Kinematics — strains and stress

1 An Eulerian strain measure

In this section we'll introduce the Eularian analogue of the Green-Lagrange strain tensor defined in class. We'll then see in a concrete example how the two behave.

You are reminded that the Green-Lagrange strain tensor $\boldsymbol{E}$ was defined as a measure of the change in lengths squared. We examined a line element (dX_1, dX_2, dX_3), of total length $d\ell$, which changed under the deformation to be (dx_1, dx_2, dx_3), of total length $d\ell'$. $\boldsymbol{E}$ was defined by the relation

$$(d\ell')^2 - (d\ell)^2 = 2\, d\boldsymbol{X}^T \boldsymbol{E} d\boldsymbol{X} \ , \tag{1}$$

that is, $\boldsymbol{E}$ is defined with respect to the material (=reference =undeformed) coordinates. It was shown in class that $\boldsymbol{E}$ can be simply expressed in terms of the deformation gradient:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \ , \tag{2}$$

where recall that $\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\varphi} = \boldsymbol{I} + \nabla_{\boldsymbol{X}}\boldsymbol{H}$. We now ask, what is the equivalent strain measure, in terms of the spatial (=laboratory =deformed) coordinates? We use the relation $d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X}$ to write $d\boldsymbol{X} = \boldsymbol{F}^{-1} d\boldsymbol{x}$ and $d\boldsymbol{X}^T = d\boldsymbol{x}^T \left(\boldsymbol{F}^{-1}\right)^T$, to obtain

$$(d\ell')^2 - (d\ell)^2 = 2\, d\boldsymbol{X}^T\, \boldsymbol{E}\, d\boldsymbol{X} = 2\, d\boldsymbol{x}^T\, \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1}\, d\boldsymbol{x} \ , \tag{3}$$

and see that we can define the Eularian analogue of $\boldsymbol{E}$ as $\boldsymbol{e} = \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1}$. $\boldsymbol{e}$ is called the *Euler-Almansi strain tensor*. Explicitly, it is given by

$$\boldsymbol{e} = \boldsymbol{F}^{-T}\, \boldsymbol{E}\, \boldsymbol{F}^{-1} = \frac{1}{2}\boldsymbol{F}^{-T}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)\boldsymbol{F}^{-1} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \ . \tag{4}$$

It is easy to see that $\boldsymbol{e}$ is symmetric. In class we've shown that if λ_i are the principal stretches, then $\boldsymbol{E}$ can be written as

$$\boldsymbol{E} = \frac{1}{2}\left(\lambda_i^2 - 1\right) \boldsymbol{M}_i \otimes \boldsymbol{M}_i \ . \tag{5}$$

where $\boldsymbol{M}_i$ are the directions (in $\boldsymbol{X}$ coordinates) of the principal stretches (that's Eq. (3.24) from the lecture notes). In much the same way, $\boldsymbol{e}$ is

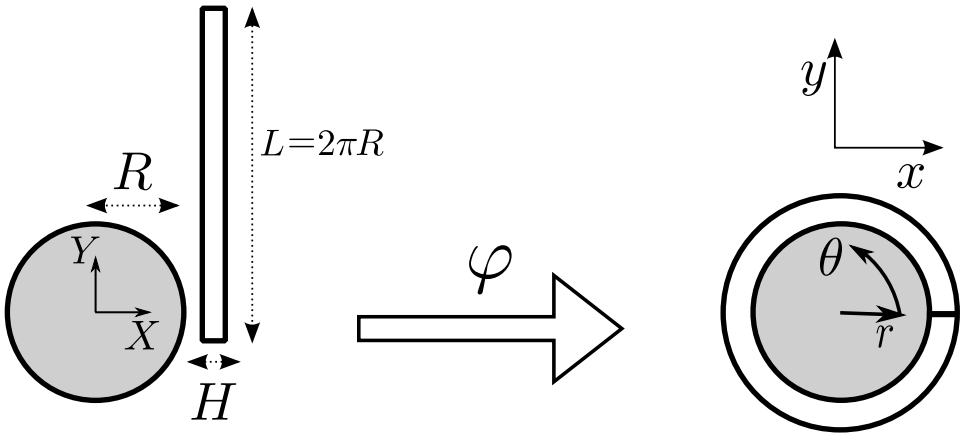
$$\boldsymbol{e} = \frac{1}{2}\left(1 - \lambda_i^{-2}\right) \boldsymbol{m}_i \otimes \boldsymbol{m}_i \ . \tag{6}$$

where $\boldsymbol{m}_i$ are the directions (in $\boldsymbol{x}$ coordinates) of the principal stretches.

As a side note, we remark that the process of taking a Lagrangian quantity and writing its Eularian analogue is called "push-forward". The inverse operation, of calculating the Lagrangian analogue of an Eulerian quantity, is called "pull-back". These are fundamental concepts in differential geometry. As above, it is generally true that pushing

forward a (covariant) Lagrangian tensor is done by multiplying from the left and right by $\boldsymbol{F}^{-T}$ and $\boldsymbol{F}^{-1}$, respectively. If you're interested in the relation between differential geometry and elasticity, have a look at Elasticity & Geometry by Basil Audoly and Yves Pomeau. Also, An introduction to differential geometry with applications to elasticity by Philippe Ciarlet is a good introduction to both elasticity and differential geometry. In addition, you might want to have a chat with Hillel Aharoni who has been working on this for years.

1.1 Application of strain measures, rotation invariance



In this exercise, we'll look at the differences between three strain tensors: the Green-Lagrange tensor $\boldsymbol{E}$, the Cauchy (linearized) tensor $\boldsymbol{\varepsilon}$, and the Euler-Almansi tensor $\boldsymbol{e}$. To this end, consider a thin rod of length $L = 2\pi R$ which is wrapped around a circle or radius R, like in the figure. By "thin" we mean $H \ll L$. The *motion* $\boldsymbol{\varphi}(X, Y)$ is defined by

$$\begin{aligned} x(X,Y) &= X\cos\left(2\pi\frac{Y}{L}\right) \ , \\ y(X,Y) &= X\sin\left(2\pi\frac{Y}{L}\right) \ , \\ \vec{\varphi}(X,Y) &= \begin{pmatrix} X\cos\left(2\pi\frac{Y}{L}\right) \\ X\sin\left(2\pi\frac{Y}{L}\right) \end{pmatrix} \ . \end{aligned} \tag{7}$$

This might be more intuitive if we introduce the shorthand notations

$$r = X \ , \qquad\qquad \theta = 2\pi\,\frac{Y}{L} = \frac{Y}{R} \ , \tag{8}$$

and then the motion takes the form

$$\vec{\varphi}(X,Y) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \ , \tag{9}$$

but we stress that r, θ here are only shorthand notations and *not coordinates* - we are strictly working in Cartesian coordinates.

Note that what we do here is not the usual scenario in this kind of problems. Usually, the motion is not given but rather has to be solved for. We usually know only the boundary conditions – the displacements or forces applied on the body's surface – and the motion in the bulk is calculated by solving the relevant continuum equations. However, since we didn't learn about these equations yet, we specify the motion at the outset. The purpose here is only to see the differences between the strain measures.

The deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}} \begin{pmatrix} x(X,Y) \\ y(X,Y) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -h\sin\theta \\ \sin\theta & h\cos\theta \end{pmatrix} , \tag{10}$$

where the notation $h = X/R$ is introduced. Note that $h - 1 = \frac{X-R}{R} \approx H/L \ll 1$ is a small number. The strain tensors are given by

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(h^2 - 1\right) \end{pmatrix} , \tag{11}$$

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left(1 - h^{-2}\right) \begin{pmatrix} \sin^2(\theta) & -\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos^2(\theta) \end{pmatrix} , \tag{12}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^T - 2\boldsymbol{I}\right) = \begin{pmatrix} \cos(\theta) - 1 & -\frac{1}{2}(h-1)\sin(\theta) \\ -\frac{1}{2}(h-1)\sin(\theta) & h\cos(\theta) - 1 \end{pmatrix} . \tag{13}$$

Note that in the expression for $\boldsymbol{E}$, h is shorthand for X/R, and θ is shorthand for $2\pi Y/R$ (incidentally, $\boldsymbol{E}$ is independent of θ). However, in the expression for $\boldsymbol{e}$, h is shorthand for $\sqrt{x^2+y^2}/R$, and θ stands for $\tan^{-1}(y/x)$. This is an important distinction – $\boldsymbol{E}$ is given in terms of X and Y, and $\boldsymbol{e}$ in terms of x and y!

Now let's analyze these expressions. We see that $\boldsymbol{E}$ is diagonal in the $\boldsymbol{X}$ coordinates and is independent of Y (or θ). $\boldsymbol{e}$ is a bit more complicated, but actually it can be written in a much simpler manner. If we define the rotation matrix $\boldsymbol{R}$ as

$$\boldsymbol{R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} , \tag{14}$$

then we can write $\boldsymbol{e}$ as

$$\boldsymbol{e} = \boldsymbol{R}^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(1 - h^{-2}\right) \end{pmatrix} \boldsymbol{R} . \tag{15}$$

So we see that like $\boldsymbol{E}$, $\boldsymbol{e}$ has one eigenvalue which is always zero, and the other eigenvalue is independent of θ (but not of y!). However, the principal directions in the spatial coordinates are rotating with θ, which is not surprising. The principal directions are indeed the polar and radial directions.

As for $\boldsymbol{\varepsilon}$, things are much less neat. Its eigenvalues are

$$\varepsilon_1 = \frac{1}{2}\left((1+h)(\cos\theta - 1)\right) , \qquad \varepsilon_2 = \frac{1}{2}\left(h - 3 + (1+h)\cos\theta\right) , \tag{16}$$

and the principal directions are also a mess.

It seems that $\boldsymbol{E}$ and $\boldsymbol{e}$ are somehow "the same" in some sense, but that $\boldsymbol{\varepsilon}$ is fundamentally different: To begin with, $\boldsymbol{E}$ and $\boldsymbol{e}$ are small everywhere (remember that $h^2-1 \approx 1-h^{-2} \approx O(H/L) \ll 1$) and the diagonal elements of $\boldsymbol{\varepsilon}$ are not; $\boldsymbol{E}$ and $\boldsymbol{e}$ have a 0 eigenvalue, while $\boldsymbol{\varepsilon}$ does not (except for $\theta = 0$ or 2π); and most importantly, $\boldsymbol{E}$ and $\boldsymbol{e}$ are θ-independent (in a proper sense) and $\boldsymbol{\varepsilon}$ is not.

This is very weird: for $X = R$ (that is, $h = 1$) we have $\boldsymbol{E} = \boldsymbol{e} = 0$, but $\boldsymbol{\varepsilon} \neq 0$!! Aren't they supposed to be equal to linear order? Or at least agree on whether they vanish or not? Which of the above is better?

The error lies in $\boldsymbol{\varepsilon}$! The system is indeed θ-invariant. The physical picture you should have in mind is that each "layer" in the rod (i.e. constant X), which was initially of length L, is stretched and attains the length $2\pi X$ in the deformed configuration. The ratio of the two lengths, $2\pi X/L = h$, is the stretch in the Y direction. In the X direction the material is not stretched. Note that the eigenvalues of $\boldsymbol{E}$, and $\boldsymbol{e}$ are exactly what you should expect, if you happen to remember Eqs. (5) and (6). Furthermore, the principal directions are also exactly what we should expect: they are indeed X and Y, and in the deformed coordinates they are $\hat{r}$ and $\hat{\theta}$. Since we work in Cartesian spatial coordinates, the principal directions are simply given by rotating the (x, y) directions by an angle θ.

So we got a good intuition as to why $\boldsymbol{E}$ and $\boldsymbol{e}$ do exactly what we expect them to, and all is well and nice. But what goes wrong in $\boldsymbol{\varepsilon}$? The answer is that the displacement gradient $\boldsymbol{F}$ is not small, due to the finite large rotations. Sadly, $\boldsymbol{\varepsilon}$ is not rotationally invariant. To see this, consider the following rigid body rotation

$$\begin{aligned} x &= X\cos\theta - Y\sin\theta \ , \\ y &= Y\cos\theta + X\sin\theta \ , \end{aligned} \tag{17}$$

we have

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\varphi} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ , \tag{18}$$

and

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^T\right) - \boldsymbol{I} = \begin{pmatrix} \cos\theta - 1 & 0 \\ 0 & \cos\theta - 1 \end{pmatrix}, \qquad \boldsymbol{E} = \boldsymbol{e} = \boldsymbol{0} \ . \tag{19}$$

That means that $\boldsymbol{\varepsilon} \neq 0$ for finite rotations. For infinitesimal rotations, $\theta \ll 1$, we have

$$\varepsilon_{xx} \simeq \varepsilon_{yy} \simeq O(\theta^2) \ , \qquad\qquad \varepsilon_{xy} = 0 \ . \tag{20}$$

So $\boldsymbol{\varepsilon} = 0$ to linear order in θ, i.e. only for *infinitesimal rotations*. This is rather trivial as $\boldsymbol{\varepsilon}$ is a linearized version of the rotationally invariant strain measures $\boldsymbol{E}$ and $\boldsymbol{e}$ (they are the same to linear order). We see that the large values of $\boldsymbol{\varepsilon}$ at θ's far from zero do not stem from physical stretches in the material, but rather from rotation of the axes, which has no physical significance.

A word of caution: Note that $\boldsymbol{\varepsilon}$ is the "plain vanilla" strain measure. If you go to Rami Levy and ask for a strain measure, this is what they'll give you. It is the absolute standard in the majority of works in linear elasticity and is often presented as the "natural" one. Beware!

2 Gauss' integral theorem for tensors (if we make it)

Finally, you know from your undergrad studies that if $\vec{u}$ is a vector field in a volume $\Omega \subset \mathbb{R}^3$, then

$$\int_\Omega \operatorname{div} \vec{u}\, dV = \int_S \vec{u} \cdot d\vec{S} \ , \tag{21}$$

where S is the surface of Ω (in mathematical notation, $S = \partial\Omega$). $d\vec{S}$ is a differential vector, perpendicular to a local surface. This is called Gauss' theorem, and it also works for tensors:

$$\int_\Omega \operatorname{div} \boldsymbol{A}\, dV = \int_{\partial\Omega} \boldsymbol{A}\, d\vec{S} \ , \tag{22}$$

where the right-hand-side should be understood as $\boldsymbol{A}$ operating as a tensor on $d\vec{S}$, exactly like the right-hand-side of (21) represented $\vec{u}$ operating as a tensor on $d\vec{S}$, i.e. the usual dot product. Both Eqs. (21) and (22) are given here without proof.

We will now see that you already know a particular case of Eq. (22). Take Ω to be a 2-dimensional sheet in a 3D space, and a vector field $\vec{u}$ on it. The boundary of Ω is now a curve, whose tangent vector will by denoted by $\vec{\ell}$, Cf. Fig. 1. For simplicity, we'll assume that Ω is confined to the $x-y$ plane, although this is not necessary. We define

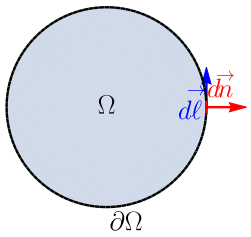


Figure 1: Illustration of a surface and boundary integration. An exemplary boundary element is marked in red, together with its normal vector $d\vec{n}$ (red) and a tangent vector $d\vec{\ell}$ (blue).

a new tensor

$$\boldsymbol{A} \equiv \boldsymbol{\mathcal{E}}\vec{u} \ , \tag{23}$$

where $\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor. Index-wise, this means $A_{ij} = \mathcal{E}_{ijk}u_k$. We'll also take $\vec{u}$ to be z-independent. We begin by calculating the left-hand-side of (22):

$$\int_\Omega \operatorname{div} \boldsymbol{A} dS = \int_\Omega \partial_j \mathcal{E}_{ijk} u_k dS = \int_\Omega \mathcal{E}_{ijk}\partial_j u_k dS = \int_\Omega \left(\vec{\nabla} \times \vec{u}\right) dS \ . \tag{24}$$

The right-hand-side gives

$$\int_{\partial\Omega} \boldsymbol{A} d\vec{n} = \int_{\partial\Omega} \mathcal{E}_{ijk} u_k dn_j = \int_{\partial\Omega} (\vec{u} \times d\vec{n}) \ . \tag{25}$$

Now, $\vec{u} \times d\vec{n}$ is a vector that is perpendicular to both $\vec{u}$ and $d\vec{n}$, that is, it is directed in the $\hat{z}$ direction. Its magnitude is $|\vec{u}||d\vec{n}| \sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{n}$. But the angle between $\vec{u}$ and $\vec{\ell}$ is $\alpha \equiv 90° - \theta$, so we can write

$$|\vec{u} \times d\vec{n}| = |\vec{u}|\, |d\vec{n}| \sin\theta = |\vec{u}| \left|d\vec{\ell}\right| \cos\alpha = \left|\vec{u} \cdot d\vec{\ell}\right| \ . \tag{26}$$

We conclude that

$$\vec{u} \times d\vec{n} = \left(\vec{u} \cdot d\vec{\ell}\right) \hat{z} \ , \tag{27}$$

The theorem (22) says that (24) and (25) are equal, so we conclude that

$$\int_{\Omega} \left(\vec{\nabla} \times \vec{u}\right) dS = \oint_{\partial\Omega} \vec{u} \cdot d\vec{l} \ , \tag{28}$$

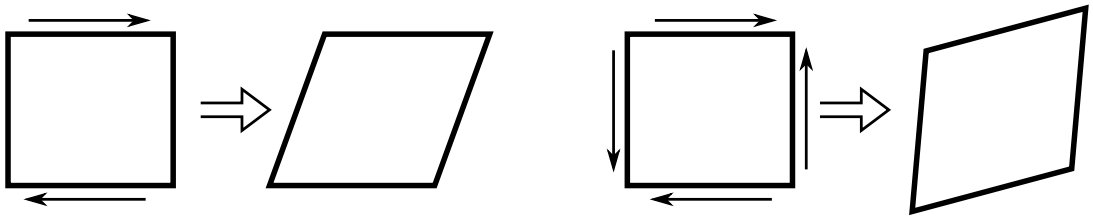
which you know well from your happy undergrad days, under the name of Stokes' Theorem (or Green's Theorem, sometimes).

3 Linearized strain under shear

We start by getting some intuition about the geometrical meaning of the linearized strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \ , \tag{29}$$

under shear. Two very common strain states are called *pure shear* and *simple shear*. Simple shear is the situation in which displacement in one direction is a linear function



(a) Simple shear (b) Pure shear

Figure 2

of the orthogonal coordinate, as seen in Fig. 2a. Pure shear is the state when the same shear is applied in both direction, as seen in Fig. 2b. In this case there are only shear strains (in some coordinate system), that is, $\boldsymbol{\varepsilon}$ is of the form

$$\varepsilon_{ij} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix} \ . \tag{30}$$

What is the relation between simple shear and pure shear? In simple shear, the displacement field is

$$x = X + \gamma Y \ , \qquad\qquad y = Y \ . \tag{31}$$

So the deformation gradient is

$$\boldsymbol{F} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \ . \tag{32}$$

This can be decomposed into a state of pure shear and infinitesimal rotation:

$$\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I} = \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 \end{pmatrix}}_{\text{pure shear}} + \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 \end{pmatrix}}_{\text{rotation}} \ . \tag{33}$$

Indeed, it is seen that Fig. 2b is a slightly rotated version of Fig. 2a.

Note that for a deformation that leaves everything in place, i.e. $\boldsymbol{x} = \boldsymbol{X}$, we have $\boldsymbol{F} = \boldsymbol{I}$ and thus $\boldsymbol{H} \equiv \boldsymbol{F} - \boldsymbol{I}$ is what quantifies the non-rigid-body deformation. What we just showed is that $\boldsymbol{F} - \boldsymbol{I}$ can be decomposed to a symmetric part, which is the strain, and an antisymmetric part, which is a rotation and does not cost energy. This is why only the symmetric part of $\boldsymbol{F}$ is used in all versions of strain measures.

Lets focus now on pure shear. We can write the pure shear deformation in one of two forms. We can write

$$\boldsymbol{F}_1 = \begin{pmatrix} 1 & \gamma/2 \\ \gamma/2 & 1 \end{pmatrix} , \tag{34}$$

or as

$$\boldsymbol{F}_2 = \begin{pmatrix} 1+\gamma/2 & 0 \\ 0 & 1-\gamma/2 \end{pmatrix} , \tag{35}$$

both shown in Fig. 3. Let us compare $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ — their trace $\operatorname{tr} \boldsymbol{F}_1 = 2 = \operatorname{tr} \boldsymbol{F}_2$ and determinant $\det \boldsymbol{F}_1 = 1 - \frac{\gamma^2}{4} = \det \boldsymbol{F}_2$ are exactly the same. This implies that these two deformations are equivalent. By diagoanlizing the two $\boldsymbol{F}$'s we observe their eigenvectors are simply rotated by $\pi/4$ (or, that one can transform one into another with similarity transformation) — they represent the same deformation. It is easy to see that both of these are equivalent to simple shear in the *linear* approximation, but when considering the full *nonlinear* deformation, simple shear *is* volume preserving, while pure shear deformations *are not*.

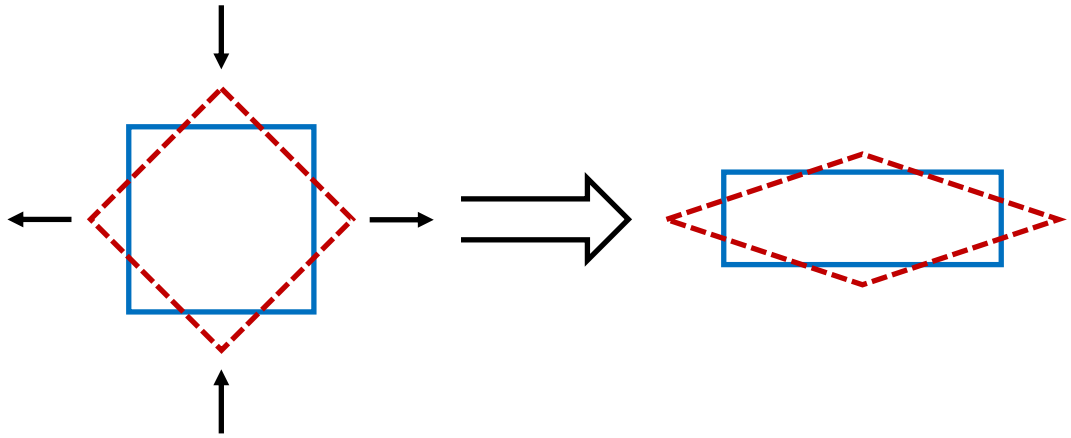


Figure 3