivatives work. This relation was used in class in deriving the mass continuity equation (Eq.(4.4) in the lecture notes).

If $\Phi(\boldsymbol{A})$ is a scalar function of a tensor, then its linear variation with respect to $\boldsymbol{A}$ is

$$\Phi(\boldsymbol{A}+d\boldsymbol{A}) = \Phi(\boldsymbol{A}) + d\Phi, \quad d\Phi = \frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} : d\boldsymbol{A} + \mathcal{O}(d\boldsymbol{A}^2) \ ,$$

and the tensor $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ is called the tensorial derivative.

Note that after a basis is chosen, the entries of $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ are given by

$$\left(\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}\right)_{ij} = \frac{\partial \Phi}{\partial A_{ij}} \ .$$

That is, if Φ is thought of as a function of $A_{11}, A_{12}, \ldots$, then the tensor $\frac{\partial \Phi}{\partial \boldsymbol{A}}$ is given, entry-wise, by the partial derivatives of Φ with respect to its arguments. Remember the definition $\boldsymbol{B} : \boldsymbol{C} \equiv \mathrm{tr}(\boldsymbol{B}\boldsymbol{C}^T)$. You may convince yourself that $\frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}}$ is indeed a tensor (i.e. that under a different choice of coordinates, the entries of $\frac{\partial \Phi}{\partial \boldsymbol{A}}$ transform as they should).

(a) Now choose $\Phi = \det$, and show that for invertible $\boldsymbol{A}$,

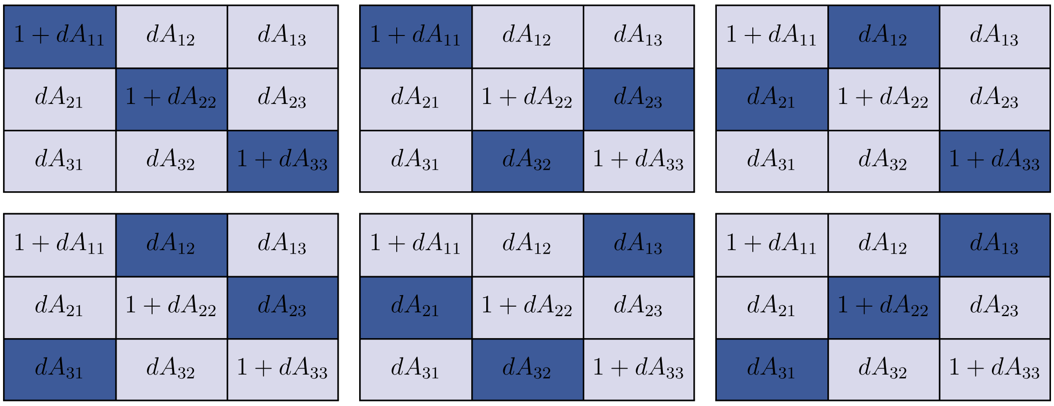
$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det(\boldsymbol{A})\boldsymbol{A}^{-T} \ ,$$

where $\boldsymbol{A}^{-T}$ denotes the inverse of the transpose (or the transpose of the inverse - they're the same). Hints: (a) Start by writing $\boldsymbol{A}+d\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{I}+\boldsymbol{A}^{-1}d\boldsymbol{A})$. (b) Keep only the part of $\det(\boldsymbol{A}+d\boldsymbol{A})$ which is linear in $d\boldsymbol{A}$.

Solution

$$\det\left[\boldsymbol{A}+d\boldsymbol{A}\right] = \det\left[\boldsymbol{A}(\boldsymbol{I}+\boldsymbol{A}^{-1}d\boldsymbol{A})\right] = \det(\boldsymbol{A})\det(\boldsymbol{I}+\boldsymbol{A}^{-1}d\boldsymbol{A})$$

Let's look at $\det(\boldsymbol{I}+\boldsymbol{A}^{-1}d\boldsymbol{A})$. The determinant is a sum of products of entries. If the product contains more than one off-diagonal term, then it will be higher than linear in $d\boldsymbol{A}$. Therefore, we can look only at products that have at most one off-diagonal term. But if you have one off-diagonal element you necessarily have at least two, so to linear order in $d\boldsymbol{A}$, we can look only at the diagonal elements. To see this, just stare for a moment at the permutations that contribute to the determinant of a 3×3 matrix:



It is easily seen that only the first permutation, i.e. the one with all diagonal elements, will contribute to linear order (and this works, of course, also for matrices which are not 3×3). So

$$\begin{aligned}\det(\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A}) &= \prod_k \left(1 + (\boldsymbol{A}^{-1} d\boldsymbol{A})_{kk}\right) + \mathcal{O}(d\boldsymbol{A}^2) = 1 + \sum_k (\boldsymbol{A}^{-1} d\boldsymbol{A})_{kk} + \mathcal{O}(d\boldsymbol{A}^2) \\ &= 1 + \operatorname{tr}\left(\boldsymbol{A}^{-1} d\boldsymbol{A}\right) + \mathcal{O}(d\boldsymbol{A}^2) \ .\end{aligned}$$

Another way to see this is to use this nice trick:

$$\begin{aligned}\det(\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A}) &= \exp\left[\log\left(\det(\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A})\right)\right] = \exp\left[\log\left(\prod_i \lambda_i\right)\right] \\ &= \exp\left[\sum_i \log \lambda_i\right] = \exp\left[\operatorname{tr}\left(\log\left(\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A}\right)\right)\right] \\ &= \exp\left[\operatorname{tr}\left(\boldsymbol{A}^{-1} d\boldsymbol{A}\right)\right] + \mathcal{O}(d\boldsymbol{A}^2) = 1 + \operatorname{tr}\left(\boldsymbol{A}^{-1} d\boldsymbol{A}\right) + \mathcal{O}(d\boldsymbol{A}^2) \ .\end{aligned}$$

where the λ_i's are the eigenvalues of $\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A}$. Either way, this tells us that the differential is

$$\begin{aligned}d(\det \boldsymbol{A}) &= \det(\boldsymbol{A} + d\boldsymbol{A}) - \det \boldsymbol{A} = \det(\boldsymbol{A})\left[\det(\boldsymbol{I} + \boldsymbol{A}^{-1} d\boldsymbol{A}) - 1\right] \\ &= \det(\boldsymbol{A}) \operatorname{tr}\left(\boldsymbol{A}^{-1} d\boldsymbol{A}\right) + \mathcal{O}(d\boldsymbol{A}^2) \equiv \det(\boldsymbol{A}) \boldsymbol{A}^{-T} : d\boldsymbol{A} + \mathcal{O}(d\boldsymbol{A}^2) \ .\end{aligned}$$

This shows that $\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det(\boldsymbol{A}) \boldsymbol{A}^{-T}$.

(b) Show that if $\boldsymbol{A}$ is a function of t, then

$$\frac{\partial}{\partial t} \Phi(\boldsymbol{A}(t)) = \frac{\partial \Phi}{\partial \boldsymbol{A}} : \partial_t \boldsymbol{A} \ .$$

Solution

$$\frac{\partial}{\partial t}\Phi(A_{ij}(t)) = \sum_{ij} \frac{\partial \Phi}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial t} = \frac{\partial \Phi(\boldsymbol{A})}{\partial \boldsymbol{A}} : \partial_t \boldsymbol{A}$$

where the first equality is a calculus rule, and the second is the definition of the double-dot product and of the tensor $\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{A}}$.

Up to now, these were general algebraic identities. Let's get down to business and look at a motion of a deformed body $\boldsymbol{x}(\boldsymbol{X})$, its deformation gradient $\boldsymbol{F}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$, the Jacobian $J(\boldsymbol{X}, t) = \det \boldsymbol{F}(\boldsymbol{X}, t)$ and the velocity field $\boldsymbol{v} = \frac{\partial \boldsymbol{x}}{\partial t}$.

(c) Show that

$$\frac{\partial \boldsymbol{F}}{\partial t} = \nabla_{\boldsymbol{X}} \boldsymbol{v} = \nabla_{\boldsymbol{x}} \boldsymbol{v}\, \boldsymbol{F}\ .$$

Note that this can be easily transformed to obtain Eq. (3.31) in the lecture notes.

Solution

In index notation:

$$\frac{\partial \boldsymbol{F}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial x_i}{\partial X_j} = \frac{\partial}{\partial X_j}\frac{\partial x_i}{\partial t} = \nabla_{\boldsymbol{X}} \boldsymbol{v}$$

Note that we commuted $\partial_{\boldsymbol{X}}$ and ∂_t. This can not be done for $\partial_{\boldsymbol{x}}$ and ∂_t, as is explained in Q2.

By the chain rule, for an arbitrary vector $\boldsymbol{q}$:

$$\nabla_{\boldsymbol{X}} \boldsymbol{q} = \frac{\partial q_i}{\partial X_j} = \frac{\partial x_k}{\partial X_j}\frac{\partial q_i}{\partial x_k} = (\boldsymbol{F})_{kj}(\nabla_{\boldsymbol{x}} \boldsymbol{q})_{ik} = \nabla_{\boldsymbol{x}} \boldsymbol{q} \boldsymbol{F}$$

This completes the derivation.

(d) Use the results of (a)-(c) to prove the desired relation:

$$\partial_t J = J \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = J\, \mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v}\ , \tag{2}$$

you might want to remind yourself that tr(grad(·))=div(·). Conclude that if a motion is volume preserving then $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$.

Solution

$$\begin{aligned}\partial_t J = \partial_t(\det \boldsymbol{F}) &= \frac{\partial \det \boldsymbol{F}}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = \det(\boldsymbol{F})\boldsymbol{F}^{-T} : \nabla_{\boldsymbol{x}} \boldsymbol{v} \boldsymbol{F} \\ &= J \operatorname{tr}\left[\boldsymbol{F}^{-T}(\nabla_{\boldsymbol{x}} \boldsymbol{v} \boldsymbol{F})^T\right] = J \operatorname{tr}\left[\boldsymbol{F}^{-T}\boldsymbol{F}^T(\nabla_{\boldsymbol{x}} \boldsymbol{v})^T\right] = J \operatorname{tr}\left[\operatorname{grad}(\nabla_{\boldsymbol{x}} \boldsymbol{v})^T\right] \\ &= J \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v}\end{aligned}$$

(e) Verify this relation by calculation of $\partial_t J$ for the motion described in Question 1, first by calculating $\partial_t J$ from the formula (2) and then by differentiating the result of 1(c).

Solution

$J = 1$ so $\partial_t J = 0$. Also,

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} = \partial_{x_1} v_1 + \partial_{x_2} v_2 = \partial_{x_1} x_2 + \partial_{x_2} x_1 = 0$$

5. Consider a material that fills the whole space, except for a spherical cavity of initial radius Q, centered at the origin. At time $t = 0$ an explosive is detonated in the cavity and its radius varies as some specified function $q(t)$, resulting in a sphero-symmetric motion. That is, the motion is given by

$$\begin{aligned}\boldsymbol{x}(t) &= \frac{r(t)}{R}\boldsymbol{X} = \frac{f(R,t)}{R}\boldsymbol{X} \ , \\ r(t) &= f(R,t) = |\boldsymbol{x}(R,t)| \ , \\ R(\boldsymbol{X}) &= |\boldsymbol{X}| \ , \\ f(R = Q, t) &= q(t) \ .\end{aligned}$$

(a) Show that the deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}} \boldsymbol{x} = \frac{\partial f}{\partial R}\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \frac{f}{R}(\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}) \ , \tag{3}$$

where $\hat{\boldsymbol{r}} = R^{-1}\boldsymbol{X} = r^{-1}\boldsymbol{x}$, and $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the spherical unit vectors.
Hints:

- For a spherically symmetric function $g(r)$, $\nabla_{\boldsymbol{X}} g = \frac{\partial g}{\partial R}\hat{\boldsymbol{r}}$.
- $\boldsymbol{I} = \sum_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ for any set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of orthonormal vectors.

Solution

Direct calculation gives simply

$$\begin{aligned}
\boldsymbol{F} = \nabla_{\boldsymbol{X}} \boldsymbol{x} = \nabla_{\boldsymbol{X}} \frac{f(\boldsymbol{X},t)}{R}\boldsymbol{X} &= \frac{\boldsymbol{X}}{R}\nabla_{\boldsymbol{X}} f + f\boldsymbol{X}\nabla_{\boldsymbol{X}}\left(\frac{1}{R}\right) + \frac{f}{R}\nabla_{\boldsymbol{X}}\boldsymbol{X} \\
&= \frac{\boldsymbol{X}}{R} \otimes \partial_R f \hat{\boldsymbol{r}} + f\boldsymbol{X} \otimes \left(-\frac{\hat{\boldsymbol{r}}}{R^2}\right) + \frac{f}{R}\boldsymbol{I} \\
&= \partial_R f \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + f\hat{\boldsymbol{r}} \otimes \left(-\frac{\hat{\boldsymbol{r}}}{R}\right) + \frac{f}{R}\left(\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right) \\
&= \nabla_{\boldsymbol{X}} \boldsymbol{x} = \frac{\partial f}{\partial R}\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \frac{f}{R}(\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}})
\end{aligned}$$

Where the 3rd line is obtained by the definition $\hat{\boldsymbol{r}} \equiv \boldsymbol{X}/R$.

(b) If the motion is isochoric (volume-preserving), show that

$$f(R,t) = \sqrt[3]{R^3 + q(t)^3 - Q^3}\ .$$

You can show that either by using Eq.(3) to calculate the volume change, or by direct computation without going knowing the explicit form of $\boldsymbol{F}$ (doing both is better!).

Solution

If the motion is volume-preserving, then

$$\det \boldsymbol{F} = \left(\frac{\partial f}{\partial R}\right)\left(\frac{f}{R}\right)^2 = 1$$

which can be written as a differential equation for f:

$$f^2 df = R^2 dR \quad \Rightarrow \quad f(R)^3 = R^3 + C$$

where C is an integration constant. Since $f(R = Q) = q$, we can get the value of C:

$$f(Q)^3 = Q^3 + C = q^3 \quad \Rightarrow \quad C = q^3 - Q^3$$

and we conclude that

$$f(R) = \left(R^3 + q^3 - Q^3\right)^{1/3}\ .$$

The other way of doing this is as follows. Before the expansion, the volume inside a sphere of radius $R > Q$ was

$$\frac{4\pi}{3}\left(R^3 - Q^3\right)\ .$$

At time t, the volume is

$$\frac{4\pi}{3}\left(f(R,t)^3 - f(Q,t)^3\right) = \frac{4\pi}{3}\left(f(R,t)^3 - q^3\right)$$

Equating the two, we have

$$f^3 = R^3 + q^3 - Q^3$$

as needed.

(c) Calculate $\boldsymbol{v}$, expressed in terms of q and $\partial_t q(t)$.

Solution

Since $\boldsymbol{x} = \frac{f(R,t)}{R}\boldsymbol{X}$, we have $\partial_t x = \partial_t f \frac{\boldsymbol{X}}{R} = \partial_t f \hat{\boldsymbol{r}}$. From our formula for f we have

$$\partial_t f = \frac{1}{3}\left(R^3 + q^3 - Q^3\right)^{-2/3} \left(3q^2\right) \partial_t q = f^{-2} q^2 \partial_t q$$

Substituting, we get

$$\boldsymbol{V}(\boldsymbol{X}, t) = \left(\frac{q}{f(|\boldsymbol{X}|, t)}\right)^2 \partial_t q \hat{\boldsymbol{r}}$$

Switching to the spatial coordinates, we simply use $|\boldsymbol{x}| = f$ to get

$$\boldsymbol{v}(\boldsymbol{x}, t) = \left(\frac{q}{|\boldsymbol{x}|}\right)^2 \partial_t q \hat{\boldsymbol{r}}$$