Non-Equilibrium Continuum Physics **HW set #3**

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Due 27.05.21 Submit as: NECP_HW3_⟨FirstName⟩_⟨LastName⟩.pdf

Elasticity and Thermo-Elasticity - Solution

1. Consider a 3D rectangular box, subject to uniaxial stress σ_0 in the z direction, as shown in Fig. 1. The faces in the x, y directions are traction-free. The rest-lengths of the boxes sides are a, b. Calculate the slope of the dashed line as a function of σ_0, E and ν. When $\sigma_0 \geq E$ something weird happens. Is linear elasticity wrong?

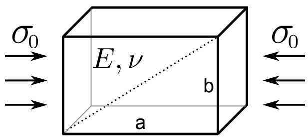


Figure 1: Box under uniaxial compression.

Solution

This is simple uniaxial compression. We solved that in class (Eq. (5.14) in the lecture notes). We have

$$\epsilon_{zz} = \frac{\Delta a}{a} = -\frac{\sigma_0}{E}, \qquad \epsilon_{yy} = \frac{\Delta b}{b} = \nu\epsilon_{zz} = \nu\frac{\sigma_0}{E}$$

The slope is thus

$$\frac{b + \Delta b}{a + \Delta a} = \frac{b}{a}\frac{1 + \nu\frac{\sigma_0}{E}}{1 - \frac{\sigma_0}{E}} \, .$$

When $\sigma_0 = E$ the box shrinks to be 2 dimensional. This is clearly unphysical - one has to remember that σ_0/E is a small parameter, and this was the assumption in the derivation of linear elasticity.

2. As a reminder, the Hertz Problem concerns two elastic bodies in contact. This problem was initially posed by Hertz when he considered Newton diffraction rings on the contact of two lenses. He solved the problem on his Xmas vacation in 1880, when he was 23 years old[1], and his treatment became canonical. In class, Eran "solved" this problem scaling-wise, and here you'll work out the solution a bit more carefully.

   The generic case is that of two paraboloids, and is justified by the assumption that near the contact point the surface of both bodies can be expanded to second order. The two bodies may

[1] What did you do by the time you were 23? I definitely did nothing as impressive.

have different radii of curvature and also different elastic properties. This case is completely tractable, and the procedure goes through reducing the problem to the case of contact between a rigid half-plane and an elastic sphere with some effective radius and elastic properties. We will not go through this reduction, and assume at the outset that this is the case - a sphere of radius R and elastic moduli E and ν is pressed against a rigid half plane with force F.

The force induces a global displacement of δ. That is, the displacement of distant points in the body is $\boldsymbol{u} \approx \delta \hat{z}$. Cylindrical symmetry tells us that the area of contact will be a circle of radius a. The problem is to find the relationship between a, F, and δ, and also to find the pressure distribution on the contact. We will assume that $a \ll R$ (otherwise the strain is not small).

(a) Assuming we have pressed the sphere down by an amount δ, creating an area of contact in a circle of radius a. What are the boundary conditions on the sphere now? Notice that we have a mixed BC - that is one BC in part of the system, and another BC in the other.

Solution

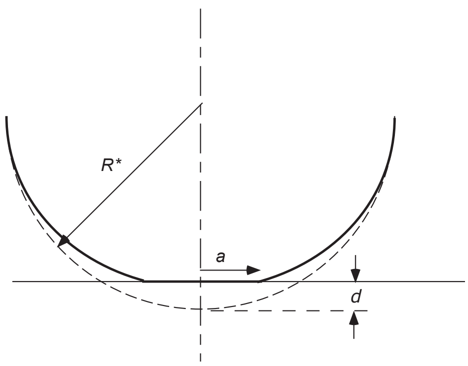
First let us consider the BC in the area which is in contact. By drawing the system, it is easily seen that

$$u_z(r, z=0) = \delta - \frac{r^2}{2R} \qquad , \qquad r < a \ , \tag{1}$$

where we defined $r = \sqrt{x^2+y^2}$. Outside of the contact area, we cannot know a-priori the displacement, only bound it

$$u_z(r^2 + (z-\delta)^2 = R^2) < \delta - \frac{r^2}{2R} \qquad , \qquad r > a \ , \tag{2}$$

which of course cannot be a BC. But not all is lost! As we know that there is no contact, for r¿a we just assume a free interface, namely $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = 0$. In spherical coordinates $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ this is can be written explicitly as $\sigma_{\rho\rho}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = \sigma_{\rho\theta}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = \sigma_{\rho\phi}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = 0$. In the next part of the question we will assume that $a \ll R$, thus in the vicinity of the contact area this can be approximated to leading order in $\frac{a}{R}$ as $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$.



(b) OK, but what next? Hertz pulled a dirty trick here. Since $a \ll R$, he assumed that the sphere can be treated as a half plane for the purposes of calculating the elastic responses. That is, he invoked the Green's function for a point load on a half surface (Eq. (5.26) in Eran's lecture notes). Following his path, use the Green's function formulation to write an integral equation for the normal stress field p_z, using the results of the previous question. Write down the results in polar coordinates, including the boundaries of integration.

Solution

Eq. (5.26) in Eran's lectures reads

$$u_z(x, y, z=0) = \frac{1-\nu^2}{\pi E} \int \frac{p_z(x', y', z'=0)dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2}} \; . \tag{3}$$

To get to a simpler (polar) representation, we can choose without loss of generality (since the system is axially symmetric) $\{x, y\} = \{r, 0\}$. We are hence left with

$$\delta - \frac{r^2}{2R} = \frac{1-\nu^2}{\pi E} \int_0^{2\pi} \int_0^a \frac{p_z(\rho)\rho d\rho d\phi}{\sqrt{\rho^2 + r^2 - 2r\rho\cos(\phi)}} \; , \tag{4}$$

applicable of course only for $r < a$.

(c) The form of p_z that solves the equation is

$$p_z(r) = p_0\sqrt{1 - \frac{r^2}{a^2}} \qquad , \qquad r < a \; , \tag{5}$$

the proof of which is very cumbersome, so I won't give here. Of course, for $r > a$ $p_z = 0$. If any of you are interested, it can be found in K. L. Johnson's Contact mechanics, which is a great book with many solutions to all kind of contact problems. Setting Eq. (5) in Eq. (5.26) in Eran's lecture notes leads to

$$u_z(r) = \frac{\pi\left(1-\nu^2\right)p_0}{4aE}\left(2a^2 - r^2\right) \qquad , \qquad r < a \; . \tag{6}$$

Now finish it off and write down a, δ, and p_0 in terms of the material parameters, the loading force F, and the radius of the sphere R. Verify that it indeed agrees with Eran's scaling analysis.

Solution

Since Eq. (6) should agree with Eq. (1), we conclude

$$\delta = \frac{\pi\left(1-\nu^2\right)p_0 a}{2E} \,, \tag{7}$$

$$\frac{\pi\left(1-\nu^2\right)p_0}{4aE} r^2 = \frac{r^2}{2R} \,. \tag{8}$$

The force can be calculated by integrating p_z

$$F = \int_{\rho<a} p_z(\rho) d^2\rho = \frac{2}{3} p_0 \pi a^2 \,. \tag{9}$$

Combining all of the above, we get

$$a = \left(\frac{3\left(1-\nu^2\right)FR}{4E}\right)^{1/3} \,, \tag{10}$$

$$\delta = \left(\frac{9\left(1-\nu^2\right)^2 F^2}{16RE^2}\right)^{1/3} \,, \tag{11}$$

$$p_0 = \left(\frac{6FE^2}{\pi^3\left(1-\nu^2\right)^2 R^2}\right)^{1/3} \,, \tag{12}$$

in accordance with Eran's scaling analysis.

(d) Small bonus - consider and tell us about your greatest achievement by the time you were 23. How does this compare with finding the solution to the elastic contact problem?

(e) Large bonus - Try to prove that the stress distribution given in Eq. (5) actually leads to the displacement given in Eq. (6). Keep in mind that it's rather long and complicated, so do it only if you really like this kind of stuff.

Solution

As this is long and complicated, with a few transformations and changed of variables, I'm not going to repeat it here, you can check it out in Contact mechanics, Sections 3.4 and 4.2.

3. A rod of radius R and length L is pointed along the z direction, and is held between two rigid walls at $z = 0$ and $z = L$. It is free of constraints in the other two dimensions. The rod is then uniformly heated by some amount ΔT. Without solving the problem completely (although you may do it, if you have nothing better to do) estimate how the

(a) stresses in the rod,
(b) strains in the rod,
(c) forces on the walls,
(d) elastic energy stored in the rod

scale with R, L and ΔT.

Solution

As is explained in the answer to the next question, warming up a material by ΔT is equivalent to increasing all its length scales by $\sim \alpha_T \Delta T$. Therefore, the situation is equivalent to putting a rod of length $L(1 + \frac{1}{3}\alpha_T \Delta T)$ between walls which are only L apart. The strain is therefore $\Delta L/L \sim \alpha_T \Delta T$ and is independent of R and L. The stress is linear in the strain and therefore has the same scaling. The elastic energy stored in the rod goes like

$$u \sim \int_V \boldsymbol{\epsilon}^2 d^3x \sim R^2 L (\alpha_T \Delta T)^2$$

The forces on the walls go like

$$F \sim \boldsymbol{\sigma} R^2 = \alpha_T \Delta T R^2$$

4. Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

Solution

If a material is heated by ΔT and all its boundaries are free, then the displacement field $\boldsymbol{r} \to (1 + \frac{\alpha_T \Delta T}{3})\boldsymbol{r}$, a simple homogeneous dilation, is a solution of the equations. Intuitively, this is so because α_T is nothing but the thermal expansion coefficient. I urge you to plug this solution into the equations and check that this is so: under the supposed deformation we have $\boldsymbol{\varepsilon} = \frac{\alpha_T \Delta T}{3}\delta_{ij}$. Simply plug that it in Hooke's law to get

$$\begin{aligned} \sigma_{ij} &= -K\alpha_T \Delta T \delta_{ij} + K \operatorname{tr} \boldsymbol{\varepsilon} \delta_{ij} + 2\mu \left(\varepsilon_{ij} - \frac{1}{3} \operatorname{tr} \boldsymbol{\varepsilon} \delta_{ij} \right) \\ &= -K\alpha_T \Delta T \delta_{ij} + K\alpha_T \Delta T \delta_{ij} + 2\mu \left(\frac{\alpha_T \Delta T}{3} \delta_{ij} - \frac{\alpha_T \Delta T}{3} \delta_{ij} \right) = 0 \end{aligned} \tag{13}$$

Therefore, when heated and free of external forces, materials simply expand by homogeneous dilation and everything will be stress free. The same goes for the hole - it will expand.

Another way to think about it - consider the piece of material that was taken out. If you heat it by the same amount as the rest, it should fit perfectly.

5. Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x, y, z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

   Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

   Guidance: Begin with Navier-Lamé equation $(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.

Solution

Since the "driving force" for the deformation is ∇T, i.e. a curl-free vector field, it is very reasonable to guess that $\vec{u}$ will also be curl-free. Otherwise this would mean that a chiral symmetry is broken. So we assume that $\vec{u}$ is curl-free, i.e. $u = \nabla\phi$. Before we plug that into the Navier-Lamé equation, we use the identity $\nabla^2\boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times (\nabla \times \boldsymbol{A})$. We get

$$K\alpha\nabla T = (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = (\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu\nabla \times \nabla \times \boldsymbol{u}$$

We now plug in $\boldsymbol{u} = \nabla\phi$, and the last term vanishes. We are left with

$$K\alpha\nabla T = (\lambda + 2\mu)\nabla\left(\nabla^2\phi\right)$$

$$\Downarrow$$

$$0 = \nabla\Big[K\alpha T - (\lambda + 2\mu)\nabla^2\phi\Big] \ .$$

This means that the term in brackets is constant, i.e. ϕ satisfies Poisson's equation:

$$\nabla^2\phi = \frac{K\alpha}{\lambda + 2\mu}T + C \ .$$

We see that C is a meaningless constant that can be swallowed into T, and since anyhow the only relevant feature of T is its gradient, we can safely set $C = 0$.

The solution of this equation you should be known from your undergrad. It is

$$\phi(\boldsymbol{r}) \sim \int \frac{T(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}' \ .$$

If $T(\boldsymbol{r})$ is localized, say a δ-function, then ϕ decays as r^{-1}, u as r^{-2} and $\boldsymbol{\varepsilon}$ as r^{-3}.