Linear Elasticity II - Waves

1 Elastic waves

I remind you that you have shown in class that the Navier-Lamé Equation,

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}+\boldsymbol{b}=\rho\partial_{tt}\boldsymbol{u} \ , \tag{1}$$

is basically two uncoupled wave equations for dilatational and shear waves. They propagate at velocities

$$c_s=\sqrt{\frac{\mu}{\rho}} \ , \qquad c_d=\sqrt{\frac{\lambda+2\mu}{\rho}} \ . \tag{2}$$

Thus, Eq. (1) can also be written as

$$(c_d^2-c_s^2)\nabla(\nabla\cdot\boldsymbol{u})+c_s^2\nabla^2\boldsymbol{u}+\boldsymbol{b}/\rho=\partial_{tt}\boldsymbol{u} \ . \tag{3}$$

The two wave speeds differ by a significant factor. c_s is always smaller than c_d and their ratio is

$$\beta\equiv\frac{c_s}{c_d}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1-2\nu}{2(1-\nu)}} \ . \tag{4}$$

For a typical value of $\nu=1/3$, this gives a ratio of $\frac{1}{2}$. This function is plotted in Fig. 1. Note that the ratio goes to 0 for $\nu\to\frac{1}{2}$. This is because incompressible materials ($\nu=1/2$) the dilatational velocity c_d diverges (as the bulk modulus K diverges). Seismographers use the difference in propagation velocity to determine the distance to an earthquake source, as is seen in Fig. 1.

1.1 Leftovers from Eran's lecture

In class, you have discussed the polarization of these two waves by writing

$$\boldsymbol{u}=g(\boldsymbol{x}\cdot\boldsymbol{n}-c\,t)\boldsymbol{a} \ , \tag{5}$$

where $\boldsymbol{n}$ is the propagation direction, $\boldsymbol{a}$ is the direction of the displacement and $|\boldsymbol{n}|=|\boldsymbol{a}|=1$. You have shown without proof that this implies

$$(c_d^2-c_s^2)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}+(c_s^2-c^2)\boldsymbol{a}=0 \ . \tag{6}$$

Eran promised that I will show how to get from the former to the latter. This is done simply by applying the differential operators to $\boldsymbol{u}$ giving

$$\nabla g=\partial_i g=g'n_i=g'\boldsymbol{n} \ , \tag{7}$$

$$\nabla\boldsymbol{u}=\partial_j u_i=\partial_j(ga_i)=g'n_ja_i=g'\boldsymbol{a}\otimes\boldsymbol{n} \ , \tag{8}$$

$$\nabla^2\boldsymbol{u}=\nabla\cdot\nabla\boldsymbol{u}=\partial_j(g'n_ja_i)=g''n_jn_ja_i=g''\boldsymbol{a} \ , \tag{9}$$

$$\nabla\cdot\boldsymbol{u}=\mathrm{tr}(\nabla\boldsymbol{u})=g'\boldsymbol{a}\cdot\boldsymbol{n} \ , \tag{10}$$

$$\nabla(\nabla\cdot\boldsymbol{u})=\partial_i(g'\boldsymbol{a}\cdot\boldsymbol{n})=g''(\boldsymbol{a}\cdot\boldsymbol{n})n_i=g''(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n} \ , \tag{11}$$

$$\partial_{tt}\boldsymbol{u}=c^2g''\boldsymbol{a} \ . \tag{12}$$

Plugging Eqs. (11)-(12) into (3) gives immediately Eq. (6).

The two waves are independent in the bulk. However, on the boundary of a body the traction-free condition $\sigma_{ij}n_j = 0$ couples between the two modes, and more modes arise with a distinct propagation velocity. These are called Rayleigh waves, and are very interesting.

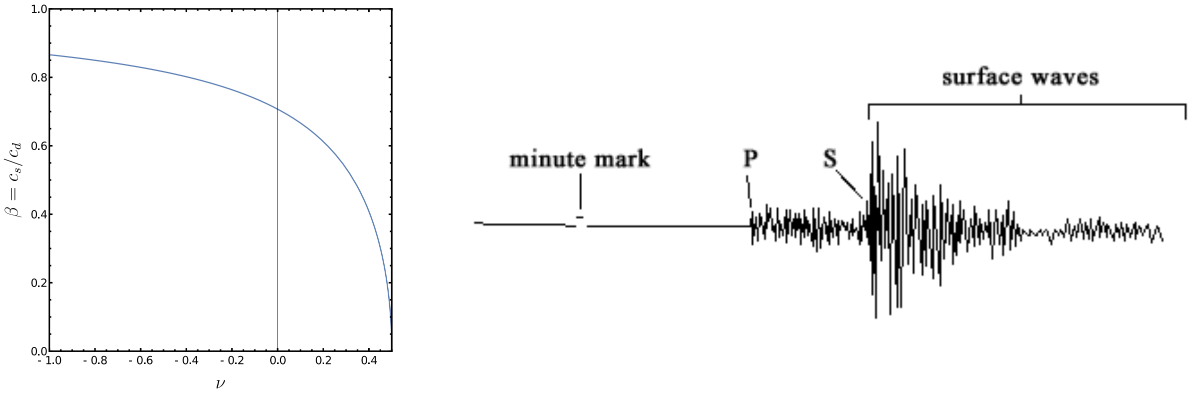


Figure 1: Left: c_s/c_d as a function of Poisson's ratio (Eq. (4)). Right: Seismograph reading of an earthquake. One can clearly see a P-wave (longitudinal) and an S-wave (transverse) arriving at different times. Later, surface waves are visible. The time difference can be used to obtain the distance from the earthquake source.

1.2 Rayleigh waves

So let's see exactly how this works. We want to look at surface waves which propagate, say, in the x-direction. To this end, consider a material that fills the lower half-space $z < 0$, and assume that

$$\boldsymbol{u} = \boldsymbol{f}(z)e^{ikx-i\omega t} \ . \tag{13}$$

If $\boldsymbol{u}$ satisfies the wave equation $(\frac{1}{c_i^2}\partial_{tt} - \nabla^2)\boldsymbol{u} = 0$, with $c_i = c_s$ or c_d, we have

$$\partial_{zz}\boldsymbol{f} = \left(k^2 - \frac{\omega^2}{c_i^2}\right)\boldsymbol{f} \ .$$

If $k^2 > \frac{\omega^2}{c_i^2}$ this gives a damped wave in the bulk. We denote

$$\boldsymbol{f}(z) = \boldsymbol{\gamma}^{(i)}e^{\eta_i z}, \qquad \eta_i = \sqrt{k^2 - \frac{\omega^2}{c_i^2}}, \quad i = s, d \ .$$

As stated above, Rayleigh waves are modes which mix dilational and shear waves. We therefore guess the ansatz

$$\boldsymbol{u} = \boldsymbol{u}^{(d)} + \boldsymbol{u}^{(s)} \ , \tag{14}$$

$$\boldsymbol{u}^{(i)} = \left(\gamma_x^{(i)}\hat{\boldsymbol{x}} + \gamma_z^{(i)}\hat{\boldsymbol{z}}\right)e^{\eta_i z + ikx - i\omega t} \ , \tag{15}$$

where $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ are dilatational and shear waves, and $\gamma_j^{(i)}$ are constants. That is, each of $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ satisfies its own wave equation,

$$(\partial_{tt} - c_s^2 \nabla^2)\boldsymbol{u}^{(s)} = 0 \qquad (\partial_{tt} - c_d^2 \nabla^2)\boldsymbol{u}^{(d)} = 0 \ . \tag{16}$$

They both oscillate with the same frequency ω (the ω of Eq. (13)). Of course, the $\boldsymbol{u}^{(i)}$ are not exactly bulk modes, because they decay exponentially with z, each over over a different length-scale η_i.

Following the discussion about the different polarizations of the different types of waves, note that we should demand

$$\vec{\nabla} \cdot \boldsymbol{u}^{(s)} = \vec{\nabla} \times \boldsymbol{u}^{(d)} = 0 \ . \tag{17}$$

Plugging the ansatz into equation (17) yields

$$\frac{\partial u_x^{(s)}}{\partial x} + \frac{\partial u_z^{(s)}}{\partial z} = \left(ik\gamma_x^{(s)} + \eta_s \gamma_z^{(s)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(s)}}{\gamma_x^{(s)}} = -i\frac{k}{\eta_s} \ , \tag{18}$$

$$\frac{\partial u_x^{(d)}}{\partial z} - \frac{\partial u_z^{(d)}}{\partial x} = \left(\eta_d \gamma_x^{(d)} - ik\gamma_z^{(d)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(d)}}{\gamma_x^{(d)}} = -i\frac{\eta_d}{k} \ . \tag{19}$$

So we write

$$\boldsymbol{u}^{(s)} = A\left(\eta_s \hat{\boldsymbol{x}} - ik\hat{\boldsymbol{z}}\right) e^{\eta_s z + ikx - i\omega t} \qquad A \in \mathbb{C} \ , \tag{20}$$

$$\boldsymbol{u}^{(d)} = B\left(ik\hat{\boldsymbol{x}} + \eta_d \hat{\boldsymbol{z}}\right) e^{\eta_d z + ikx - i\omega t} \qquad B \in \mathbb{C} \ , \tag{21}$$

We now want to demand that the boundary is traction-free. That is, we want to impose $\sigma_{ij}\big|_{z=0} n_j = 0$, where n_j is the local normal to the deformed surface. In principle, $\hat{n}$ also changes because the surface deforms. However, since $\boldsymbol{\sigma}$ is already first-order in the deformation, we are allowed to take the zeroth order of $\hat{n}$, that is, we can take $\hat{n} = \hat{z}$. Therefore, imposing the traction-free boundary conditions means $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ on $z = 0$. This translates via Hooke's law to

$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu\left(\partial_z u_x + \partial_x u_z\right) = 0 \ , \tag{22}$$

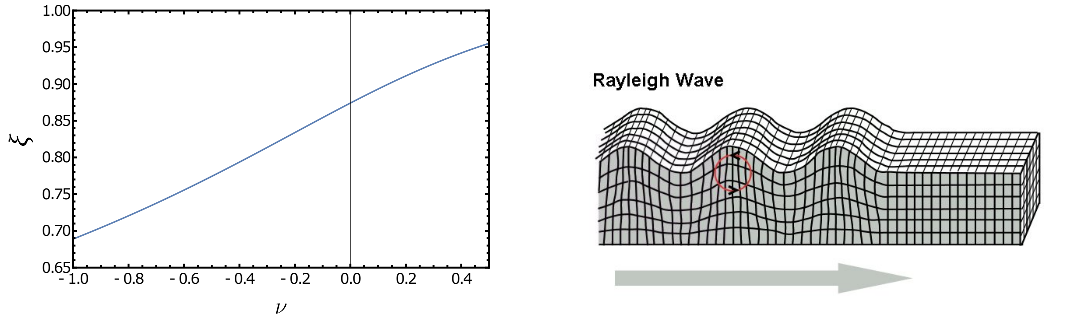
$$\sigma_{zz} = (2\mu + \lambda)\varepsilon_{zz} + \lambda\varepsilon_{xx} = (2\mu + \lambda)\partial_z u_z + \lambda\partial_x u_x = \frac{c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x}{\rho} = 0 \ . \tag{23}$$

We now plug Eqs. (20)-(21) into (22)-(23). This is some uninteresting but necessary algebra. Eq. (22) is relatively simple:

$$0 = \partial_z u_x + \partial_x u_z = \left(\eta_s^2 A + i\eta_d k B\right) + \left(k^2 A + i\eta_d k B\right) = \left(\eta_s^2 + k^2\right) A + 2i\eta_d k B \ , \tag{24}$$

Eq. (23) requires some simplification in order to be sensible:

$$\begin{aligned} 0 &= c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x \\ &= c_d^2\left(\eta_d^2 B - ik\eta_s A\right) - \left(c_d^2 - 2c_s^2\right)\left(ik\eta_s A - k^2 B\right) \\ &= \left(\frac{c_d^2}{c_s^2}\left(\eta_d^2 - k^2\right) + 2k^2\right) B - 2ik\eta_s A \end{aligned} \tag{25}$$



(a) Numerical solution for ξ. (b) A Rayleigh wave (from wikipedia).

Figure 2: Stuff about Rayleigh waves.

But since $(k^2-\eta_i^2)c_i^2=\omega^2$ is the same for both i's, we can replace $\left(\eta_d^2-k^2\right)c_d^2$ in the last equation by $\left(\eta_s^2-k^2\right)c_s^2$ and get

$$\left(\eta_s^2+k^2\right)B-2ik\eta_s A=0\ , \tag{26}$$

Eq. (24) together with (26) form a linear set of equations:

$$\begin{pmatrix} k^2+\eta_s^2 & 2ik\eta_d \\ -2i\eta_s k & k^2+\eta_s^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=0$$

The condition for a non-trivial solution to exist is det $=0$, that is $\left(k^2+\eta_s^2\right)^2=4k^2\eta_s\eta_d$. Plugging in $\eta_i^2=k^2-\left(\frac{\omega}{c_i}\right)^2$ and squaring, this gives

$$\left(2k^2-\frac{\omega^2}{c_s^2}\right)^4=16\,k^4\left(k^2-\frac{\omega^2}{c_s^2}\right)\left(k^2-\frac{\omega^2}{c_d^2}\right)\ . \tag{27}$$

This is the dispersion relation for Rayleigh waves (sometimes this equation is called the Rayleigh equation). It is a very nice and simple dispersion relation because...it is linear! Huh! you didn't see that coming now, did you? Divide both sides by k^8 to get

$$\left(2-\left(\frac{\omega}{kc_s}\right)^2\right)^4=16\left(1-\left(\frac{\omega}{kc_s}\right)^2\right)\left(1-\left(\frac{\omega}{kc_s}\right)^2\left(\frac{c_s}{c_d}\right)^2\right)\ .$$

Denoting the dimensionless phase velocity $\xi=\frac{\omega}{kc_s}=\frac{c_{ph}}{c_s}$ and remembering our definition $\beta=\frac{c_s}{c_d}$ (cf. Eq. (4)), this turns to

$$\left(2-\xi^2\right)^4-16\left(1-\xi^2\right)\left(1-\beta^2\xi^2\right)=0\ .$$

So knowing β, which is a material parameter that equals $\sqrt{\frac{1-2\nu}{2(1-\nu)}}$ gives the (physically unique) solution for ξ and thus completely defines the **linear** dispersion relation $\omega=\xi c_s k$. the solution is shown in Fig. 2a, and it is seen that the wave speed, ξc_s, is somewhat slower than c_s.

1.2.1 Some remarks regarding Rayleigh waves

- Dilational and shear waves travel at two different speeds. Nevertheless, Rayleigh waves couple the two (!) to create a different mode that travels at a third speed (!!), and all this is within a linear theory (!!!).
- The coupling comes from the traction-free boundary condition.
- A single Rayleigh mode with k, ω is a combination of two evanescent bulk modes with the same ω, but different k.
- The bulk modes are evanescent because the velocity of the Rayleigh mode is slower than c_s and c_d. This makes η_s, η_d real. Otherwise, the modes will not be localized on the surface.
- Rayleigh waves are surface waves.Therefore, their magnitude decreases only as $1/\sqrt{r}$ rather than the bulk $1/r$. In large earthquakes, some Rayleigh waves circle the earth a few times before dissipating!
- They are confined to propagate on the surface and decay exponentially with depth. Therefore, the amplitude of earthquake-generated Rayleigh waves is generally a decreasing function of the depth of the earthquake's hypocenter (origin/focus).
- The particle trajectories in a Rayleigh wave are elliptic, much like in ocean surface waves.