Coarse graining (and some leftovers)

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1 Symmetry of Cauchy's stress tensor

In this section, we'll see why the Cauchy stress tensor must be symmetric. We'll do this in two ways: the first is intuitive and physically transparent, and the second is a bit technical and uses the machinery of continuum theories. I hope that you'll learn to appreciate both.

The first, "easier", way is as follows. Examine the torque applied on a small cube of linear size L. The torque $\boldsymbol{\tau} \sim \boldsymbol{R} \times \boldsymbol{F}$ scales as $(\sigma_{xy} - \sigma_{yx}) L^3$ while the moment of inertia $I \sim mr^2$ goes like $ML^2 \simeq \rho L^5$. Therefore, as one takes smaller and smaller cubes the angular acceleration diverges unless $\boldsymbol{\sigma}$ is symmetric. Since we do not want to consider infinite torques, it is clear that the stress must be symmetric.

The second proof was not shown in class, but is given here for completeness. For the second proof, we will need to use Reynold's transport theorem. In class you have proven that

$$\frac{D}{Dt} \int_\Omega \psi(\boldsymbol{x}, t) d\boldsymbol{x}^3 = \int_\Omega \left[\partial_t \psi(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} \cdot \left(\psi(\boldsymbol{x}, t) \boldsymbol{v}(\boldsymbol{x}, t) \right) \right] d\boldsymbol{x}^3 \ . \tag{1}$$

You'll be glad to know that there is a more useful version of this theorem. Since in the theorem ψ can by any field, one can replace it by $\rho\psi$. Then, using Leibniz's rule for the divergence $\nabla \cdot (f\boldsymbol{g}) = f\nabla \cdot \boldsymbol{g} + \boldsymbol{g} \cdot \nabla f$, we get (I omit all the arguments of the functions for readability. Remember that everything is a function of $(\boldsymbol{x}, t)$):

$$\begin{aligned}
\frac{D}{Dt} \int_\Omega \rho\psi d\boldsymbol{x}^3 &= \int_\Omega \left[\partial_t(\rho\psi) + \nabla_{\boldsymbol{x}} \cdot (\rho\psi\boldsymbol{v}) \right] d\boldsymbol{x}^3 \\
&= \int_\Omega \left[\rho \left(\partial_t \psi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \psi \right) + \psi \left(\partial_t \rho + \nabla_{\boldsymbol{x}} \cdot (\rho\boldsymbol{v}) \right) \right] d\boldsymbol{x}^3 \ .
\end{aligned} \tag{2}$$

Note that the expression in the first brackets is exactly $\frac{D}{Dt}\psi$. Also, the term in the second brackets vanishes identically due to mass conservation (Eq. (4.5) in Eran's lecture notes). Thus, we conclude that Reynold's theorem can be reformulated in a more pleasant way:

$$\frac{D}{Dt} \int_\Omega \rho(\boldsymbol{x}, t) \psi(\boldsymbol{x}, t) d\boldsymbol{x}^3 = \int_\Omega \rho(\boldsymbol{x}, t) \frac{D}{Dt} \psi(\boldsymbol{x}, t) d\boldsymbol{x}^3 \ . \tag{3}$$

Very loosely speaking, this means that in the material coordinates, the operator $\frac{D}{Dt}(\cdot)$ commutes with the operator $\int_\Omega \rho(\boldsymbol{x}, t)(\cdot) d\boldsymbol{x}^3$. Remember that ψ can be a tensor of any rank.

Physically, the symmetry of Cauchy's stress tensor is the local version of the conservation of angular momentum. To see this, we first define the total angular momentum $\boldsymbol{J}$ (do not confuse with the Jacobian J):

$$\boldsymbol{J} = \int_\Omega \rho(\boldsymbol{x}, t) \boldsymbol{r} \times \boldsymbol{v}(\boldsymbol{x}, t) d\boldsymbol{x} \ , \tag{4}$$

analogous to $m\boldsymbol{r} \times \boldsymbol{v}$. We apply Reynold's theorem to this definition, getting

$$\frac{D\boldsymbol{J}}{Dt} = \int_\Omega \rho\,(\dot{\boldsymbol{r}} \times \boldsymbol{v} + \boldsymbol{r} \times \dot{\boldsymbol{v}})\,d\boldsymbol{x} \;=\; \int_\Omega \rho\,\boldsymbol{r} \times \dot{\boldsymbol{v}}\,d\boldsymbol{x}\;, \tag{5}$$

where we used the fact that $\dot{\boldsymbol{r}} \times \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{v} = 0$. Newton's second law says that

$$\frac{D\boldsymbol{J}}{Dt} = \int_\Omega (\boldsymbol{r} \times \boldsymbol{b})d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{r} \times \boldsymbol{t})ds\;, \tag{6}$$

where $\boldsymbol{b}, \boldsymbol{t}$ are the body force and traction fields, respectively. We use the relation $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, and equate Eqs. (5) and Eqs. (6):

$$\int_\Omega \boldsymbol{r} \times [\rho\partial_t\boldsymbol{v} - \boldsymbol{b}]\,d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{r} \times \boldsymbol{\sigma}\boldsymbol{n})ds = 0\;. \tag{7}$$

Here we use a little lemma:

Lemma: Let $\boldsymbol{u}, \boldsymbol{A}$ be vector and tensor fields defined in the region Ω. Then

$$\int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{A}\boldsymbol{n}\,ds = \int_\Omega \left[\boldsymbol{\mathcal{E}} : (\operatorname{grad}\boldsymbol{u})\boldsymbol{A}^T + \boldsymbol{u} \times \operatorname{div}\boldsymbol{A} \right] dv\;, \tag{8}$$

where $\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor. To see this, write in index notation

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{A}\boldsymbol{n}\,ds &= \int_{\partial\Omega} \mathcal{E}_{ijk}u_j(\boldsymbol{A}\boldsymbol{n})_k\,ds = \int_{\partial\Omega} \mathcal{E}_{ijk}u_j A_{kl}n_l\,ds \\
&= \int_{\partial\Omega} (\boldsymbol{\mathcal{E}} : \boldsymbol{u}\boldsymbol{A})_{il}\,n_l\,ds = \int_\Omega \operatorname{div}(\boldsymbol{\mathcal{E}} : \boldsymbol{u}\boldsymbol{A})\,dv \\
&= \int_\Omega \partial_l\,(\mathcal{E}_{ijk}u_j A_{kl})\,dv = \int_\Omega \Big[\underbrace{\mathcal{E}_{ijk}(\partial_l u_j)A_{kl}}_{\boldsymbol{\mathcal{E}}:(\operatorname{grad}\boldsymbol{u})\boldsymbol{A}^T} + \underbrace{\mathcal{E}_{ijk}u_j(\partial_l A_{kl})}_{\boldsymbol{u}\times\operatorname{div}\boldsymbol{A}} \Big]dv\;.
\end{aligned} \tag{9}$$

We now plug that into (7) to get

$$\int_\Omega \boldsymbol{r} \times [\rho\partial_t\boldsymbol{v} - \boldsymbol{b} - \operatorname{div}\boldsymbol{\sigma}]\,dv - \int_\Omega \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^T\,dv = 0\;. \tag{10}$$

The first integrand vanishes identically, as this is an equation of motion. The second one can be integrated on an arbitrary volume and so we see that $\boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^T = 0$, or in other words, $\boldsymbol{\sigma}$ is symmetric.

2 Equations of motion in the reference configuration

This section deals with formulating the equation of motion in the material coordinates rather than in the spatial ones. For further reading, see pg. 146 in Holzapfel (it's in the library).

As Eran stressed in class, this step is crucial because in a generic problem we do not know in advance what is the deformed configuration and therefore it is very useful to describe the motion in the undeformed coordinates. We remark again that in standard

linear elasticity the two sets of coordinates are the same to linear order, so this distinction is not emphasized in this kind of treatments.

The Piola-Kirchoff stress tensor was defined in class by the relation

$$\boldsymbol{T} = \boldsymbol{P}d\boldsymbol{S} = \boldsymbol{\sigma}d\boldsymbol{s} = \boldsymbol{t} \ , \tag{11}$$

where $\boldsymbol{t}$ is the infinitesimal forces in the spatial coordinate and $\boldsymbol{T}$ is its (fictitious) correspondent in the material coordinates. How does $d\boldsymbol{S}$ relate to $d\boldsymbol{s}$? Consider an arbitrary line element $d\boldsymbol{X}$ going through $d\boldsymbol{S}$. The spanned volume is $dV = d\boldsymbol{S} \cdot d\boldsymbol{X}$. Correspondingly, in the deformed coordinates we have $dv = d\boldsymbol{s} \cdot d\boldsymbol{x}$. By definition of the Jacobian, we know that the ratio of the volumes is $dv = J\,dV$. Since $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$ we have

$$dX_i F_{ji} ds_j = dx_j ds_j = dv = JdV = JdX_i dS_i \ . \tag{12}$$

Since $d\boldsymbol{X}$ was arbitrary, we get $ds_j F_{ji} = JdS_i$ or in more convenient notation

$$\boldsymbol{F}^T d\boldsymbol{s} = Jd\boldsymbol{S} \ , \qquad d\boldsymbol{s} = J\boldsymbol{F}^{-T}d\boldsymbol{S} \ . \tag{13}$$

Plugging that into (11) we have

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \ . \tag{14}$$

So now we know how $\boldsymbol{P}$ relates to $\boldsymbol{\sigma}$. But what are its equations of motion? For this we need 3 lemmas:

1. Piola's identity: $\nabla_{\boldsymbol{X}} \cdot \left(J\boldsymbol{F}^{-T}\right) = 0$.

2. For every two tensors $\boldsymbol{A}, \boldsymbol{B}$, we have $\mathrm{div}(\boldsymbol{A}\boldsymbol{B}) = (\mathrm{grad}\,\boldsymbol{A}) : \boldsymbol{B} + \boldsymbol{A}\,\mathrm{div}\,\boldsymbol{B}$.

3. For every tensor $\boldsymbol{A}$ we have $\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{A} = (\mathrm{grad}_{\boldsymbol{X}}\,\boldsymbol{A}) : \boldsymbol{F}^{-T}$.

The proofs of these lemmas are trivial:

1. Integrate $\nabla_{\boldsymbol{X}} \cdot \left(J\boldsymbol{F}^{-T}\right)$ over an arbitrary volume Ω_0:

$$\int_{\Omega_0} \nabla_{\boldsymbol{X}} \cdot \left(J\boldsymbol{F}^{-T}\right) d^3\boldsymbol{X} = \int_{\partial\Omega_0} J\boldsymbol{F}^{-T}d\boldsymbol{S} = \int_{\partial\Omega} d\boldsymbol{s}$$
$$= \int_{\partial\Omega} \boldsymbol{I}d\boldsymbol{s} = \int_{\Omega}(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{I})d^3\boldsymbol{x} = 0 \ . \tag{15}$$

2. $\partial_j(A_{ik}B_{kj}) = \partial_j A_{ik}B_{kj} + A_{ik}\partial_j B_{kj}$.

3. $\frac{\partial A_{ij}}{\partial x_j} = \frac{\partial X_k}{\partial x_j}\frac{\partial A_{ij}}{\partial X_k}$.

Using these lemmas, and defining the *reference body force* by $\boldsymbol{B}(\boldsymbol{X}, t) \equiv J(\boldsymbol{X}, t)\boldsymbol{b}(\boldsymbol{x}, t)$, we get

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} = \nabla_{\boldsymbol{X}} \cdot \left(\boldsymbol{\sigma}J\boldsymbol{F}^{-T}\right) = \nabla_{\boldsymbol{X}}\boldsymbol{\sigma} : \left(J\boldsymbol{F}^{-T}\right) + \boldsymbol{\sigma}\underbrace{\nabla_{\boldsymbol{X}} \cdot \left(J\boldsymbol{F}^{-T}\right)}_{=0}$$
$$= J\nabla_{\boldsymbol{X}}\boldsymbol{\sigma} : \boldsymbol{F}^{-T} = J\nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} = J\left(\rho\dot{\boldsymbol{v}} - \boldsymbol{b}\right)$$
$$\Rightarrow \qquad \boxed{\rho_0\dot{\boldsymbol{V}} = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} + \boldsymbol{B}} \tag{16}$$

Note the resemblance to the equation of motion in the deformed coordinates:

$$\rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} + \boldsymbol{b} \ . \tag{17}$$

One more point: having Eq. (16) is not enough in order to formulate a problem in the $\boldsymbol{X}$ coordinates. We also need to transform the boundary conditions to the material coordinates in order to fully define the problem. If the boundary conditions are forces, then they have to be transformed to the fictitious material coordinates forces. In the case of free boundary conditions (i.e. zero tractions) it is easy - they remain free.

3   Coarse graining

The main purpose of this TA is to see how to develop a continuum theory to describe the deformation of a discrete system, a process known as "coarse graining". This is also the first time in the course that we'll develop a *constitutive relation* from microscopic physics, i.e. calculate how the stress depends on the deformation. The system we'll examine is a two-dimensional triangular lattice of masses and springs, shown in Fig. 1. We assume that the springs are usual linear springs with an identical spring constant k and that in the undeformed state all the springs are at their rest-lengths, which we denote by a.
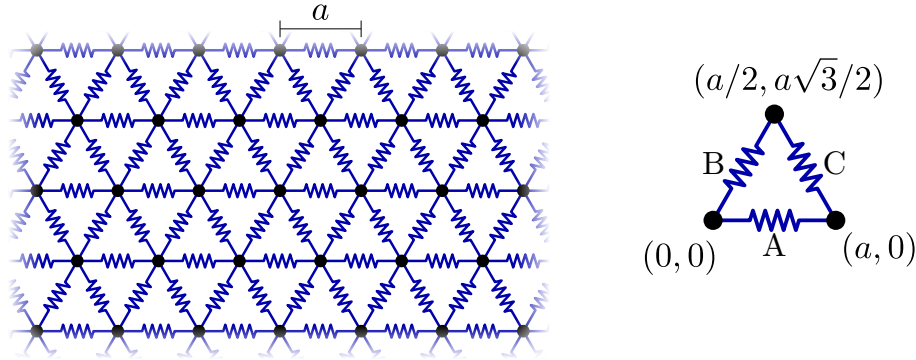

Figure 1: Left: A triangular lattice with a lattice constant a. Right: the unit cell with the coordinates of the vertices.

The central quantity we will want to calculate is the energetic cost of deformation. As you've heard from Eran in the last lecture, stresses and strains are intimately related and the observable that relates the two is the energy. Formally, stress and strain are a conjugate variable pair (like many other pairs you know of: pressure-volume, magnetic field-magnetization, chemical potential-particle number etc.).

Assume the material undergoes some motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$. A crucial requirement in coarse graining is that the observables of interest vary slowly in space, much slower than the relevant microscopic degrees of freedom. Formally, this means that the Fourier decomposition of $\boldsymbol{\varphi}$ only has contributions from wavevectors $\boldsymbol{q}$ satisfying $|\boldsymbol{q}a| \ll 1$ (you may have seen a similar requirement when you learned about the Debye model for the phonon contribution to the specific heat). Thus, we assume that we can analyze each

small portion of the lattice separately and write the energy density u as a function of the *local values* of the deformation (and its gradients)

$$u(\boldsymbol{r}) = u\Big(\boldsymbol{\varphi}(\boldsymbol{r}), \boldsymbol{F}(\boldsymbol{r}), \nabla \boldsymbol{F}, \dots \Big) \ . \tag{18}$$

Note that in Eran's notes energy density is defined per unit mass and here I use densities per unit volume (=area). The two are simply related by a factor ρ. Finally, we can then express the total energy of the system as a volume integral $\mathcal{U} = \int u(\boldsymbol{r}) d^3 \boldsymbol{r}$.

So let's calculate u defined in Eq. (18). Since we do not consider any external fields, the energy must be independent of global translation, i.e. $\boldsymbol{\varphi}$ itself. Also, we consider here only the contribution of $\boldsymbol{F}$ to the energy and neglect higher gradients. This is an "extra" approximation which follows the same spirit of the coarse-graining: if $|\boldsymbol{q}a| \ll 1$ then two adjacent masses will experience roughly the same $\boldsymbol{F}$, and the approximation becomes better when $|\boldsymbol{q}a|$ becomes smaller. Note that it is perfectly "kosher" to consider also terms like $\nabla \boldsymbol{F}$, $\nabla^2 \boldsymbol{F}$, etc, but that would require the theory to include explicitly a microscopic lengthscale, something that we would like to avoid at this point.

Examine the unit cell that in the undeformed state is located at $\boldsymbol{X} = 0$. Its vertices are at $\boldsymbol{X}^{(1)} = (0,0)$, $\boldsymbol{X}^{(2)} = (a,0)$ and $\boldsymbol{X}^{(3)} = (a/2, a\sqrt{3}/2)$. After the deformation, their positions are

$$\boldsymbol{x}(\boldsymbol{X}) \approx \boldsymbol{F}\boldsymbol{X} + \mathcal{O}(a^2) \ , \boldsymbol{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{x}^{(2)} = a\begin{pmatrix} F_{1,1} \\ F_{2,1} \end{pmatrix}, \boldsymbol{x}^{(3)} = \frac{a}{2}\begin{pmatrix} F_{1,1} + \sqrt{3}F_{1,2} \\ F_{2,1} + \sqrt{3}F_{2,2} \end{pmatrix} \ . \tag{19}$$

The deformed lengths of the three springs denoted by A,B,C in the figure are

$$L_A = \left|\boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}\right| = \left|(aF_{1,1}, aF_{2,1})\right| = a\sqrt{F_{1,1}{}^2 + F_{2,1}{}^2} \ , \tag{20}$$

$$L_B = \left|\boldsymbol{x}^{(3)} - \boldsymbol{x}^{(1)}\right| = \frac{a}{2}\sqrt{\left(F_{1,1} + \sqrt{3}F_{1,2}\right)^2 + \left(\sqrt{3}F_{2,2} + F_{2,1}\right)^2} \ , \tag{21}$$

$$L_C = \left|\boldsymbol{x}^{(3)} - \boldsymbol{x}^{(2)}\right| = \frac{a}{2}\sqrt{\left(F_{1,1} - \sqrt{3}F_{1,2}\right)^2 + \left(\sqrt{3}F_{2,2} - F_{2,1}\right)^2} \ . \tag{22}$$

The total energy is given by

$$Su = \frac{1}{2}k\left(L_A - a\right)^2 + \frac{1}{2}k\left(L_B - a\right)^2 + \frac{1}{2}k\left(L_C - a\right)^2 \ , \tag{23}$$

where $S = a^2\sqrt{3}/2$ is twice the area of the unit cell[1]. It's easy to see that if $\boldsymbol{F} = \boldsymbol{I}$ then $L_i = a$ and clearly the energy will vanish, as expected.

Let us examine this energy function. The first thing we note about u is that it doesn't look like a "tensor function", i.e. it seems to depend on all the entries of $\boldsymbol{F}$ in some kind of a nasty manner that cannot be written in a nice geometrical form like $f(\mathrm{tr}\,\boldsymbol{F}, \det \boldsymbol{F}, \boldsymbol{F} : \boldsymbol{F}, \cdots)$. Second, it seems that, unlike what we said in the last TA session, it seems to depend on both the symmetric and the antisymmetric parts of $\boldsymbol{F}$.

So let's see if we can write u in a way that makes some more sense. We'll denote the line segments that connect the vertices by $d\boldsymbol{X}^{(A)}$, $d\boldsymbol{X}^{(B)}$ and $d\boldsymbol{X}^{(C)}$ and their deformed

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[1] Since every spring is shared between two cells.

counterparts as $d\boldsymbol{x}^{(A)} = \boldsymbol{F}d\boldsymbol{X}^{(A)}$, $d\boldsymbol{x}^{(B)} = \boldsymbol{F}d\boldsymbol{X}^{(B)}$ and $d\boldsymbol{x}^{(C)} = \boldsymbol{F}d\boldsymbol{X}^{(C)}$. The deformed lengths are

$$L_\alpha^2 = d\boldsymbol{x}^{(\alpha)} \cdot d\boldsymbol{x}^{(\alpha)} = d\boldsymbol{X}^{(\alpha)T}\boldsymbol{F}^T\boldsymbol{F}d\boldsymbol{X}^{(\alpha)} = \boldsymbol{F}^T\boldsymbol{F} : (d\boldsymbol{X}^{(\alpha)} \otimes d\boldsymbol{X}^{(\alpha)}) . \qquad (24)$$

Starting to look familiar? (I remind you of the definition of the Green-Lagrange strain tensor $\boldsymbol{E} \equiv \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$). In order to write the energy, it seems that the matrices $\boldsymbol{M}^{(\alpha)} \equiv d\boldsymbol{X}^{(\alpha)} \otimes d\boldsymbol{X}^{(\alpha)}$ will be handy. Explicitly, they read

$$\boldsymbol{M}^{(A)} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \boldsymbol{M}^{(B)} = \frac{a^2}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \quad \boldsymbol{M}^{(C)} = \frac{a^2}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} . \quad (25)$$

Note that $\operatorname{tr}\boldsymbol{M}^{(\alpha)} = a^2$ and $\sum \boldsymbol{M}^{(\alpha)} = \frac{3}{2}a^2\boldsymbol{I}$. We are now ready to write the energy:

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\begin{aligned}
Su &= \frac{1}{2}k\left(L_A - a\right)^2 + \frac{1}{2}k\left(L_B - a\right)^2 + \frac{1}{2}k\left(L_C - a\right)^2 \\
&= \frac{1}{2}k\left[L_A^2 + L_B^2 + L_C^2 - 2a(L_A + L_B + L_C) + 3a^2\right] \\
&= \frac{1}{2}k\left[\boldsymbol{F}^T\boldsymbol{F} : \left(\sum_\alpha \boldsymbol{M}^{(\alpha)}\right) - 2a\sum_\alpha \sqrt{\boldsymbol{F}^T\boldsymbol{F} : \boldsymbol{M}^{(\alpha)}} + 3a^2\right] \\
&= \frac{1}{2}k\left[\frac{3}{2}a^2 \operatorname{tr}(\boldsymbol{F}^T\boldsymbol{F}) - 2a\sum_\alpha \sqrt{\boldsymbol{F}^T\boldsymbol{F} : \boldsymbol{M}^{(\alpha)}} + 3a^2\right] \\
&= \frac{1}{2}k\left[\frac{3}{2}a^2 \operatorname{tr}(2\boldsymbol{E} + \boldsymbol{I}) - 2a\sum_\alpha \sqrt{(2\boldsymbol{E} + \boldsymbol{I}) : \boldsymbol{M}^{(\alpha)}} + 3a^2\right] \Rightarrow \\
u &= k\sqrt{3}\left[\operatorname{tr}\boldsymbol{E} - \frac{2}{3}\sum_\alpha \sqrt{1 + 2\boldsymbol{E} : \tilde{\boldsymbol{M}}^{(\alpha)}} + 2\right] ,
\end{aligned}
\qquad (26)
$$

where we defined $\tilde{\boldsymbol{M}}^{(\alpha)} = a^{-2}\boldsymbol{M}^{(\alpha)}$ and in the last transition we divided by S.

So that's about it. This is the energy function in its full glory. Two things are worth noting about this energy function: First, you can already see that it depends only on $\boldsymbol{E}$ which is rotationally invariant and symmetric. That is, the continuum level quantity $\boldsymbol{E}$ emerges naturally from the discrete analysis. Second, note that although we are using strictly linear springs whose energies are quadratic in their elongation the energy function is far from being a simple quadratic function. This is because the geometry itself introduces nonlinearities.

Let's take this one step further, and develop u in orders of $\boldsymbol{E}$ to see if we can get understand this better. The zeroth order clearly vanishes, which is a good sign that we didn't have any mistakes. The first order should also vanish (why?), and indeed

$$u^{(1)} = k\sqrt{3}\left[\operatorname{tr}\boldsymbol{E} - \frac{2}{3}\sum_\alpha \left(1 + \boldsymbol{E} : \tilde{\boldsymbol{M}}^{(\alpha)}\right) + 2\right] = k\sqrt{3}\left[\operatorname{tr}\boldsymbol{E} - \frac{2}{3}\boldsymbol{E} : \sum_\alpha \tilde{\boldsymbol{M}}^{(\alpha)}\right] = 0$$

So in all subsequent orders we can worry only about the sqrt term. To second order we

have (recall that $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$)

$$u^{(2)} = \frac{k}{\sqrt{3}} \sum_\alpha \left(\boldsymbol{E} : \tilde{\boldsymbol{M}}^{(\alpha)} \right)^2 = \frac{k}{\sqrt{3}} \sum_\alpha E_{ij} E_{kl} \tilde{M}_{ij}^{(\alpha)} \tilde{M}_{kl}^{(\alpha)} = \frac{1}{2} E_{ij} E_{kl} C_{ijkl} \tag{27}$$

$$C_{ijkl} \equiv \frac{2k}{\sqrt{3}} \sum_\alpha \tilde{M}_{ij}^{(\alpha)} \tilde{M}_{kl}^{(\alpha)} = \left(\frac{2k}{a^4\sqrt{3}} \sum_\alpha d\boldsymbol{X}^{(\alpha)} \otimes d\boldsymbol{X}^{(\alpha)} \otimes d\boldsymbol{X}^{(\alpha)} \otimes d\boldsymbol{X}^{(\alpha)} \right)_{ijkl} \tag{28}$$

$\boldsymbol{C}$ is called the stiffness tensor and you've heard a lot about it in Eran's lecture. Note that the fact that to quadratic order $u = E_{ij} E_{kl} C_{ijkl}$ for *some* tensor $\boldsymbol{C}$ is generally true in the framework of linear elasticity. Also, the fact that C_{ijkl} is symmetric under any of the transpositions $i \leftrightarrow j$, $k \leftrightarrow l$ and $ij \leftrightarrow kl$ (and any compositions of them) also generally holds. However, our $\boldsymbol{C}$ has some extra symmetries on top of that. First, note that $C_{ikjl} \propto \sum_\alpha dX_i^{(\alpha)} dX_j^{(\alpha)} dX_k^{(\alpha)} dX_l^{(\alpha)}$ so the tensor is fully symmetric to any permutation of $ijkl$. But there's even more than that: In fact, explicit calculation shows that

$$C_{ijkl} = \frac{\sqrt{3}}{4} k \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} \right) , \tag{29}$$

which is, pretty surprisingly, an *isotropic tensor*. That is, to leading order the elastic response of triangular lattice is isotropic, although the lattice is clearly not invariant to rotations. Explicitly, we have

$$u(\boldsymbol{E}) = \frac{\sqrt{3}k}{8} E_{ij} E_{kl} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} \right) = \frac{\sqrt{3}}{8} k \left((\operatorname{tr} \boldsymbol{E})^2 + 2\operatorname{tr}(\boldsymbol{E}^2) \right) . \tag{30}$$

As explained at length by Eran last week, the relation between the stress and the strain is $\boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}} = C_{ijkl}\varepsilon_{kl}$ (note that I switched from $\boldsymbol{E}$ to $\boldsymbol{\varepsilon}$, which is OK because we're in the linear approximation within which the two are identical). Thus,

$$\sigma_{ij}(\boldsymbol{\varepsilon}) = \frac{\sqrt{3}k}{4} \varepsilon_{kl} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} \right) = \frac{\sqrt{3}k}{4} \left(\delta_{ij} \operatorname{tr} \boldsymbol{\varepsilon} + 2\varepsilon_{ij} \right) ,$$
$$\boldsymbol{\sigma} = \frac{\sqrt{3}k}{2} (\operatorname{tr} \boldsymbol{\varepsilon}) \boldsymbol{I} + \frac{\sqrt{3}k}{4} \boldsymbol{\varepsilon} , \tag{31}$$

As you'll see next week, the form $\boldsymbol{\sigma} = \lambda(\operatorname{tr} \boldsymbol{\varepsilon}) \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}$ for two constants λ, μ is the most general constitutive relation for linear isotropic materials. λ and μ are called Lamé coefficients (μ is also called the shear modulus). Note that the microscopic lengthscale a does not appear explicitly in the theory, a common situation in continuum theories (but if we were to include also strain-gradient terms, we could not have avoided having a in our equations).