Linear elasticity I

This TA session is the first of three (we'll make a brief deviation to linear thermo-elasticity, but then return) in which we'll dive deep into linear elasticity theory. Linear elasticity by itself can be the topic of a year-long course, and in these TA's we'll try to convey a significant fraction of the richness of the theory. While Eran will continue to explore more complex constitutive behaviors (non-linear elasticity, visco-elasticity, and so on), in the next few TA sessions we'll stay in the linear realm and try to acquire the knowledge we believe is fundamental for a well-educated physicist.

1 Green's function for an infinite medium

It seems that the time is ripe to fully and completely solve a problem, with all the 2π's and everything, without resorting to hand waving and scaling arguments. While the emphasis will still be on the structure of the problem, we think it will be instructive, at least once, to write down a problem and solve it exactly.

A nice problem to consider is the response of an infinite linear isotropic homogeneous elastic medium to a localized force $\vec{f} = F_i\delta(\vec{r})$, i.e. finding the Green's function of an infinite medium.

We define the Green function (matrix) $G_{ij}(\vec{r}_1, \vec{r}_2)$ as the displacement in the i direction at the point $\vec{r}_1$ as a response to a localized force in the j direction applied at $\vec{r}_2$. For homogeneous materials we know that $G_{ij}(\vec{r}_1, \vec{r}_2) = G_{ij}(\vec{r}_1 - \vec{r}_2)$. We therefore denote $\vec{r} = \vec{r}_1 - \vec{r}_2$. You all know well that, within the linear elastic theory, this will allow us to solve the problem of an arbitrary force distribution $f(\vec{r})$ by convolving $f(\vec{r})$ with the Green function.

Conceptually, the structure is the following. We would like to find a displacement field $u_i(\vec{r})$, from which we calculate

$$u_i \quad \Rightarrow \quad \varepsilon_{ij} \quad \Rightarrow \quad \sigma_{ij} \quad \Rightarrow \quad \partial_j\sigma_{ij} = \delta(\vec{r}) \ ,$$

but of course, we will want to do the whole thing backwards. I stress (no pun intended[1]) that we already know how to express div($\boldsymbol{\sigma}$) in terms of u_i - this is what we called the Navier-Lamé equation. But for completeness, let's do it again:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} = \lambda\partial_k u_k\delta_{ij} + \mu\left(\partial_i u_j + \partial_j u_i\right) \ ,$$

$$\partial_j\sigma_{ij} = \partial_j\left(\lambda\partial_k u_k\delta_{ij} + \mu\left(\partial_i u_j + \partial_j u_i\right)\right) = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i \ ,$$

which is nothing but the u-dependent term of the Navier-Lamé equation. Since the equation is linear, it seems right to solve the problem by Fourier transform. We use the conventions

$$u_i(\vec{q}) = \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{r}} u_i(\vec{r}) \ , \tag{1}$$

$$u_i(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3\vec{q}\, e^{-i\vec{q}\cdot\vec{r}} u_i(\vec{q}) \ . \tag{2}$$

[1] Just kidding, of course it's intended.

The equation we want to transform is

$$(\lambda+\mu)\partial_j\partial_i u_j + \mu\partial_j\partial_j u_i = -F_i\delta(\vec{x}) \ , \tag{3}$$

which readily gives

$$-(\lambda+\mu)q_j q_i\, u_j - \mu\, q_j q_j\, u_i = -F_i \ . \tag{4}$$

This is a matrix equation:

$$[(\lambda+\mu)q_j q_i + \mu\, q_k q_k \delta_{ij}]\, u_j = F_i \ . \tag{5}$$

Or even more explicitly:

$$\begin{pmatrix} (\lambda+\mu)\,q_1^2+\mu|\vec{q}|^2 & (\lambda+\mu)\,q_1 q_2 & (\lambda+\mu)\,q_1 q_3 \\ (\lambda+\mu)\,q_1 q_2 & (\lambda+\mu)\,q_2^2+\mu|\vec{q}|^2 & (\lambda+\mu)\,q_2 q_3 \\ (\lambda+\mu)\,q_1 q_3 & (\lambda+\mu)\,q_2 q_3 & (\lambda+\mu)\,q_3^2+\mu|\vec{q}|^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \ . \tag{6}$$

This matrix can be inverted by using any of your favorite methods, giving

$$u_i = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_k q_k} - \frac{1}{2(1-\nu)}\frac{q_i q_j}{(q_k q_k)^2}\right] F_j \ . \tag{7}$$

Where we used $\nu = \frac{\lambda}{2(\lambda+\mu)}$. In other words, we have found the Fourier representation of the Green function:

$$G_{ij}(\vec{q}) = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_k q_k} - \frac{1}{2(1-\nu)}\frac{q_i q_j}{(q_k q_k)^2}\right] \ . \tag{8}$$

We now need to preform the inverse Fourier transform. We'll begin with the first term, and do it in spherical coordinates with the q_z direction parallel to $\vec{r}$:

$$\frac{1}{(2\pi)^3}\int d^3\vec{q}\,\frac{e^{-i\vec{q}\cdot\vec{x}}}{q^2} = \frac{1}{(2\pi)^3}\int \frac{e^{-iqr\cos\theta}}{q^2} q^2 \sin\theta\, d\theta\, dq\, d\phi = \frac{1}{(2\pi)^2}\int e^{-iqr\cos\theta}\sin\theta\, d\theta\, dq$$
$$= \frac{1}{(2\pi)^2}\int \frac{e^{iqr}-e^{-iqr}}{iqr} dq = \frac{2}{(2\pi)^2}\int_0^\infty \frac{\sin(qr)}{qr} dq = \frac{1}{4\pi r} \ .$$

If this result surprises you, maybe you should remind yourself of the first linear field theory that you met in your life - Poisson's equation for a point charge $\nabla^2\phi = \delta(\vec{r})$. I'll let you complete the analogy by yourselves.

For the second term, we use a dirty trick:

$$\begin{aligned}\mathcal{F}^{-1}\left\{\frac{q_i q_j}{(q_k q_k)^2}\right\} &= -\frac{1}{2}\mathcal{F}^{-1}\left\{q_i \frac{\partial}{\partial q_j}\left(\frac{1}{q_k q_k}\right)\right\} = \frac{i}{2}\frac{\partial}{\partial x_i}\mathcal{F}^{-1}\left\{\frac{\partial}{\partial q_j}\left(\frac{1}{q_k q_k}\right)\right\} \\ &= \frac{i}{2}\frac{\partial}{\partial x_i}\left(-i x_j \mathcal{F}^{-1}\left\{\frac{1}{q_k q_k}\right\}\right) = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{x_j}{4\pi r}\right) \\ &= \frac{1}{2}\left(\frac{\delta_{ij}}{4\pi r} - \frac{x_i x_j}{4\pi r^3}\right) \ .\end{aligned} \tag{9}$$

Plugging in (8) we get

$$G_{ij}(\vec{r}) = \frac{1}{16(1-\nu)\pi\mu r}\left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{r^2}\right] \ . \tag{10}$$

A more elegant way to go, is to write G_{ij} as gradients of r (I mean $|r|$, not $\vec{r}$):

$$G_{ij}(\vec{r}) = \frac{1}{8\pi\mu}\left[\partial_k\partial_k r\,\delta_{ij} - \frac{1}{2(1-\nu)}\partial_i\partial_j r\right] = \frac{1}{8\pi\mu}\left[\boldsymbol{I}\,\nabla^2 r - \frac{\nabla\nabla r}{2(1-\nu)}\right] \ . \tag{11}$$

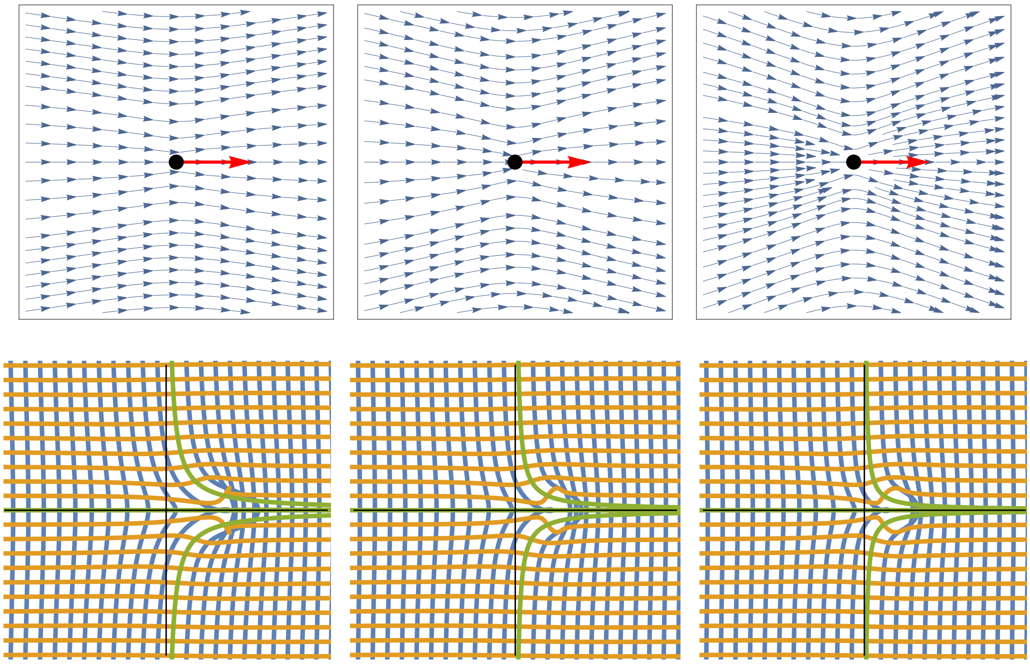

Figure 1: Top: Displacement field lines in the $x - y$ plane for point $\vec{F} = F\hat{x}$ in the horizontal direction. From left to right, with $\nu = 0$, 0.33, 0.5. Bottom: deformation of a regular mesh under this motion. Note that crossing of two lines of the same color is physically forbidden (why?).

1.1 Notes about the solution

1. The displacements go as $1/r$, which means that the strain/stress go as $1/r^2$. Therefore the elastic energy density, which goes like ε^2, goes like $1/r^4$ and *its integral diverges*. This is much like the case of electrostatics, where the total energy of the electrical field of a point charge diverges.

2. The scaling $u \sim 1/r$ could also have been obtained from simple dimensional analysis. It is quite common that dimensional considerations in elasticity take the "dimension" from the shear modulus μ, and then there's an unknown (and usually uninteresting) function of ν.

3. You might have noticed that G_{ij} is a symmetric matrix. This might look at first glance as a trivial property that stems from the translational symmetry or rotational symmetry (=isotropy), but this is not the case. This symmetry property does not stem from any simple argument (that I can think of). Instead, this symmetry is a special case of a more general property that is called *reciprocity*. For a general linear elastic solid, and by general I mean that $C_{ijkl}(\boldsymbol{r})$ can have any symmetry and can even depend on space, the static Green function satisfies

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}') = G_{ji}(\boldsymbol{r}', \boldsymbol{r}) \ . \tag{12}$$

The proof of this general property is given in Sec. 4 of this file. I'm not sure I'll have time to go over it in class, but I wanted you to be able to read it even if we don't, in case you're interested.

2 Hooke's law, stiffness, and compliance

Hooke's law is

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \left(\lambda + \frac{2}{3}\mu\right)\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij}\right) . \tag{13}$$

Let's write it explicitly, to get a better feel of what's going on

$$\begin{aligned}
\sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{xx} + \lambda\left(\varepsilon_{yy} + \varepsilon_{zz}\right) , \\
\sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{zz}\right) , \quad (14) \\
\sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) ,
\end{aligned}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij}, \qquad i \neq j \tag{15}$$

or in matrix form

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} = \frac{2\mu}{1-2\nu}\begin{pmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix} . \tag{16}$$

The shear terms $i \neq j$ have a simple dependence, while the others are a bit more complicated. This equation has the general form of $\boldsymbol{\sigma} =$
$bmC\boldsymbol{\epsilon}$, where $\boldsymbol{C}$ is called the stiffness tensor.

Let's try to invert these relations to find $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ - that is, let's find the compliance matrix for an isotropic linear elastic material. The same considerations that we used to derive Eq. (13) (i.e. that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are related by a 4-rank isotropic tensor) allow us to write

$$\varepsilon_{ij} = a\,\sigma_{kk}\delta_{ij} + b\,\sigma_{ij} , \tag{17}$$

so finding the compliance reduces to finding a, b. If $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, then Eq.(13) and (17)[2] reduce to, respectively,

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} , \tag{18}$$

$$\varepsilon_{ij} = b\,\sigma_{ij} , \tag{19}$$

so we immediately find $b = (2\mu)^{-1}$. Taking the trace of Eq. (13) and (17) gives, respectively

$$\mathrm{tr}(\boldsymbol{\sigma}) = (3\lambda + 2\mu)\,\mathrm{tr}(\boldsymbol{\varepsilon}) , \tag{20}$$

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = (3a + b)\,\mathrm{tr}(\boldsymbol{\sigma}) , \tag{21}$$

[2] Recall that $\mathrm{tr}\,\boldsymbol{\sigma} \propto \mathrm{tr}\,\boldsymbol{\varepsilon}$

which tells us that

$$
\begin{aligned}
3a + b = (3\lambda + 2\mu)^{-1} \quad &\Rightarrow \quad a = \frac{1}{3}\left(\frac{1}{3\lambda + 2\mu} - b\right) = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \Rightarrow \\
\varepsilon_{ij} &= -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\mathrm{tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \ .
\end{aligned}
\tag{22}
$$

Writing explicitly, we have

$$
\begin{aligned}
\varepsilon_{xx} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{xx} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{yy} + \sigma_{zz}\right) \ , \\
\varepsilon_{yy} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{yy} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{xx} + \sigma_{zz}\right) \ , \\
\varepsilon_{zz} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{zz} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{xx} + \sigma_{yy}\right) \ ,
\end{aligned}
\tag{23}
$$

$$
\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} \qquad \text{for } i \neq j \ . \tag{24}
$$

As discussed in class, the term in parentheses in Eq.(23) is the inverse of the Young's modulus, and for uniaxial stress it reads

$$
E = \sigma_{xx}/\varepsilon_{xx} = \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)^{-1} = \mu\frac{3\lambda + 2\mu}{\lambda + \mu} \ . \tag{25}
$$

It is the microscopic analogue of the spring constant. If the uniaxial stress is in the x direction, then we have

$$
\varepsilon_{yy} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{xx} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}E\varepsilon_{xx} = -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_{xx} \ , \tag{26}
$$

and as discussed in class, $-\varepsilon_{yy}/\varepsilon_{xx}$ is known as the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$. Rewriting Eqs. (23) with these quantities yields a much nicer expression:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \ , \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \ , \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \ , \\
\varepsilon_{ij} &= \frac{1 + \nu}{E}\sigma_{ij} \quad \text{for} \quad i \neq j \ ,
\end{aligned}
\tag{27}
$$

or in matrix form

$$
\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{zy} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{pmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{pmatrix} \ . \tag{28}
$$

This is just an inversion of Eq. (16) into the form $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{S} = \boldsymbol{C}^{-1}$ is the compliance tensor (if you noticed that $\boldsymbol{C}$ is called the stiffness tensor and $\boldsymbol{S}$ is called the compliance tensor and wondered about it, this is not a mistake and there is no intention to confuse you. It is a long-time convention that cannot be reverted anymore). A useful table with all the conversions is found in Wikipedia. We are now ready to perform the reduction to 2D.

3 2D elasticity

As shown in class, the field equation of elasticity is the Navier-Lamé equation

$$\rho\partial_{tt}\boldsymbol{u} = (\lambda + \mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{b}\ . \tag{29}$$

It is notoriously difficult to solve. However, things become much simpler in 2D. There are 3 generic ways in which one can obtain an effectively 2D elastic system, by ignoring the z dimension:

1. **Plane stress**: when $\sigma_{zi} = 0$. This typically holds in very thin systems (in the z direction).
2. **Plane strain**: when the system is translationally invariant in z, and therefore ∂_z of anything vanishes. This typically holds in very thick (in the z direction) systems.
3. **Anti plane - scalar elasticity**: If the motion is only in z and does not depend on z. This is mainly a pedagogical example, though some physical examples exist, mainly thin sheets and mode-III fracture (tearing).
4. **"Flatland"**: If the world truly is two dimensional. We will not treat this case as it is a bit delicate.

The fourth case is a bit delicate, and we will not discuss it in the course. We'll now develop the theory for the first two cases, and Eran will demonstrate in class the formalism of scalar elasticity (the third case).

To see how one reduces elasticity to 2 dimensions, let us explicitly write Hooke's law (27) in terms of μ and ν. We note that although the stiffness matrix is a 4-rank tensor, it can be represented as a 6 by 6 matrix by rearranging the entries as in Eq. (16).

3.1 Plane-stress

We first consider objects that are thin in one dimension, say z, and are deformed in the xy-plane. What happens in the z-direction? Since the two planes $z = 0$ and $z = h$ (where h is the thickness which is much smaller than any other lengthscale in the problem) are traction-free, we approximate $\sigma_{zz} = 0$ everywhere (an approximation that becomes better and better as $h \to 0$). Similarly, we have $\sigma_{zy} = \sigma_{zx} = 0$. We can therefore set $\sigma_{zz} = \sigma_{zy} = \sigma_{zx} = 0$ in Eq. (28) to obtain

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix}\ , \tag{30}$$

and

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx}+\sigma_{yy}\right) \ . \tag{31}$$

To obtain the plane-stress analog of Eq. (29), the Navier-Lamé equation, we need to invert Eq. (30), obtaining

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} \ . \tag{32}$$

Note that the last equation *can not be obtained from Eq. (16)* by simply removing columns and rows. We can now substitute the last relation in the 2D momentum balance equation $\nabla\cdot\boldsymbol{\sigma} = \rho\partial_{tt}\boldsymbol{u}$ (we stress again that $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are already 2D here). The resulting 2D equation reads

$$\left[\frac{\nu E}{1-\nu^2}+\mu\right]\nabla\left(\nabla\cdot\boldsymbol{u}\right)+\mu\nabla^2\boldsymbol{u} = \rho\partial_{tt}\boldsymbol{u} \ , \tag{33}$$

which is identical in form to Eq. (29) simply with a renormalized λ

$$\lambda\to\tilde{\lambda} = \frac{\nu E}{1-\nu^2} = \frac{2\nu\mu}{1-\nu} = \frac{2\lambda\mu}{\lambda+2\mu} \ . \tag{34}$$

The shear modulus μ remains unchanged

$$\tilde{\mu} = \mu = \frac{E}{2(1+\nu)} \ . \tag{35}$$

Finally, we can substitute $\sigma_{xx}(x,y)$ and $\sigma_{yy}(x,y)$ inside Eq. (31) to obtain $\varepsilon_{zz}(x,y)$. Note that $u_z(x,y,z) = \varepsilon_{zz}(x,y)z$ is linear in z.

3.2 Plane-strain

We now consider objects that are very thick in one dimension, say z, and are deformed in the xy-plane with no z dependence. These physical conditions are termed plane-strain and are characterized by $\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$. Eliminating these components from Eq. (16) we obtain

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{2\mu}{1-2\nu}\begin{pmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} \ , \tag{36}$$

and

$$\sigma_{zz}(x,y) = \frac{2\mu\nu}{1-2\nu}\left[\varepsilon_{xx}(x,y)+\varepsilon_{yy}(x,y)\right] \ . \tag{37}$$

We can now substitute Eq. (36) in the 2D momentum balance equation $\nabla\cdot\boldsymbol{\sigma} = \rho\partial_{tt}\boldsymbol{u}$ (where again $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are 2D). The resulting 2D equation is identical to Eq. (29), both in form and in the elastic constants. With the solution at hand, we can use Eq. (37) to calculate $\sigma_{zz}(x,y)$. Finally, we note that Eq. (36) can be inverted to

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{2\mu}\begin{pmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} \ , \tag{38}$$

which can not be simply obtained from Eq. (28) by eliminating columns and rows. Using the last relation we can rewrite Eq. (37) as

$$\sigma_{zz}(x,y) = \nu\left[\sigma_{xx}(x,y) + \sigma_{yy}(x,y)\right] \ . \tag{39}$$

In summary, we see that in both plane-stress and plane-strain cases we can work with 2D objects instead of their 3D counterparts, which is a significant simplification. This allows to use the heavy mathematical tools available in 2D: complex analysis and conformal mapping. One has, though, to be careful with the elastic constants as explained above.

4 Clapeyron's & Betti's theorems and reciprocity (time permitting)

Clapeyron's theorem is a nice result in static elasticity that says that "the total elastic energy stored in a body is equal to half the work done by the external forces computed assuming these forces had remained constant from the initial state to the final state". You already know a trivial version of this theorem, in the context of the energy stored in a simple spring. Assume you load a given spring with a force F. The energy stored in the spring is

$$\mathcal{U} = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(kx)x = \tfrac{1}{2}Fx \ . \tag{40}$$

If you assume that during the elongation of the spring the force had the constant value F (although the force clearly started from zero and ramped up to F) you get that the total work done was Fx. The real energy, however, is exactly one half that value.

The proof is pretty simple. Assume an elastic body is subject to volume forces $\boldsymbol{b}(\boldsymbol{r})$ and surface tractions $\boldsymbol{t}(\boldsymbol{r})$. The total elastic energy in the body is

$$\begin{aligned}\mathcal{U} &= \int_\Omega \tfrac{1}{2}\sigma_{ij}(\boldsymbol{r})\varepsilon_{ij}(\boldsymbol{r})d^3\boldsymbol{r} \stackrel{(1)}{=} \tfrac{1}{2}\int_\Omega \sigma_{ij}\partial_i u_j \, d^3\boldsymbol{r} = \tfrac{1}{2}\int_\Omega \Big[\partial_i(\sigma_{ij}u_j) - u_j\partial_i\sigma_{ij}\Big]\, d^3\boldsymbol{r} \\ &\stackrel{(2)}{=} \tfrac{1}{2}\left[\int_{\partial\Omega} \sigma_{ij}u_j n_i \, d^2\boldsymbol{r} + \int_\Omega b_j u_j \, d^3\boldsymbol{r}\right] = \tfrac{1}{2}\left[\int_{\partial\Omega} t_j u_j \, d^2\boldsymbol{r} + \int_\Omega b_j u_j \, d^3\boldsymbol{r}\right] \ ,\end{aligned} \tag{41}$$

where in the transition (1) we used the symmetry of $\boldsymbol{\sigma}$ and in (2) we used Gauss' theorem and the equilibrium condition $\partial_j\sigma_{ij} + b_i = 0$.

Betti's theorem (sometimes called "Betti's reciprocal theorem") is a general important result about the energetics in static elasticity. It is useful in theoretical analyses (e.g. for proving the uniqueness of solutions to the static Navier-Lamè equation) and is also for practical use if average quantities are calculated (you might have an exercise about that in the HW).

Suppose that when a set of body and traction forces $\boldsymbol{b}^{(1)}$ and $\boldsymbol{t}^{(1)}$ is applied to a body, the resulting deformation field is $\boldsymbol{u}^{(1)}$, and the stress and strain fields are $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\varepsilon}^{(1)}$. Suppose also that when a different set of body and traction forces, $\boldsymbol{b}^{(2)}$ and $\boldsymbol{t}^{(2)}$, is applied then the resulting deformation field is $\boldsymbol{u}^{(2)}$, and the stress and strain fields are $\boldsymbol{\sigma}^{(2)}$ and $\boldsymbol{\varepsilon}^{(2)}$. Betti's theorem states that

$$\int_{\partial\Omega} t_i^{(1)}u_i^{(2)} d^2\boldsymbol{r} + \int_\Omega b_i^{(1)}u_i^{(2)} d^3\boldsymbol{r} = \int_{\partial\Omega} t_i^{(2)}u_i^{(1)} d^2\boldsymbol{r} + \int_\Omega b_i^{(2)}u_i^{(1)} d^3\boldsymbol{r} \ . \tag{42}$$

That is, the work done by the set (1) through the displacements produced by the set (2) is equal to the work done by the set (2) through the displacements produced by the set (1).

To see this, consider the total energy of the system if both sets of forces are applied:

$$\mathcal{U} = \frac{1}{2}\int_\Omega \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}\right)\left(\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}\right) d^3\boldsymbol{r} \tag{43}$$

$$= \frac{1}{2}\int_\Omega \left[\sigma_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \sigma_{ij}^{(2)}\varepsilon_{ij}^{(2)} + \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} + \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}\right] d^3\boldsymbol{r} \tag{44}$$

$$= \mathcal{U}^{(1)} + \mathcal{U}^{(2)} + \frac{1}{2}\int_\Omega \Big[\underbrace{\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}}_{A} + \underbrace{\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}}_{B}\Big] d^3\boldsymbol{r}\ . \tag{45}$$

The first two terms are the energies of each of the "pure modes" and the integral is the "interaction energy". Using exactly the same arguments as we used in Eq. (41) it is possible (and easy) to show that term A equals the left-hand-side of Eq. (42) and term B equals the right-hand-side. However, terms A and B are equal since

$$A = \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} = C_{ijkl}\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} = C_{klij}\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} = \varepsilon_{kl}^{(1)}\sigma_{kl}^{(2)} = B\ , \tag{46}$$

where the symmetry $C_{ijkl} = C_{klij}$ was used, and the theorem is proven. Note that we did not invoke isotropy or translational invariance. The symmetry $C_{ijkl} = C_{klij}$ is a thermodynamic one that holds for the most general linear elastic case.

With Betti's theorem it is very easy to prove the reciprocity property, Eq. (12). If the set (1) is a point force $\boldsymbol{F}^{(1)}$ applied at $\boldsymbol{r}^{(1)}$ and set (2) is a point force $\boldsymbol{F}^{(2)}$ applied at $\boldsymbol{r}^{(2)}$ then

$$\begin{aligned} b_i^{(1)} &= F_i^{(1)}\delta(\boldsymbol{r} - \boldsymbol{r}^{(1)}) & b_i^{(2)} &= F_i^{(2)}\delta(\boldsymbol{r} - \boldsymbol{r}^{(2)}) \\ u_i^{(1)}(\boldsymbol{r}) &= G_{ij}(\boldsymbol{r}, \boldsymbol{r}^{(1)})F_j^{(1)} & u_i^{(2)}(\boldsymbol{r}) &= G_{ij}(\boldsymbol{r}, \boldsymbol{r}^{(2)})F_j^{(2)}\ . \end{aligned} \tag{47}$$

Homogeneous boundary conditions on $\partial\Omega$ mean either $\boldsymbol{u} = 0$ or $\boldsymbol{t} = 0$, or that some portions of the surface is subject to that and other portions to the other. In any case, this means that the boundary integrals in Eq. (42) vanish identically. Then, using the properties of the delta function, we immediately get

$$F_i^{(1)}F_j^{(2)}G_{ij}(\boldsymbol{r}^{(1)}, \boldsymbol{r}^{(2)}) = F_i^{(2)}F_j^{(1)}G_{ij}(\boldsymbol{r}^{(2)}, \boldsymbol{r}^{(1)})\ , \tag{48}$$

and since $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{r}^{(1)}$ and $\boldsymbol{r}^{(2)}$ were arbitrary the reciprocal theorem of Eq. (12) is obtained.